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HADRONIC MATHEMATICS, MECHANICS AND CHEMISTRY

Volume III:

Iso-, Geno-, Hyper-Formulations for Matter and Their Isoduals for Antimatter

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This volume is dedicated to

Professor Alvyn van der Merwe

in recognition of his Editorship of Foundations of Physics in disrespect of organized accademic, financial and ethnic interests in science that, otherwise, would have suppressed the birth of undesired advances in human knowledge.

Contents

Foreword

Mathematics is a subject which possibly finds itself in a unique position in academia in that it is viewed as both an Art and a Science. Indeed, in different universities, graduates in mathematics may receive Bachelor Degrees in Arts or Sciences. This probably reflects the dual nature of the subject. On the one hand, it may be studied as a subject in its own right. In this sense, its own beauty is there for all to behold; some as serene as da Vinci's "Madonna of the Rocks", other as powerful and majestic as Michelangelo's glorious ceiling of the Sistine Chapel, yet more bringing to mind the impressionist brilliance of Monet's Water Lily series. It is this latter example, with the impressionists interest in light, that links up with the alternative view of mathematics; that view which sees mathematics as the language of science, of physics in particular since physics is that area of science at the very hub of all scientific endeavour, all other branches being dependent on it to some degree. In this guise, however, mathematics is really a tool and any results obtained are of interest only if they relate to what is found in the real world; if results predict some effect, that prediction must be verified by observation and/or experiment. Again, it may be remembered that physics is really a collection of related theories. These theories are all manmade and, as such, are incomplete and imperfect. This is where the work of Ruggero Santilli enters the scientific arena.

Although "conventional wisdom" dictates otherwise, both the widely accepted theories of relativity and quantum mechanics, particularly quantum mechanics, are incomplete. The qualms surrounding both have been muted but possibly more has emerged concerning the inadequacies of quantum mechanics because of the people raising them. Notably, although it is not publicly stated too frequently, Einstein had grave doubts about various aspects of quantum mechanics. Much of the worry has revolved around the role of the observer and over the question of whether quantum mechanics is an objective theory or not. One notable contributor to the debate has been that eminent philosopher of science, Karl Popper. As discussed in my book, "Exploding a Myth", Popper preferred to refer to the experimentalist rather than observer, and expressed the view that that person played the same role in quantum mechanics as in classical mechanics. He felt, therefore, that such a person was there to test the theory. This is totally opposed to the Copenhagen Interpretation which claims that "objective reality has evaporated" and "quantum mechanics does not represent particles, but rather our knowledge, our observations, or our consciousness, of particles". Popper points out that, over the years, many eminent physicists have switched allegiance from the pro-Copenhagen view. In some ways, the most important of these people was David Bohm, a greatly respected thinker on scientific matters who wrote a book presenting the Copenhagen view of quantum mechanics in minute detail. However, later, apparently under Einstein's influence, he reached the conclusion that his previous view had been in error and also declared the total falsity of the constantly repeated dogma that the quantum theory is complete. It was, of course, this very question of whether or not quantum mechanics is complete which formed the basis of the disagreement between Einstein and Bohr; Einstein stating "No", Bohr "Yes".

However, where does Popper fit into anything to do with Hadronic Mechanics? Quite simply, it was Karl Popper who first drew public attention to the thoughts and ideas of Ruggero Santilli. Popper reflected on, amongst other things, Chadwick's neutron. He noted that it could be viewed, and indeed was interpreted originally, as being composed of a proton and an electron. However, again as he notes, orthodox quantum mechanics offered no viable explanation for such a structure. Hence, in time, it became accepted as a new particle. Popper then noted that, around his (Popper's) time of writing, Santilli had produced an article in which the "first structure model of the neutron" was revived by "resolving the technical difficulties which had led, historically, to the abandonment of the model". It is noted that Santilli felt the difficulties were all associated with the assumption that quantum mechanics applied within the neutron and disappeared when a generalised mechanics is used. Later, Popper goes on to claim Santilli to belong to a new generation of scientists which seemed to him to move on a different path. Popper identifies quite clearly how, in his approach, Santilli distinguishes the region of the arena of incontrovertible applicability of quantum mechanics from nuclear mechanics and hadronics. He notes also his most fascinating arguments in support of the view that quantum mechanics should not, without new tests, be regarded as valid in nuclear and hadronic mechanics.

Ruggero Santilli has devoted his life to examining the possibility of extending the theories of quantum mechanics and relativity so that the new more general theories will apply in situations previously excluded from them. To do this, he has had to go back to the very foundations and develop new mathematics and new mathematical techniques. Only after these new tools were developed was he able to realistically examine the physical situations which originally provoked this lifetime's work. The actual science is his, and his alone, but, as with the realization of all great endeavours, he has not been alone. The support and encouragement he has received from his wife Carla cannot be exaggerated. In truth, the scientific achievements of Ruggero Santiili may be seen, in one light, as the results of a team effort; a team composed of Ruggero himself and Carla Gandiglio in Santilli. The theoretical foundations of the entire work are contained in this volume; a volume which should be studied rigorously and with a truly open mind by the scientific community at large. This volume contains work which might be thought almost artistic in nature and is that part of the whole possessing the beauty so beloved of mathematicians and great artists. However, the scientific community should reserve its final judgement until it has had a chance to view the experimental and practical evidence which may be produced later in support of this elegant new theoretical framework.

Jeremy Dunning-Davies,
Physics Department,
University of Hull,
England.
September 8, 2007

Preface

In Volume I we have identified and denounced scientific imbalance of historical proportions caused by organized academic, financial and ethnic interests on Einsteinian theories via the abuse of academic credibility and public funds to impose the validity of time reversal invariant doctrines for the treatment of irreversible events, including energy releasing processes.

A primary scope of this Volume III is the presentation of the lifelong research by the author on the generalization (called *lifting*) of Einstein's special and general relativities, quantum mechanics and quantum chemistry into forms that are structurally irreversible in time, that is, irreversible for all possible Lagrangians and Hamiltonians, since the latter are known as being all reversible.

It is evident that a task of this type cannot be achieved without the prior lifting of the entire mathematics used in the 20-th century physics, since the latter is all structurally reversible. In turn, as soon as this problem is addressed, the transition from the 20-th century reversible mathematics to its irreversible covering soon emerges as being excessive, particularly for non-mathematically oriented readers.

The latter occurrence has suggested the presentation of a progressive transition from the 20-th century mathematics to a first generalization, today known as *Santilli isomathematics,* where the prefix "iso" is intended in the Greek sense of being "axiom preserving"; the latter mathematics is then lifted into a single-valued structurally irreversible form known as *Santilli genomathematics,* where the prefix "geno" is intended in the Greek meaning of inducing new axioms; and, finally, the latter is lifted into the most general known mathematics, that of multi-valued irreversible type known as *Santilli's hypermathematics.* The corresponding mathematics for antimatter are characterized by the isodual map of Volume II.

Once the above mathematics are known, the construction of the corresponding broader relativities is elementary, yielding formulations today known as *Santilli iso-, geno-, and hyper-relativities* for matter and their isoduals for antimatter. The construction of the underlying iso-, geno-, and hyper-mechanics for matter and their isoduals for antimatter is equally elementary.

This yields a progression of formulations each one being a covering of the preceding one, for the quantitative, axiomatically consistent and invariant representation of matter and of antimatter in conditions of progressively increasing complexity.

As we shall see, besides resolving the historical imbalance on irreversibility, iso-, geno-, and hyper-formulations allow the resolution of numerous additional scientific imbalances of the 20-th century caused by adapting nature to preferred theories for evident personal gains, rather than adapting the theories to new physical reality, as done in these volumes.

Ruggero Maria Santilli
January 19, 2008

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This legal notice has been made necessary because, as shown in Section 1.5, the author has been dubbed "the most plagiarized scientist of the 20-th century," as it is the case of the thousands of papers in deformations published without any quotation of their origination by the author in 1967. These, and other attempted paternity frauds, have forced the author to initiate legal action reported in web site [1].

In summary, honest scientists are encouraged to copy, and/or study, and/or criticize, and/or develop, and/or apply the formulations presented in these volumes in any way desired without any need of advance authorization by the copyrights owner, under the sole conditions of implementing standard ethical rules 2A, 2B, 2C. Dishonest academicians, paternity fraud dreamers, and other schemers are warned that legal actions to enforce scientific ethics are already under way [1], and will be continued after the author's death.

In faith

Ruggero Maria Santilli

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Tarpon Springs, Florida, U. S. A.

October 11, 2007

[1] International Committee on Scientific Ethics and Accountability
http://www.scientificethics.org

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- The late Paul A. M. Dirac, for supporting in a short but memorable meeting reviewed in Section 6.2.8, nonunitary liftings of his celebrated equation (today known as Dirac-Santilli isotopic, genotopic and hyperstructural equations) for the representation of an electron within the hyperdense medium inside the proton, with particular reference to the development of a new mathematics eliminating the vexing divergencies in particle physics, since Dirac spent his last years in attempting the elimination of divergencies amidst strong opposition by organized interests on quantum chromodynamical theologies;

- The late British philosopher Karl Popper, for his strong support in the construction of hadronic mechanics, as shown in the Preface of his last book *Quantum Theory and the Schism in Physics;*

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Chapter 3

LIE-ISOTOPIC BRANCH OF HADRONIC MECHANICS AND ITS ISODUAL

3.1 INTRODUCTION

3.1.1 Conceptual Foundations

As recalled in Chapter 1, the systems generally considered in the 20-th century are the conventional *exterior dynamical systems,* consisting of closed-isolated and reversible systems of constituents approximated as being point-like while moving in vacuum under sole action-at-a-distance potential interactions, as typically represented by planetary and atomic systems.

More technically, we can say that *exterior dynamical systems are characterized by the exact invariance of the Galilean symmetry for the nonrelativistic case and Poincaré symmetry for relativistic treatments,* with the consequential verification of the well known ten total conservation laws.

In this chapter we study the more general *interior dynamical systems of extended particles* and, separately, of *extended antiparticles,* consisting of systems that are also closed-isolated, thus verifying the same ten total conservation laws of the exterior systems, yet admit additional internal force of nonlocal-integral and nonpotential type due to actual contact and/or mutual penetration of particles, as it is the case for the structure of planets at the classical level (see Figure 3.1), and the structure of hadrons, nuclei, stars, and other systems at the operator level (see Figure 3.2).

To avoid excessive complexity, the systems considered in this chapter will be assumed to be *reversible,* that is, invariant under time reversal. The open-irreversible extension of the systems will be studied in the next chapter.

The most important methodological differences between exterior and interior systems are the following:

1) Exterior systems are completely represented with the knowledge of only *one* quantity, the Hamiltonian, while the representation of interior systems requires

the knowledge of the Hamiltonian for the potential forces, plus additional quantities for the representation of nonpotential forces, as done in the *true Lagrange and Hamilton equations,* those with external terms,

$$\frac{d}{dt}\frac{\partial L(t,r,v)}{\partial v_a^k} - \frac{\partial L(t,r,v)}{\partial r_a^k} = F_{ak}(t,r,v), \tag{3.1.1a}$$

$$\frac{dr_a^k}{dt} = \frac{\partial H(t,r,p)}{\partial p_{ak}}, \quad \frac{dp_{ak}}{dt} = -\frac{\partial H(t,r,p)}{\partial r_a^k} + F_{ak}(t,r,p), \tag{3.1.1b}$$

$$L = \Sigma_a \frac{1}{2} \times m_a \times v_{ak} \times v_a^k - V(t,r,v), \tag{3.1.1c}$$

$$H = \Sigma_a \frac{p_{ak} \times p_{ak}}{2 \times m_a} + V(t,r,p), \tag{3.1.1d}$$

$$V = \Sigma_a U(t,r)_{ak} \times v_a^k + U_o(t,r), \tag{3.1.1e}$$

$$F(t,r,v) = F(t,r,p/m), \tag{3.1.1f}$$

$$a = 1,2,3,\ldots,N; \quad k = 1,2,3.$$

Consequently, by their very conception, interior systems are structurally beyond the representational capability of classical and quantum Hamiltonian mechanics, in favor of covering disciplines.

2) Exterior systems are of *Keplerian type*, while interior systems are not, since they do not admit a Keplerian center (see, again, Figures 3.1 and 3.2). Consequently, also by their very conception, interior systems cannot be characterized by the Galilean and Poincaré symmetries in favor of covering symmetries.

3) Exterior systems are local-differential, that is, they describe a finite set of isolated points, thus being fully treatable with the mathematics of the 20-th century, beginning with conventional local-differential topologies. By contrast, interior systems are nonlocal-integral, that is, they admit internal interactions over finite surfaces or volumes that cannot be consistently reduced to a finite set of isolated points. Consequently, interior systems cannot be consistently treated via the mathematics of classical and quantum Hamiltonian mechanics in favor of a basically new mathematics.

4) The time evolution of the Hamiltonian treatment of exterior systems characterizes a *canonical transformation* at the classical level, and a *unitary transformation* at the operator level, that we shall write in the unified form

$$U \times U^\dagger = U^\dagger \times U = I, \tag{3.1.2}$$

where $\times$ represents the usual (associative) multiplication.[1] By contrast, the time evolution of interior systems, being non-Hamiltonian, characterizes *noncanonical transformations* at the classical level and *nonunitary transformations* at the

[1] Since we shall use several types of multiplications, to avoid confusions, it is essential to identify the assumed multiplication in any mathematical treatment.

Figure 3.1. A view of Jupiter, a most representative interior dynamical system, where one can see with a telescope the dramatic differences with exterior systems, such as internal exchanges of linear and angular momentum always in such a way to verify total conservation laws. As repeatedly stated in the literature on hadronic mechanics, the structure of Jupiter has been assumed as fundamental for the construction of new structure models of hadrons, nuclei and stars, and the development of their new clean energies and fuels.

operator level, that we shall jointly write

$$U \times U^\dagger \neq I. \tag{3.1.3}$$

In particular, the noncanonical-nonunitary character is necessary to exit from the class of equivalence of classical and quantum Hamiltonian theories.

5) The *invariance* (rather than "covariance") of exterior systems under the Galilean or Poincaré symmetry has the fundamental implication of preserving the basic units, predicting the same numerical values under the same conditions at different times, and admitting all conditions needed for consistent applications of the theory to experimental measurements. By comparison, the loss of the Galilean and Poincaré invariance, combined with the necessary noncanonical-nonunitary structure of interior systems, activates the *theorems of catastrophic mathematical and physical inconsistencies* studied in Chapter 1 whenever treated with the mathematics of canonical-unitary theories.

In this chapter we report the rather long scientific journey that lead to a mathematically and physically consistent, classical and operator treatment of interior dynamical systems via the *isotopic branch of hadronic mechanics for matter,* and the *isodual isotopic branch for antimatter* including the resolution of all the above problems.

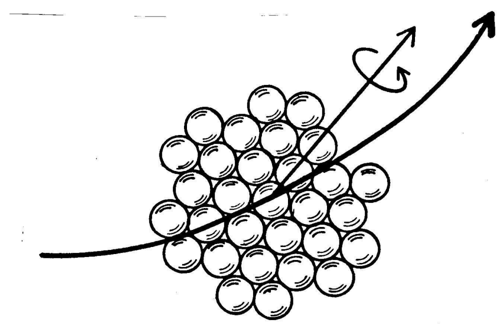

Figure 3.2. A schematic view of nuclei as they are in the physical reality, bound states of extended particles without a Keplerian center, under which conditions quantum mechanics cannot possibly be exact due to the breaking of the fundamental Galilean and Poincaré symmetries in favor of covering theories. As we shall see in this chapter, even though these breakings are small (because nucleons are in conditions of mutual penetration in nuclei of about 10^{-3} parts of their volumes), said breakings permit the prediction and industrial development of new clean energies and fuels that are prohibited by the exact validity of quantum mechanics.

Besides a number of experimental verifications reviewed in this chapter, the achievement of a consistent treatment of interior systems offers basically new structure models of hadrons, nuclei, stars, Cooper pairs, molecules and other interior structures. In turn, these new models permit quantitative studies of new clean energies and fuels already under industrial, let alone scientific development.

Stated in a nutshell, a primary aim of this chapter is to show that the assumption of a final character of quantum mechanics and special relativity beyond the conditions of their original conception (isolated point particles in vacuum) is the primary origin of the current alarming environmental problems.

The reader should be aware that, nowadays, the literature on hadronic mechanics is rather vast, having surpassed the mark of 15,000 pages of published research. As such, to avoid a prohibitive length, *the presentation in this chapter is restricted to the outline of the origination* of each topic and of the *most important developments.* Scholars interested in a comprehensive list of literature are suggested to consult the quoted references as well as those of Chapter 1.

Also to avoid a prohibitive length, the presentation of this chapter is restricted to studies of direct relevance for hadronic mechanics, namely, research fundamentally dependent on a generalization of the basic unit. The quotation of related studies not fundamentally dependent on the generalization of the basic unit cannot be reviewed for brevity.

3.1.2 Closed Non-Hamiltonian Systems

The first step in the study of hadronic mechanics is the dispelling of the belief that nonpotential forces, being nonconservative, do not permit total conservation laws, namely, that the external terms in the analytic equations (3.1.1) solely applies for open-nonconservative systems, such as an extended object moving within a resistive medium considered as external.

This belief was disproved, apparently for the first time, by Santilli in monographs [1,2]. Ref. [1] presented a comprehensive treatment of the integrability conditions for the existence of a potential or a Hamiltonian, *Helmholtz's conditions of variational selfadjointness,* according to which the total force is divided into the following two components

$$F(t,r,p,\ldots) = F^{SA}(t,r,p) + F^{NSA}(t,r,p,\ldots), \tag{3.1.4}$$

where the selfadjoint(SA) component F^{SA} admits a potential and the nonselfadjoint (NSA) component F^{NSA} does not.

We should also recall for clarity that, to be Newtonian as currently understood, a force should solely depend on time t, coordinates r and velocity $v = dr/dt$ or momenta $p = m \times v$, $F = F(t,r,v)$. Consequently, forces depending on derivatives of the coordinates of order bigger than the first, such as forces depending on the acceleration $F = F(t,r,v,a)$, $a = dv/dt$, are not generally considered Newtonian forces.

Ref. [2] then presented the broadest possible realization of the conditions of variational selfadjointness via analytic equations derivable from a variational principle, and included the first known identification of *closed non-Hamiltonian systems* (Ref. [2], pages 233–236), namely, systems that violate the integrability conditions for the existence of a Hamiltonian, yet verify all ten total conservation laws of conventional Hamiltonian systems.

Let us begin by recalling the following well known property:

THEOREM 3.1.1: Necessary and sufficient conditions for a system of N particles to be closed, that is, isolated from the rest of the universe, are that the following ten conservation laws are verified along an actual path

$$\frac{dX_i(t,r,p)}{dt} = \frac{\partial X_i}{\partial b^\mu} \times \frac{db^\mu}{dt} + \frac{\partial X_i}{\partial t} = 0, \tag{3.1.5a}$$

$$X_1 = E_{tot} = H = T + V, \tag{3.1.5b}$$

$$(X_2, X_3, X_4) = \mathbf{P}_{tot} = \Sigma_a \mathbf{p}_a, \tag{3.1.5c}$$

$$(X_5, X_6, X_7) = \mathbf{J}_{tot} = \Sigma_a \mathbf{r_a} \wedge \mathbf{p}_a, \tag{3.1.5d}$$

$$(X_8, X_9, X_{10}) = \mathbf{G}_{Tot} = \Sigma_a (m_a \times \mathbf{r}_a - t \times \mathbf{p}_a), \tag{3.1.5e}$$

$$i = 1, 2, 3, \ldots, 10; \quad k = 1, 2, 3; \quad a = 1, 2, 3, \ldots, N.$$

It is also well known that Galilean or Poincaré invariant systems do verify the above conservation laws since the X_i quantities are the generators of the indicated symmetries. However, in this case all acting forces are derivable from a potential and the systems are Hamiltonian.

Assume now the most general possible dynamical systems, those according to the true Lagrange's and Hamilton equations (3.1.1) where the selfadjoint forces are represented with the Lagrangian or the Hamiltonian and the nonselfadjoint forces are external.

DEFINITION 3.1.1 [2]: Closed-isolated non-Hamiltonian systems of particles are systems of $N \geq 2$ particles with potential and nonpotential forces characterized by the following equations of motion

$$\frac{db_a^\mu}{dt} = \begin{pmatrix} dr_a^k/dt \\ dp_{ka}/dt \end{pmatrix} = \begin{pmatrix} p_{ak}/m_a \\ F_{ka}^{SA} + F_{ka}^{NSA} \end{pmatrix}, \tag{3.1.6}$$

verifying all conditions (3.1.5), where the term "non-Hamiltonian" denotes the fact that the systems cannot be entirely represented with the Hamiltonian, thus requiring additional quantities, such as the external terms.

The case $n = 2$ is exceptional, yet it admits solutions, and closed non-Hamiltonian systems with $N = 1$ evidently cannot exist (because a single free particle is always Hamiltonian).

Closed non-Hamiltonian systems can be classified into:

CLASS α: systems for which Eqs. (3.1.5) are first integrals;

CLASS β: systems for which Eqs. (3.1.5) are invariant relations;

CLASS γ: systems for which Eqs. (3.1.5) are subsidiary constraints.

The case of closed non-Hamiltonian systems of antiparticles are defined accordingly.

The study of closed non-Hamiltonian systems of Classes β and γ is rather complex. For the limited scope of this presentation it is sufficient to see that interior systems of Class α exist.

THEOREM 3.1.2 [2]: Necessary and sufficient conditions for the existence of a closed non-Hamiltonian systems of Class α are that the nonselfadjoint forces verify the following conditions:

$$\sum_a \mathbf{F}_a^{NSA} \equiv 0, \tag{3.1.7a}$$

$$\sum_a \mathbf{p}_a \otimes \mathbf{F}_a^{NSA} \equiv 0, \tag{3.1.7b}$$

$$\sum_a \mathbf{r}_a \wedge \mathbf{F}_a^{NSA} \equiv 0. \tag{3.1.7c}$$

Proof. Consider first the case $N > 2$ and assume first for simplicity that $\mathbf{F}_a^{SA} = 0$. Then, the first nine conservation laws are verified when

$$\frac{\partial X_i}{\partial p_{ka}} \times F_{ka}^{NSA} \equiv 0, \tag{3.1.8}$$

in which case the 10-th conservation law, Eq. (3.1.5e), is automatically verified, and this proves the *necessity* of conditions (3.1.7) for $N > 2$.

The sufficiency of the conditions is established by the fact that Eqs. (3.1.7) consist of seven conditions on $3N$ unknown functions F_{ka}^{NSA}. Therefore, a solution always exists for $N \geq 3$.

The case $N = 2$ is special inasmuch as motion occurs in a plane, in which case Eqs. (3.1.7) reduce to *five* conditions on *four* functions $\mathbf{F}_{ka}^{NSA}$, and the system appears to be overdetermined. Nevertheless, solutions always exist because the verification of the first four conditions (3.1.5) automatically implies the verification of the last one, Eqs. (3.1.5e). As shown in Ref. [2], Example 6.3, pages 272–273, a first solution is given by the *non-Newtonian force*

$$\mathbf{F}_1^{NSA} = -\mathbf{F}_2^{NSA} = K \times a = K \times \frac{dv}{dt}, \tag{3.1.9}$$

where K is a constant. Another solution is given by

$$\mathbf{F}_1^{NSA} = -\mathbf{F}_2^{NSA} = M \times \frac{dr}{dt} \times \phi(M \times \dot{r} + V), \quad M = \frac{m_1 \times m_2}{m_1 + m_2}. \tag{3.1.10}$$

Other solutions can be found by the interested reader. The addition of a non-null selfadjoint force leaves the above proof unchanged. **q.e.d.**

The search for other solutions is recommended to readers interested in acquiring a technical knowledge of hadronic mechanics because such solutions are indeed useful for applications. A general solution of Eqs. (3.1.7), as well as of their operator counterpart and of their isodual images for antimatter will be identified later on in this chapter after the identification of the applicable mathematics.

It should be noted that the proof of Theorem 3.1.2 is not necessary because the existence of closed non-Hamiltonian systems is established by visual observations (Figure 3.1). At any rate, the representation of Jupiter's structure via one single function, the Lagrangian or the Hamiltonian, necessarily implies the belief in the perpetual motion within physical media, due to the necessary condition that constituents move inside Jupiter with conserved energy, linear momentum and angular momentum.

As recalled in Chapter 1, whenever exposed to departures from closed Hamiltonian systems, a widespread posture is the claim that the non-Hamiltonian character of the systems is "illusory" (*sic*) because, when the systems are reduced to their elementary constituents, all nonpotential forces "disappear" (*sic*) and conventional Hamiltonian disciplines are recovered in full.

The political-nonscientific character of the above posture is established by the following property of easy proof by any graduate student in physics:

THEOREM 3.1.3 [3]: A classical non-Hamiltonian system cannot be consistently reduced to a finite number of quantum mechanical point-like particles and, vice-versa, a finite ensemble of quantum mechanical point-like particles cannot consistently characterize a classical non-Hamiltonian system.

The above property establishes that, rather than being "illusory," *nonpotential effect originate at the deepest and most elementary level of nature.* The property also establishes the need for the identification of methods suitable for the invariant treatment of classical and operator non-Hamiltonian systems in such a way to constitute a covering of conventional Hamiltonian treatments.

This chapter is devoted to the mathematical theoretical and experimental study of classical and operator interior system of particles and antiparticles, their experimental verifications and their novel applications.

3.1.3 Need for New Mathematics

By following the main guidelines of hadronic mechanics, we adapt the mathematics to nature, rather than adapting nature to preferred mathematics. For this purpose, we shall seek a mathematics capable of representing the following main features of interior dynamical systems:

1) Points have no dimension and, consequently can only have action-at-a-distance potential interactions. Therefore, the first need for the new mathematics is the representation of the *actual, extended, generally nonspherical shape of the wavepackets and/or of the charge distribution of the particles considered,* that we shall assume in this monograph for simplicity to have the shape of spheroidal ellipsoids with diagonal form

$$Shape_a = Diag.(n_{a1}^2, n_{a2}^2, n_{a3}^2), \quad a = 1, 2, 3, \ldots, N, \tag{3.1.11}$$

with more general non-diagonal expressions not considered for simplicity, where $n_{a1}^2, n_{a2}^2, n_{a3}^2$ represent the semiaxes of the spheroidal ellipsoids assumed as *deviation* from, or normalized with respect to the perfect sphericity

$$n_{a1}^2 = n_{a2}^2 = n_{a3}^2 = 1. \tag{3.1.12}$$

The n's are called *characteristic quantities* of the particles considered. It should be stressed that, contrary to a rather popular belief, *the n-quantities are not parameters because they represent the actual shape as derived from experimental measurements.*

To clarify this important point, by definition a "parameter" can assume any value as derived form the fit of experimental data, while this is not the case for the characteristic quantities here considered. As an example, the use for the n's of value of the order of 10^{-16} cm to represent a proton would have no physical value because the proton charge distribution is a spheroidal ellipsoid of the order of 10^{-13} cm.

2) Once particles are assumed as being extended, there is the consequential need to represent their *density.* This task can be achieved via a fourth set of quantities

$$Density_a = n_{a4}^2, \tag{3.1.13}$$

representing the *deviation* of the density of the particle considered from the density of the vacuum here assumed to be one,

$$n_{Vacuum,4}^2 = 1. \tag{3.1.14}$$

Again, n_4 is not a free parameter because its numerical value is fixed by experimental data. As an example for the case of a hadron of mass m and radius r we have the density

$$n_4^2 = \frac{m \times c^2}{\frac{4}{3} \times \pi \times r^3}, \tag{3.1.15}$$

thus establishing that n_{a4} is not a free parameter capable of assuming.

Predictably, most nonrelativistic studies can be conducted with the sole use of the space components characterizing the shape. Relativistic treatments require the additional use of the density as the forth component, resulting in the general form

$$(Shape - Density)_a = Diag.(n_{a1}^2, n_{a2}^2, n_{a3}^2, n_{a4}^2), \quad a = 1, 2, 3, \ldots, N. \tag{3.1.16}$$

3) Perfectly rigid bodies exist in academic abstractions, but not in the physical reality. Therefore, the next need is for a meaningful representation of the *deformation of shape* as well as *variation of density* that are possible under interior conditions. This is achieved via the appropriate functional dependence of the characteristic quantities on the energy E_a, linear momentum p_a, pressure P and other characteristics, and we shall write

$$n_{ak} = n_{ak}(E, p, P, \ldots), \quad k = 1, 2, 3, 4. \tag{3.1.17}$$

The reader is suggested to meditate a moment on the fact that Lagrangian or Hamiltonian theories simply cannot represent the actual shape and density of particles. The impossibility of representing deformations of shapes and variations of density are well known, since the pillar of contemporary relativities, the rotational symmetry, is notoriously incompatible with the theory of elasticity.

4) Once particles are represented as they are in the physical reality (extended, nonspherical and deformable), there is the emergence of the following new class of interactions nonexistent for point-particles (for which reason these interactions have been generally ignored throughout the 20-th century), namely, interactions of:

I) *contact type,* that is, due to the actual physical contact of extended particle; consequently, of

II) *zero range type,* since all contacts are dimensionless; consequently of

III) *nonpotential type,* that is, not representable with any possible action-at-a-distance potential; consequently, of

IV) *non-Hamiltonian type,* that is, not representable with any Hamiltonian; consequently, of

V) *noncanonical type* at the classical level and *nonunitary type at the operator level;* as well as of

VI) *nonlinear type,* that is, represented via nonlinear differential equations, such as depending on power of the wavefunction greater than one; and, finally, of

VII) *nonlocal-integral type.* Interactions among point-particles are local-differential, that is, reducible to a finite set of isolated points, while contact interactions among extended particles and/or their wavepackets are, by conception, nonlocal-integral in the sense of being dependent on a finite surface or volume that, as such, cannot be reduced to a finite set of isolated points (see Figure 3.3).

5) Once the above new features of interior systems have been identified, there is the need not only of their mathematical representation, but above all of their *invariant representation* in order to avoid the theorem of catastrophic inconsistencies of Chapter 1.

As an illustration, Coulomb interactions have reached their towering position in the physics of the 20-th century because *the Coulomb potential is invariant under the basic symmetries of physics,* thus predicting the same numerical values under the same conditions at different times with consequentially consistent physical applications. The same occurs for other interactions derivable from a potential (except gravitation represented with curvature as shown in Section 1.4).

Along the same lines, any representation of the extended, nonspherical and deformable character of particles, their densities and their novel nonlinear, nonlocal and nonpotential interactions cannot possibly have physical value unless it is also *invariant,* and not "covariant," again, because the latter would activate the theorems of catastrophic inconsistencies of Chapter 1.

It should be indicated that an extensive search conducted by the author in 1978–1983 in the advanced libraries of Cambridge, Massachusetts, identified nu-

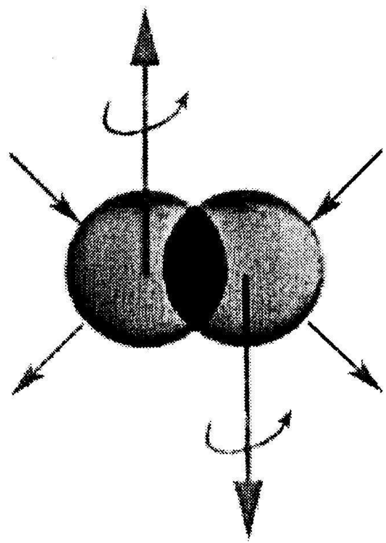

Figure 3.3. A schematic view of the fundamental interactions studied in this monograph, those originating from deep wave-overlappings of the wavepackets of particles also for the case with point-like charge as occurring in electron valence bonds, Cooper pairs in superconductivity, Pauli's exclusion principle, and other basic structures. These interactions have been ignored throughout the 20-th century, resulting in the problematic aspects or sheer inconsistencies identified in Chapter 1. As we shall see in this chapter, the representation of the new interactions here depicted with generalized units of type (3.1.19) permits the achievement of the first known, exact and invariant representation of molecular data and other data that have escaped an exact and invariant representation via quantum mechanics for about one century.

merous integral geometries and other nonlocal mathematics. However, none of them verifies all the following conditions necessary for physical consistency:

CONDITION 1: *The new nonlocal-integral mathematics must admit the conventional local-differential mathematics as a particular case under a well identified limit procedure,* because new physical advances must be a covering of preceding results. This condition alone is not verified by any integral mathematics the author could identify.

CONDITION 2: *The new nonlocal-integral mathematics must permit the clear separation of the contributions of the new nonlocal-integral interactions from those of local-differential interactions.* This second condition too was not met by any of the integral mathematics the author could identify.

CONDITION 3: *The new nonlocal-integral mathematics must permit the invariant formulation of the new interactions.* This latter condition was also violated by all integral mathematics the author could identify, thus ruling them out in a final form for consistent physical applications.

After clarifying that the mathematics needed for the correct treatment of interior systems was absent, the author was left with no other choice than that of constructing the needed mathematics. After extensive search, Santilli [4,5] suggested as the *only* possible or otherwise known solution, the invariant representation of nonlinear, nonlocal and nonpotential interactions via a generalization of the trivial unit of conventional theories. The selection was based on the fact that, whether conventional or generalized, the unit is the basic invariant of any theories. We reach in this way the following:

FUNDAMENTAL ASSUMPTION OF HADRONIC MECHANICS [4-10]: The actual, extended, nonspherical and deformable shape of particles, their variable densities and their nonlinear, nonlocal and nonpotential interactions can be invariantly represented with a generalization of the basic spacetime unit of conventional Hamiltonian theories

$$I = Diag.(1, 1, 1, 1), \tag{3.1.18}$$

into nowhere singular, sufficiently smooth, most general possible integro-differential forms, today called "Santilli isounit", of the type here expressed for simplicity for the case of two particles:

$$\hat{I} = \hat{I}^\dagger = \hat{I}_{1-2} = Diag.(n_{11}^2, n_{12}^2, n_{13}^2, n_{14}^2)\times$$
$$\times Diag.(n_{21}^2, n_{22}^2, n_{23}^2, n_{24}^2)\times$$
$$\times e^{\Gamma(t,r,\psi,\psi^\dagger,\dots)\times\int dr^3\times\psi^\dagger(r)\times\psi(r)} = 1/\hat{T} > 0, \tag{3.1.19}$$

with trivial generalizations to multiparticle and nondiagonal forms, where the n_{ak}^2 *represents the semiaxes of the spheroidal shape of particle* a, n_{a4}^2 *represents its density, the expression* $\Gamma(t, r, \psi, \psi, \dots)$ *represents the nonlinearity of the interaction and* $\int dr^3 \times \psi^\dagger(r) \times \psi(r)$ *provides a simple representation of its nonlocality. The corresponding features of antiparticles are represented by Santilli's isodual isounit*

$$\hat{I}^d = -\hat{I}^\dagger = -\hat{I} < 0, \tag{3.1.20}$$

and mixed states of particles and antiparticles are represented by the tensorial product of the corresponding units and their isoduals.

Explicit examples of classical (operator) systems with nonpotential forces represented via generalized units will be given in Section 2.3 (Section 2.4).

As we shall see, the entire structure of hadronic mechanics follows uniquely and unambiguously from the assumption of the above basic unit. As a matter of fact, some of the main features of hadronic mechanics can already be derived from the above basic assumption.

First, the maps, called in the literature *Santilli liftings*

$$I \ \rightarrow \ \hat{I}, \ \ I^d \ \ \rightarrow \ \ \hat{I}^d; \tag{3.1.21}$$

(where $I^d = -I$ is the isodual unit of Chapter 2 [8]) require two corresponding generalizations of the totality of the mathematical and physical formulations of conventional classical and quantum Hamiltonian theories without any exception known to this author (to avoid catastrophic inconsistencies).

As we shall see in this chapter, even basic notions such as trigonometric functions, Fourier transforms, differentials, etc. have to be lifted into two forms admitting the new quantity $\hat{I}$ and $\hat{I}^d$ as the correct left and right units.

In view of the assumed Hermiticity and positive-definiteness of $\hat{I}$, the resulting new mathematics is called in the literature *Santilli's isotopic mathematics* or *isomathematics* for short, with the corresponding *isodual isomathematics* for antimatter in interior conditions. The resulting new physical formulations are known as *Santilli isotopic mechanics* or *isomechanics* for short for the case of particles, with the *isodual isomechanics* for antiparticles.

Again in view of the fact that $\hat{I}$ is Hermitian and positive-definite, at the abstract, realization-free level there is no topological difference between I and $\hat{I}$ and, for this reason $\hat{I}$ is called *Santilli isotopic unit* or *isounit* for short.

Consequently, the new mathematical and physical formulations are expected to be *new realizations of the same axioms of conventional Hamiltonian mechanics,* and they should not be intended as characterizing "new theories" since they do not admit new abstract axioms. This illustrates the name of *isotopic mathematics* from the Greek meaning of preserving the topology.[2]

Finally, Santilli isounit $\hat{I}$ identifies in full the *covering* nature of isomechanics over conventional mechanics, as well as the type of resulting covering. This covering character is illustrated by the fact that at sufficiently large mutual distances of particles the integral in the exponent of Eq. (3.1.19) is null

$$\lim_{r \gg 1\,\mathrm{Fm}} \int dr^3 \times \psi^\dagger(r) \times \psi(r) = 0, \tag{3.1.22}$$

in which case the actual shape of particles has no impact in the interactions and the generalized unit recovers the conventional unit[3]

$$\lim_{r \gg 1\,\mathrm{Fm}} \hat{I} = I = Diag.(1,\,1,\,1,\,1), \tag{3.1.23}$$

[2]When $\hat{I}$ is no longer Hermitian, we have the more general *genotopic mathematics* studied in Chapter 4.
[3]When the exponent of Eq. (3.1.19) is null, that is, when the mutual distances of particles are large, the characteristic quantities are constant and, consequently, terms such as $Diag.(n_{11}^{-2},\,n_{12}^{-2},\,n_{13}^{-2},n_{14}^{-2})$ factor out of all equations, resulting in reduction (3.1.23).

under which limit hadronic mechanics recovers conventional quantum mechanics identically and uniquely.

The above limits also identify the important feature according to which *hadronic mechanics coincides with quantum mechanics for all mutual distances of particles sufficiently bigger than their wavepackets, while at mutual distances below that value hadronic mechanics provides a generally small corrections to quantum mechanics* (see Figure 3.3).

In this chapter we review the long and laborious scientific journey by mathematicians, theoreticians and experimentalists (see the bibliography of Chapter 1) for the achievement of maturity of formulation of the isotopic branch of hadronic mechanics, its experimental verification, its novel industrial applications, and its isodual for antimatter.

We shall begin with a review of recent developments in the construction of isomathematics that have occurred following the publication of the second edition of Vol. I of this series in 1995 [6] since these developments have important implications. We shall then identify the recent developments in physical theories occurred since the second edition of Vol. II of this series [7]. We shall then review the novel industrial applications developed since the appearance of Volumes I and II.

It should be noted that in this chapter we shall merely present recent developments. As a consequence, Volumes I and II of this series [6,7] remain useful for all detailed aspects that will not be repeated in this final volume.

A primary motivation of this volume is to present *industrial applications.* Consequently, we have selected the simplest possible mathematical treatment accessible to any experimentalists. Readers interested in utmost mathematical rigor are suggested to consult the specialized mathematical literature in the field.

Finally, the literature on the mathematics, physics and chemistry of classical and quantum Hamiltonian theories is so vast to discourage discriminatory quotations. For this reason, unless there is a contrary need, we shall abstain from quotations of works on pre-existing methods since their knowledge is a pre-requisite for the understanding of this monograph in any case.

3.2 ELEMENTS OF SANTILLI'S ISOMATHEMATICS AND ITS ISODUAL

3.2.1 Isounits, Isoproducts and their Isoduals

As indicated earlier, *Santilli isotopic mathematics*, [4–10] or *isomathematics* for short, is characterized by the map, called *lifting,* of the trivial unit $I = +1$ into a generalized unit $\hat{I}$

N-dimensional unit

$$I = +1 \rightarrow \hat{I}(t, r, p, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \dots), \tag{3.2.1}$$

or, more generally, by the lifting of N-dimensional units

$$I = (I^i_j) = Diag.(1, 1, 1, \ldots),\ i, j = 1, 2, \ldots, N$$

of conventional Hamiltonian theories[4] into a nowhere singular, Hermitian and positive-definite, matrix $\hat{I}$ of the same dimension N whose elements $\hat{I}^i_j$ have an arbitrary, nonlinear and integral dependence on time t, space coordinates r, momenta p, wavefunctions ψ, their derivatives $\partial\psi$, and any other needed quantity [*loc. cit.*]

$$I \ = \ (I^i_j) = Diag.(1, 1, \ldots) > 0 \rightarrow$$
$$\rightarrow \hat{I} \ = \ (\hat{I}^j_i) \ = \ \hat{I}(t, r, p, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \ldots) \ = \ 1/\hat{T} \ > \ 0. \tag{3.2.2}$$

Isomathematics can then be defined as the lifting of all possible branches of mathematics with left and right unit I into forms admitting $\hat{I}$ as the new left and right unit.

Recall that I is the right and left unit under the conventional *associative product* $A \times B = AB$, where A, B are generic quantities (e.g., numbers, vector-fields, operators, *etc.*) for which $I \times A = A \times I = A$ for all element A of the considered set.

It is easy to see that $\hat{I}$ cannot be a unit under the same product because $\hat{I} \times A \neq A$. Therefore, for consistency, the conventional associative product $A \times B$ must be lifted into the new form first proposed by Santilli in Ref. [5] of 1978,

$$A \times B \rightarrow A \hat{\times} B = A \times \hat{T} \times B = A \times (1/\hat{I}) \times B, \tag{3.2.3}$$

where $\hat{T}$ is fixed for the set considered, under which product $\hat{I}$ is indeed the correct left and right new unit,

$$I \times A = A \times I = A \rightarrow \hat{I} \hat{\times} A = A \hat{\times} \hat{I} = A, \tag{3.2.4}$$

for all elements A of the considered set. In this case (only) $\hat{I}$ is called *Santilli's isotopic unit*, or *isounit* for short, and $\hat{T}$ is called *Santilli's isotopic element,* or *isoelement* for short.

Isomathematics was first submitted by Santilli in memoirs [*loc. cit.*] of 1978 and then worked out in various additional contributions by the same author, as well as by numerous mathematicians and theoreticians (see the references of Chapter 1 as well as of this section).

[4]For instance, Hamiltonian theories in 3-dimensional Euclidean space are based on the unit $I = Diag.(1, 1, 1)$ of the rotational and Euclidean symmetries, while Hamiltonian theories in Minkowski space are based on the unit $I = Diag.(1, 1, 1, 1)$ that is at the foundation of Lie's theory of the Lorentz and Poincaré symmetries.

The most salient feature of Santilli's liftings (3.2.2) and (3.2.3) is that they are *axiom preserving,* from which feature they derived their name "isotopic" [*loc. cit.*], recently contracted to the prefix "iso."

In fact, $\hat{I}$ preserves the basic topological characteristics of I. Therefore, isomathematics is expected to provide *new realizations* of the abstract axioms of the mathematics admitting I as left and right unit. In particular, the preservation of the original abstract axioms is an important guiding principle in the consistent construction of isomodels and their applications.

At this introductory stage the axiom-preserving character of generalized product (3.2.3) is easily verified by the fact that it preserves all basic axioms of the original product. In fact, the isoproduct verifies the *right and left isoscalar laws*

$$n\hat{\times}(A\hat{\times}B) = (n\hat{\times}A)\hat{\times}B, \qquad (3.2.5a)$$

$$(A\hat{\times}B)\hat{\times}n = A\hat{\times}(B\hat{\times}n), \qquad (3.2.5b)$$

the *right and left isodistributive laws*[5]

$$A\hat{\times}(B + C) = A\hat{\times}B + A\hat{\times}C, \qquad (3.2.6a)$$

$$(A + B)\hat{\times}C = A\hat{\times}C + B\hat{\times}C, \qquad (3.2.6b)$$

and the *isoassociative law*

$$A\hat{\times}(B\hat{\times}C) = (A\hat{\times}B)\hat{\times}C. \qquad (3.2.7)$$

A verification of the preservation of the axioms of all subsequent constructions is crucial for a serious study and application of hadronic mechanics.

The simplest method for the construction of isomathematics as needed for various applications is given by the use of a positive-definite N-dimensional *noncanonical transform* at the classical level or a *nonunitary transform* at the operator level, here written in the unified form

$$U \times U^\dagger \neq I, \qquad (3.2.8)$$

and its identification with the basic isounit of the theory

$$\hat{I} = U \times U^\dagger = 1/\hat{T} > 0, \qquad (3.2.9)$$

realization first introduced by Santilli in Ref. [6,7] of 1993.

[5]The reader should keep in mind that the verification of the right and left scalar and distributive laws are necessary for any product to characterize an *algebra* as commonly understood in contemporary mathematics.

In this case, the Hermiticity of $\hat{I}$ is guaranteed because of the property,

$$(U \times U^\dagger)^\dagger = U \times U^\dagger. \tag{3.2.10}$$

Therefore, realization (3.2.9) of the isounit only requires that $U \times U^\dagger$ be a positive-definite N-dimensional matrix other than the unit matrix, from which the nowhere singularity follows, e.g., via condition

$$\mathrm{Det}(U \times U^\dagger) > 0, \neq I. \tag{3.2.11}$$

Once the fundamental realization (3.2.9) is assumed, the construction of isomathematics follows in a simple, unique and unambiguous way. In fact, *isomathematics can be constructed by submitting conventional mathematics with left and right unit I to said noncanonical-nonunitary transform,* with very few exception, such as the isodifferential calculus that escapes construction via noncanonical-nonunitary transforms.

To begin, the isounit itself is simply given by said noncanonical-nonunitary transform of the conventional unit,

$$I \rightarrow U \times I \times U^\dagger = \hat{I}, \tag{3.2.12}$$

the isoproduct too is simply given by said noncanonical-nonunitary transform of the conventional product

$$\begin{aligned} A \times B \rightarrow U \times (A \times B) \times U^\dagger = \\ = (U \times A \times U^\dagger) \times (U \times U^\dagger)^{-1} \times (U \times B \times U^\dagger) = \\ = \hat{A} \times \hat{T} \times \hat{B} = \hat{A} \hat{\times} \hat{B}, \end{aligned} \tag{3.2.13}$$

and the same simple transform holds for the construction of other aspects of isomathematics, as illustrated in this section.

As a matter of fact, the use of the above transform provides a method for the construction of isomathematics that is more rigorous than empirical liftings. For instance, by comparing Eqs. (3.2.3) and (3.2.13), we see that the lifting of the unit $I \rightarrow \hat{I} = U \times I \times U^\dagger$ implies not only the lifting of the associative product $\times \rightarrow \hat{\times} = \times (U \times U^\dagger)^{-1} \times$, but also the lifting of all elements of the set considered, $A \rightarrow \hat{A} = U \times A \times U^\dagger$.

In view of the above, the claim often expressed in the nontechnical physics literature that "the mathematics of hadronic mechanics is too difficult to comprehend" is just a case of venturing judgment without any serious knowledge of the topic.

The reader should be aware that other generalizations of the associative product, such as

$$A \bigotimes B = \hat{T} \times A \times B, \tag{3.2.14a}$$

$$A \bigodot B = A \times B \times \hat{T}, \tag{3.2.14b}$$

are unacceptable because they violate either the right or the left distributive and scalar laws, thus being unable to characterize an algebra. As such, liftings (3.2.14) are not isotopic in Santilli's sense [*loc. cit.*].

Examples of isounits have been given in Section 3.1.3. Additional examples will be provided in Sections 3.3 and 3.4. Note that, since they are Hermitian by assumption, isounits can always be diagonalized into the form of type (3.1.19).

Santilli isodual isomathematics [6–10] is the image of isomathematics under the anti-isomorphic *isodual map* of an arbitrary quantity

$$\begin{aligned} A(t, r, p, \psi, \psi^\dagger, \ldots) &\rightarrow A^d(t^d, r^d, p^d, \psi^d, \psi^{\dagger d}, \ldots) \\ &\rightarrow -A^\dagger(-t, -r^t, -p^t, -\psi^\dagger, -\psi^\dagger, \ldots), \end{aligned} \tag{3.2.15}$$

(where t denotes transposed) first submitted by Santilli in Ref. [8] of 1985 (see also Chapter 2).

The basic quantity of isodual isomathematics is then the *isodual isounit*

$$\hat{I}^d = -\hat{I}^\dagger(-t, -r^\dagger, -p^\dagger, -\psi^\dagger, -\partial\psi^\dagger, \ldots) = 1/\hat{T}^d. \tag{3.2.16}$$

Similarly, we have the *isodual isoproduct*

$$B^\dagger \times \hat{T}^d \times A^\dagger = B^\dagger \hat{\times}^d A^\dagger, \tag{3.2.17}$$

under which $\hat{I}^d$ is indeed the right and left unit,

$$\hat{I}^d \hat{\times}^d A = A \hat{\times}^d \hat{I}^d = A, \tag{3.2.18}$$

for all A of the considered set.

Note that, *isodual map (3.2.15) must be applied for consistency to the totality of quantities of isomathematics as well as of their operations.* As an illustration, the application of the isodual map only to the quantities A, B of a product $A \times B$ and not to the product itself $\times$, leads to a host of inconsistencies.

For this and other reasons the conventional associative product is written in this monograph with the explicit notation $A \times B$ rather than the conventional notation AB. In fact, the latter would lead to gross misunderstandings and inconsistencies under the various liftings of hadronic mechanics.

Also, the construction of isomathematics is indeed recommended for physicists to be done via a noncanonical-nonunitary transform (3.2.9), while the construction of isodual isomathematics is recommended via the isodual map (3.2.15) and not via the use of an anti-isomorphic transform.

In fact, the use of anti-isomorphic transforms causes ambiguities in the very central issue, the achievement of equivalence of the isodual operator theory with charge conjugation due to ambiguities and other technical aspects. In turn, this occurrence illustrates the significance and uniqueness of Santilli isodual map (3.2.15).

Note also that isodual isomathematics preserves the axioms, not of conventional mathematics, but of the isodual mathematics of Chapter 2, that with the simplest possible isounit unit $I^d = -I$.

Needless to say, mathematicians do not need the above elementary construction of isomathematics and its isodual since they can be formulated on abstract realization-free grounds from basic axioms.

3.2.2 Isonumbers, Isofields and their Isoduals

The first necessary isotopic lifting following that of the basic unit and product, is that of ordinary numbers. The resulting new numbers were first presented by Santilli at the 1980 meeting in Clausthal, Germany, on *Differential Geometric Methods in Mathematical Physics* and then published in a variety of papers, such as Ref. [8] of 1985, Vols. [15,16] of 1991, memoir [9] of 1993 and other works. A comprehensive presentation is available in Vol. I [6] of 1995 that also presents industrial applications of the new numbers for cryptograms and other fields. As a result of these contributions the new numbers are today known as *Santilli's isonumbers.*

The new numbers have also been studied by various authors. An important contribution has been made by E. Trell [11] in 1998 consisting in a proof of Fermat's celebrated theorem that is the simplest on record and, therefore, credibly conceivable by Fermat (as compared to other proof requiring mathematics basically unknown during Fermat's time). Unfortunately, Fermat left no record of the proof of his celebrated theorem and, therefore, there is no evidence that Fermat first studied numbers with arbitrary units. Nevertheless, Trell's proofs of Fermat's theorems remains the most plausible known to this author for being conceived during Fermat's time.

Numerous additional studies on isonumbers have been conducted by other authors. For a complete bibliography we refer interested readers to the monograph on *Santilli isonumber theory* by C.-X. Jiang [12] of 2002. Additional studies on isonumbers have occurred for their use as basis of other isostructures. Related references will be quoted in the appropriate subsequent sections.

Santilli's isonumbers have also been subjected to a generalization called *pseudo-isonumbers* identified in Ref. [9] and studies by various authors, including N. Kamiya [13] and others. However, the latter generalization violates the axioms of a field and, as such, it cannot be used for hadronic mechanics.

The reader should be aware that in this section we merely present the minimal possible properties of isonumbers sufficient for industrial applications.

Let us consider: the field $R(n, +, \times)$ of *real numbers* n with ordinary sum $+$ and product $\times$; the field $C(c, +, \times)$ of *complex numbers* $c = n_1 + i \times n_2$ where i is the imaginary unit and $n_1, n_2 \in R$; and the field $Q(q, +, \times)$ of *quaternions* $q = i_o + i_1 \times n_1 + i_2 \times n_2 + i_3 \times n_3$, where i_o is the 2-dimensional unit matrix, $i_k, k = 1, 2, 3$ are Pauli's matrices and $n_1, n_2, n_3 \in R$. These fields are hereon represented with the unified notation[6]

$$F(a, +, \times) : a = n,\ c,\ q, \tag{3.2.19}$$

In this section we present first the simplest possible method for the lifting of numbers via the use of a positive-definite (thus invertible) noncanonical-nonunitary transform identified with Santilli's isounit

$$I \rightarrow \hat{I} = U \times I \times U^\dagger = 1/\hat{T} > 0, \quad U \times U^\dagger \neq I. \tag{3.2.20}$$

We shall then pass to a mathematical presentation.

The isotopic lifting of ordinary numbers is easily achieved via the above map resulting in *Santilli isonumbers* for the characterization of *matter*

$$a \rightarrow \hat{a} = U \times a \times U^\dagger = a \times (U \times U^\dagger) = a \times \hat{I}, \tag{3.2.21}$$

and related *isoproduct*

$$a \times b \rightarrow U \times (a \times b) \times U^\dagger = \hat{a} \times \hat{T} \times \hat{b} = \hat{a} \hat{\times} \hat{b}, \tag{3.2.22}$$

under which $\hat{I}$ is the correct right and left isounit, Eq. (3.2.4), with the element *isozero* coinciding with the ordinary zero

$$0 \rightarrow \hat{0} = U \times 0 \times U^\dagger \equiv 0, \tag{3.2.23}$$

and, consequently, the *isosum* coinciding with the ordinary sum,

$$a + b \rightarrow U \times (a + b) \times U^\dagger = \hat{a} \hat{+} \hat{b} \equiv \hat{a} + \hat{b}. \tag{3.2.24}$$

The above liftings result in: *Santilli isofield* $\hat{R}(\hat{n}, \hat{+}, \hat{\times})$ of *isoreal isonumbers*; the isofield $\hat{C}(\hat{c}, \hat{+}, \hat{\times})$ of *isocomplex isonumbers*; and the isofield $\hat{Q}(\hat{q}, \hat{+}, \hat{\times})$ of *isoquaternionic isonumbers*; hereon represented with the unified notation

$$\hat{F}(\hat{a}, \hat{+}, \hat{\times}), \quad \hat{a} = \hat{n}, \hat{c}, \hat{q}. \tag{3.2.25}$$

Needless to say, the liftings of the unit and of the product require a corresponding lifting of all conventional operations of numbers depending on the

[6]Octonions are not considered "numbers" because they violate the associativity property of the axioms of a field.

multiplication. By using the above noncanonical-nonunitary map, one can easily prove the *isopowers*

$$\hat{a}^{\hat{n}} = \hat{a} \hat{\times} \hat{a} \hat{\times} \ \ldots \ \hat{\times} \hat{a} \ \ (n \ \text{ times}) = a^n \times \hat{I}. \tag{3.2.26}$$

An important particular case is the property that *isopowers of the isounits reproduce the isounit identically,*

$$\hat{I}^{\hat{n}} = \hat{I} \hat{\times} \hat{I} \hat{\times} \ldots \hat{\times} \hat{I} \equiv \hat{I}. \tag{3.2.27}$$

Similarly we have the *isosquare isoroot*

$$\hat{a}^{\widehat{1/2}} = a^{1/2} \times \hat{I}^{1/2}; \tag{3.2.28}$$

the *isoquotient*

$$\hat{a} \hat{/} \hat{b} = (\hat{a}/\hat{b}) \times \hat{I} = (a/b) \times \hat{I}; \tag{3.2.29}$$

and the *isonorm*

$$\hat{|}\hat{a}\hat{|} = |a| \times \hat{I}, \tag{3.2.30}$$

where $|a|$ is the conventional norm. All these properties were first introduced by Santilli in Refs. [6–9]. The reader can now easily construct the desired isotopic image of any other operation on numbers.

Despite their simplicity, isonumbers are nontrivial. As an illustration, the assumption of the isounit $\hat{I} = 3$ implies that "2 multiplied by 3" = 18, while 4 becomes a prime number.

The best way to illustrate the nontriviality of the new numbers is to indicate the **industrial applications of Santilli's isonumbers,** that are a primary objective of this monograph as indicated earlier.

To begin, *all* applications of hadronic mechanics are based on isonumbers, and they will be presented later on in this chapter. In addition to that, *Santilli's isonumbers have already found a direct industrial application consisting of the isotopic lifting of cryptograms used by the industry to protect secrecy, including banks, credit cards. etc.* This industrial application was first presented by Santilli in Appendix 2.C of the second edition of Vol. I [6] of 1995, and will be reviewed later on in this chapter.

At this moment we merely mention that all cryptograms based on the multiplication depend on only one value of the unit, the quantity +1 dating back to biblical times. A mathematical theorem establishes that a solution of any cryptogram can be identified in a finite period of time. As a result of this occurrence, banks and other industries are forced to change continuously their cryptograms to properly protect their secrecy.

By comparison, *Santilli's isocryptograms* are based on the isoproduct and, as such, they admit an *infinite number of possible isounits,* such as, for instance, the values

$$\hat{I} = 7.2; \ \ 0.98364; \ \ 236; \ \ 1,293' \ 576; \ \ \text{etc.} \tag{3.2.31}$$

Consequently, it remains to be seen whether Santilli isocryptograms can be broken in a finite period of time under the availability of an infinite number of possible isounits.

Independently from that, with the use of isocryptograms banks and other industries do not have to change the entire cryptogram for security, but can merely change the value of the isounit to keep ahead of possible hackers, and even that process can be computerized for frequent automatic changes of the isounit, with clearly added safety.

Finally, another application of Santilli isocryptograms permitted by their simplicity is their use to protect the access to personal computers.

It is hoped this illustrates the industrial significance of Santilli isonumbers *per se,* that is, independently from their basic character for hadronic mechanics.

We now pass to a mathematical presentation of the new numbers.

DEFINITION 3.2.1 [9]: Let $F = F(a, +, \times)$ be a field of characteristic zero as per Definition 2.1.1. Santilli's isofields are rings $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times})$ with: elements

$$\hat{a} = a \times \hat{I}, \tag{3.2.32}$$

where $a \in F$, $\hat{I} = 1/\hat{T}$ is a positive-definite quantity generally outside F and $\times$ is the ordinary product of F; the isosum $\hat{+}$ coincides with the ordinary sum $+$,

$$\hat{a} \hat{+} \hat{b} \equiv \hat{a} + \hat{b}, \ \ \forall\, \hat{a}, \hat{b} \in \hat{F}, \tag{3.2.33}$$

consequently, the element $\hat{0} \in \hat{F}$ coincides with the ordinary $0 \in F$; and the isoproduct $\hat{\times}$ is such that $\hat{I}$ is the right and left isounit of $\hat{F}$,

$$\hat{I} \hat{\times} \hat{a} = \hat{a} \hat{\times} \hat{I} \equiv \hat{a}, \ \ \forall\, \hat{a} \in \hat{F}. \tag{3.2.34}$$

Santilli's isofields verify the following properties:

1) For each element $\hat{a} \in \hat{F}$ there is an element $\hat{a}^{-\hat{1}}$, called isoinverse, for which

$$\hat{a} \hat{\times} \hat{a}^{-\hat{1}} = \hat{I}, \ \ \forall\, \hat{a} \in \hat{F}; \tag{3.2.35}$$

2) The isosum is isocommutative

$$\hat{a} \hat{+} \hat{b} = \hat{b} \hat{+} \hat{a}, \tag{3.2.36}$$

and isoassociative

$$(\hat{a} \hat{+} \hat{b}) + \hat{c} = \hat{a} \hat{+} (\hat{b} \hat{+} \hat{c}), \ \ \forall \hat{a}, \hat{b}, \hat{c} \in \hat{F}; \tag{3.2.37}$$

3) The isoproduct is not necessarily isocommutative

$$\hat{a} \hat{\times} b \neq \hat{b} \hat{\times} \hat{a}, \tag{3.2.38}$$

but isoassociative

$$\hat{a}\hat{\times}(\hat{b}\hat{\times}\hat{c}) = (\hat{a}\hat{\times}\hat{b})\hat{\times}\hat{c}, \quad \forall \hat{a}, \hat{b}, \hat{c} \in \hat{F}; \tag{3.2.39}$$

4) The set $\hat{F}$ is closed under the isosum,

$$\hat{a}\hat{+}\hat{b} = \hat{c} \in \hat{F}, \tag{3.2.40}$$

the isoproduct,

$$\hat{a}\hat{\times}\hat{b} = \hat{c} \in \hat{F}, \tag{3.2.41}$$

and right and left isodistributive compositions,

$$\hat{a}\hat{\times}(\hat{b}\hat{+}\hat{c}) = \hat{d} \in \hat{F}, \tag{3.2.42a}$$

$$(\hat{a}\hat{+}\hat{b})\hat{\times}c = \hat{d} \in \hat{F}, \quad \forall \hat{a}, \hat{b}, \hat{c}, \hat{d} \in \hat{F}; \tag{3.2.42b}$$

5) The set $\hat{F}$ verifies the right and left isodistributive law

$$\hat{a}\hat{\times}(\hat{b}\hat{+}\hat{c}) = (\hat{a}\hat{+}\hat{b})\hat{\times}\hat{c} = \hat{d}, \ \forall \hat{a}, \hat{b}, \hat{c}, \hat{d} \in \hat{F}. \tag{3.2.43}$$

Santilli's isofields are called of the first (second) kind when $\hat{I}$ is (is not) an element of F.

The basic axiom-preserving character of the isotopies of numbers is illustrated by the following:

LEMMA 3.2.1 [9]: Isofields of first and second kind are fields (namely, they verify all axioms of a field).

Note that the isotopic lifting does indeed change the *operation* of the multiplication but not that of the sum because the isotopies here considered do change the multiplicative unit I, but not the additive unit 0, Eq. (3.2.23). This is a crucial property of hadronic mechanics best illustrated by the following property:

LEMMA 3.2. [9]: Nontrivial liftings of the additive unit 0 *and related sum violates the axioms of a field (for which reason, they are called "pseudoisofields")*

In fact, suppose that one wants to change the value of the element 0, e.g.,

$$0 \to \hat{0} = K \neq 0, \ \ K \in F. \tag{3.2.44}$$

Then, for $\hat{0}$ to remain the new additive unit, one must alter the sum into a new form admitting $\hat{0}$ as left and right additive unit, e.g.,

$$a\hat{+}b = a + (-\hat{0}) + b, \tag{3.2.45}$$

under which

$$a\hat{+}\hat{0} = \hat{0}\hat{+}a \equiv a, \quad \forall a \in F. \tag{3.2.46}$$

However, there is no single lifting of the product such that

$$\hat{0}\hat{\times}a \neq \hat{0}, \quad \forall a \in F, \tag{3.2.47}$$

under which there is the loss of the distributive axiom of a field, i.e.,

$$(a\hat{+}b) \times c \neq a \times c \ \hat{+} \ b \times c. \tag{3.2.48}$$

In turn, the loss of the distributive law causes very serious physical inconsistencies, such as preventing experimental applications of the theory. Therefore, *being axiom-preserving, hadronic mechanics is solely based on the isotopic lifting of the multiplicative unit and related product, but not on any lifting of the additive unit and related sum.*

Santilli's isodual isonumbers for the characterization of *antimatter* can be uniquely and unambiguously characterized via the isodual map (3.2.15). They are characterized by the *additive and multiplicative isodual isounit*

$$\hat{0} \rightarrow \hat{0}^d \equiv 0, \tag{3.2.49a}$$

$$\hat{I}^d = -\hat{I} < 0, \tag{3.2.49b}$$

where one should recall that $\hat{I}$ is real valued and positive-definite, thus Hermitian. Isodual isonumbers are then explicitly given by

$$\hat{a}^d = -\hat{a}^\dagger = -\hat{I} \times \hat{a}^\dagger. \tag{3.2.50}$$

The isodual isonumbers were first introduced by Santilli in Ref. [8] of 1985, treated mathematically in Ref. [9] of 1993 and studied extensively in Vol. I of this series [6].

The use of the same isodual map then identifies the *isodual isosum*

$$\hat{a}^d \hat{+}^d \hat{b}^d = \hat{a}^d + \hat{b}^d, \tag{3.2.51}$$

the *isodual isoproduct*

$$(\hat{a}\hat{\times}\hat{b})^d = \hat{b}^d \times^d \hat{T}^d \times^d \hat{A}^d = -\hat{b}^d \hat{\times} \hat{a}^d = -\hat{b}^\dagger \hat{\times} \hat{a}^\dagger, \tag{3.2.52}$$

and the *isodual isonorm*

$$\hat{|}\hat{a}\hat{|}^d = -\hat{|}\hat{a}\hat{|} = -|a| \times \hat{I}. \tag{3.2.53}$$

that is always *negative-definite.*

The above liftings result in: *Santilli's isodual isofield* $\hat{R}^d(\hat{n}^d, \hat{+}^d, \hat{\times}^d)$ of *isodual isoreal isonumbers*; the isodual isofield $\hat{C}^d(\hat{c}^d, \hat{+}^d, \hat{\times}^d)$ of *isodual isocomplex isonumbers*; and the isodual isofield $\hat{Q}^d(\hat{q}^d, \hat{+}^d, \hat{\times}^d)$ of *isodual isoquaternionic isonumbers*; hereon represented with the unified notation

$$\hat{F}^d(\hat{a}^d, \hat{+}^d, \hat{\times}^d), \hat{a}^d = \hat{n}^d, \hat{c}^d, \hat{q}^d. \tag{3.2.54}$$

DEFINITION 3.2.3 [9]: Let $\hat{F}(\hat{a}, \hat{+}, \hat{\times})$ be an isofield as per Definition 3.2.1. Then Santilli isodual isofields $\hat{F}^d(\hat{a}^d, \hat{+}^d, \hat{\times}^d)$ are the image of $\hat{F}$ under the isodual map (3.2.15).

LEMMA 3.2.3 [9]: Isodual isofields are fields (that is, they verify all axioms of a field).

LEMMA 3.2.4 [9]: Isodual isofields are anti-isomorphic to isofields.

As we shall see in this chapter, the latter property, jointly with the anti-isomorphic character of the isodual map, will result to be crucial for a consistent treatment of antimatter composed of extended particles with potential and non-potential internal forces.

The above properties establish the fact (first identified in Ref. [8]) that, by no means, the axioms of a field require that the multiplicative unit to be the trivial unit $+1$, because the basic unit can be a negative-definite quantity -1 as it occurs for the isodual mathematics of Chapter 2, an arbitrary positive-definite quantity $\hat{I} > 0$ as occurring in isomathematics, or an arbitrary negative-definite quantity $\hat{I}^d < 0$ as it occurs for the isodual isomathematics.

The reader should be aware that an in depth knowledge of Santilli's isonumbers and their isoduals requires an in depth study of memoir [9] or of Chapter 2 of Vol. I of this series, Ref. [6], and that an in depth knowledge of Santilli's isonumbers theory requires a study of Jiang's monograph [12].

Finally, the reader should meditate a moment on the viewpoint expressed several times in this writing to the effect that *there cannot be really new physical theories without new mathematics, and there cannot be really new mathematics without new numbers.* The basic novelty of hadronic mechanics can, therefore, be reduced to the novelty of Santilli's isonumbers.

By remembering that all "numbers" have been fully identified centuries ago, the novelty of hadronic mechanics can be reduced to the discovery that the axioms of conventional fields admit new realizations with nonsingular, but otherwise arbitrary multiplicative units.

3.2.3 Isospaces and Their Isoduals

Following the lifting of units, products and fields, the next necessary lifting is that of N-dimensional *metric or pseudo-metric spaces* with local coordinates r and Hermitian, thus diagonalized metric m over a field F, here written in the unified notation

$$S(r, m, F): \ r = (r^k), \ m = [m_{ij}(r, \ldots)] = Diag.(m_{11}, m_{22}, \ldots, m_{NN}), \quad i, j, k = 1, 2, \ldots, N, \tag{3.2.55}$$

basic invariant

$$r^2 = (r^i \times m_{ij} \times r^j) \times I = (r^t \times m \times r) \times I \quad \in \quad F(a, +, \times), \tag{3.2.56}$$

(where t stands for transposed) and fundamental N-dimensional unit[7]

$$I = Diag.(1, 1, \ldots, 1). \tag{3.2.57}$$

As now familiar, isotopies are based on the lifting of the above N-dimensional unit via a positive-definite noncanonical-nonunitary transform in the same dimension with an otherwise unrestricted functional dependence

$$I = Diag.(1, 1, \ldots, 1) \to \hat{I}(t, r, p, \psi, \psi^\dagger, \ldots) = U \times I \times U^\dagger = 1/\hat{T} > 0, \tag{3.2.58}$$

The above liftings requires that of spaces $S(r, m, R)$ into *isotopic spaces,* or *isospaces* for short, for the treatment of *matter,* hereon denoted $\hat{S}(\hat{r}, \hat{M}, \hat{F})$, where $\hat{r}$ denotes the *isocoordinates,* and $\hat{M}$ denotes the *isometric* defined on the isofields $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times})$ of Section 3.2.2.

Isospaces were first proposal by Santilli in Ref. [14] of 1983 for the axiom-preserving isotopies of the Minkowskian spacetime and special relativity that are at the foundations of hadronic mechanics. Isospaces were then used by Santilli for the liftings of the various spacetime and internal symmetries (such as $SU(2)$, $SO(3)$, $SO(3.1)$, $SL(2.C)$, $G(3.1)$, $P(3.1)$, $SU(3)$, etc.) as studied later on in this chapter.

A comprehensive presentation of isospaces first appeared in monographs [15,16] of 1991 and in the first edition of Volumes I and II of this series, Ref. [6,7] of 1993 (see the second edition of 1995 for various upgradings). A mathematical study of isospaces by Santilli was presented in memoir [10] of 1996. In view of all these contributions, the new spaces are today known as *Santilli's isospaces.*

Following the appearances of these contributions, isospaces have been also studied by a number of authors for both mathematical and physical applications to be studied in subsequent sections, including the definition of isocontinuity,

[7] The basic character of the unit should be recalled here. For the case of the three-dimensional Euclidean space, $I = Diag.(1, 1, 1)$ is not only the basic geometric unit, but also the unit of the entire Lie theory of the rotational and Euclidean symmetries. Similarly, for the case of the Minkowski spacetime, the unit $I = Diag.(1, 1, 1, 1)$ is at the foundations of the entire Lie theory for the Lorentz and Poincaré symmetries. We begin to see in this way the far reaching implications of isotopic generalization of the basic unit.

isotopology, isomanifolds, etc. The related literature will be presented in the appropriate subsequent sections.

In this section we identify the basic notions of Santilli isospaces. Specific types of isospaces needed for applications will be studied in subsequent sections.

The coordinates r of ordinary spaces $S(r,m,F)$ are defined on the base field $F = F(a,+,\times)$, thus being real numbers for $F = R$, complex numbers for $F = C$ and quaternionic numbers for $F = Q$.

Consequently, the *isocoordinates* $\hat{r}$ on isospaces $\hat{S}(\hat{r},\hat{m},\hat{F})$ must be defined on the isofields $\hat{F} = \hat{F}(\hat{a},\hat{+},\hat{\times})$, namely, must be *isonumbers* and, more particularly, be isoreal isonumbers for $\hat{F} = \hat{R}$, isocomplex isonumbers for $\hat{F} = \hat{C}$, and isoquaternionic isonumbers for $\hat{F} = \hat{Q}$.

Since isocoordinates are isonumbers, they can be easily constructed via the same lifting used for isonumbers, resulting in the simple definition

$$r \to \hat{r} = U \times r \times U^\dagger = r \times (U \times U^\dagger) = r \times \hat{I}. \tag{3.2.59}$$

Similarly, the metric m on $S(r,m,F)$ is an ordinary matrix in N-dimension whose elements m_{ij} are functions defined on the base field F, thus being real, complex or quaternionic functions depending on the corresponding character of F.

As we shall see shortly, a necessary condition for $\hat{S}(\hat{r},\hat{M},\hat{F})$ to preserve the geometric axioms of $S(r,m,F)$ (that is, for $\hat{S}$ to be an isotope of S), is that, when the unit is lifted in the amount $I \to \hat{I} = 1/\hat{T}$, the metric is lifted by the *inverse* amount $m \to \hat{m} = \hat{T}\times m$, thus yielding the transform (where the diagonal character of m is taken into account)

$$\begin{aligned} m \to U^{\dagger -1} \times m \times U^{-1} &= (U \times U^\dagger)^{-1} \times m = \\ &= \hat{T} \times m = (\hat{m}_{ij}) = (\hat{T}_i^k \times m_{kj}), \end{aligned} \tag{3.2.60}$$

However, in this case the elements $\hat{m}_{ij}$ are not properly defined on $\hat{S}$ because they are not isonumbers on $\hat{F}$. For this purpose, the correct definition of the *isometric* is given by

$$\hat{M} = \hat{m} \times \hat{I} = (\hat{m}_{ij} \times \hat{I}) = (\hat{m}_{ij}) \times \hat{I}. \tag{3.2.61}$$

As we shall see in the next section, the above definition is independently confirmed by the isotopies of matrices. We, therefore, have the following

DEFINITION 3.2.3 [14]: Let $S(r,m,F)$ be an N-dimensional metric or pseudometric space with contravariant coordinates $r = (r^k)$, metric $m = (m_{ij})$ and invariant $r^2 = (r_k \times r^k) \times I = (r^i \times m_{ij} \times r^j) \times I$ over a field F with trivial unit I. Then, Santilli's isospaces are the N-dimensional isovector spaces

$$\hat{S}(\hat{r},\hat{M},\hat{F}): \quad \hat{r} = (\hat{r}^k) = (r^k) \times \hat{I} \ \in \hat{F}, \tag{3.2.62a}$$

$$\hat{M} = (\hat{T} \times m) \times \hat{I} = (T_i^k \times m_{ki}) \times \hat{I} \in \hat{F}, \quad \hat{M}^{ij} = [(\hat{M}_{pq})^{-1}]^{ij} \in \hat{F}, \quad (3.2.62b)$$

$$\hat{r}^k = \hat{M}^{ki} \hat{\times} \hat{r}_i = \hat{m}^{ki} \times r_i \times \hat{I}, \quad \hat{r}_k = \hat{M}_{ki} \hat{\times} \hat{r}^i = \hat{m}_{ki} \times r^i \times \hat{I}, \quad (3.2.62c)$$

$$\hat{r}^{\hat{2}} = \hat{r}^k \hat{\times} \hat{r}_k = \hat{r}^i \hat{\times} \hat{M}_{ij} \hat{\times} \hat{r}^j = (r^i \times \hat{m}_{ij} \times r^j) \times \hat{I} \in \hat{F}, \quad (3.2.62d)$$

$$i, j, k, p, q = 1, 2, \ldots, N,$$

and its projection on the original space $S(r, m, F)$, is characterized by

$$\hat{S}(r, \hat{m}, F): \quad r = (r^k) = (r^k) \times I \in F; \quad (3.2.63a)$$

$$\hat{m} = \hat{T} \times m = (\hat{T}_i^k \times m_{kj}) \in F, \quad \hat{m}^{ij} = [(\hat{m}_{ps})^{-1}]^{ij} \in F, \quad (3.2.63b)$$

$$r^k = \hat{m}^{ki} \times r_i \in R, \quad r_k = \hat{m}_{ki} \times r^1 \in F, \quad (3.2.63c)$$

$$r^2 = r^i \times \hat{m}_{ij} \times r^j) \times I = r^i \times (\hat{T}_i^k \times m_{kj}) \times r^j) \times I \in F. \quad (3.2.63d)$$

As one can see, expression (3.2.62) is the proper formulation of the isoinvariant on isospaces over the base isofield, and we shall write $\hat{S}(\hat{r}, \hat{M}, \hat{F})$, while expression (3.2.63) is the "projection" of the preceding space in the original space S, and we shall write $\hat{S}(r, \hat{m}, F)$, because the latter space is defined with conventional coordinates, units and products over the conventional field F by construction.

It should be stressed that *isospaces are mathematical spaces and, therefore, all physical calculations and applications will be done in the projection of isospaces over conventional spaces.* In fact, experimental measurements and events can only occur in our space time. Therefore, all physical applications of isospaces can only occur in their projection in our spacetime.

A simple visual inspection of invariants (3.2.56) and (3.2.62) establish the following

THEOREM 3.2.1 [10]: All line elements of metrics or pseudo-metric spaces with metric m and unit I, and all their isotopes possess the following invariance property

$$I \to \hat{I} = n^2 \times I, \quad m \to \hat{m} = n^{-2} \times m, \quad (3.2.64)$$

where n is a non-null parameter.

This property too will soon acquire fundamental character, since it permits the identification, for the first time, of the property that *the Galilean and Poincaré symmetries are "eleven" dimensional,* and not ten-dimensional as believed throughout the 20-th century.

In particular, the 11-th invariance is "hidden" in conventional line elements and will permit the first and perhaps only known, axiomatically consistent grand unification of electroweak and gravitational interactions, as studied later on in this chapter.

The nontriviality of isospaces is then expressed by the following

THEOREM 3.2.2 [14]: Even though preserving all topological properties of m (from the positive-definiteness of $\hat{I}$), the projection $\hat{m}$ of the isometric $\hat{M}$ on $\hat{S}$ over $\hat{F}$ into the original space S over F acquires an unrestricted functional dependence on any needed local variables or quantities,

$$\hat{M} \rightarrow \hat{m} = \hat{m}(t, r, p, \psi, \psi^{\dagger}, \ldots). \tag{3.2.65}$$

As we shall see, the above property has truly fundamental implications, since it will permit the first and only known *geometric unification of the Minkowskian and Riemannian geometries with the consequential unification of special and general relativities,* and other applications of manifestly fundamental nature.

By recalling that the basic invariant r^2 represents the square of the "distance" in S, from Eqs. (3.2.56) and (3.2.62) we derive the following additional property

THEOREM 3.2.3 [6,7,10]: The basic invariant of a metric or pseudometric space has the structure:

$$Invariant \;\; = \;\; [Length]^2 \times [Unit]^2 \tag{3.2.66}$$

The above property will soon have deep geometric implications, such as permitting different shapes, sizes and dimension for the same object under inspection by different observers, all in a way compatible with our sensory perception.

Note that invariant structure (3.2.66) is indeed new because identified for the first time by the isotopies, since the multiplication of the invariant by the unit is trivial for conventional studies and, as such, it was ignored.

It is now important to indicate the *differences between Santilli isospaces $\hat{S}(\hat{r}, \hat{M}, \hat{F})$ or $\hat{S}(r, \hat{m}, F)$ and deformed spaces* that, as well known, are given by the sole deformations of the metric, for which we use the notation $S(r, \hat{m}, F)$.

It is easy to see that *deformed spaces $S(r, \hat{m}, F)$ have a conventional noncanonical or nonunitary structure,* thus activating the theorems of catastrophic inconsistencies of Section 3.4. By comparison, Santilli isospaces have been constructed precisely to resolve these catastrophic inconsistencies via the reconstruction of canonicity or unitarity on isospaces over isofields.

Moreover, *deformed metric spaces $S(r, \hat{m}, F)$ necessarily break the symmetries of the original spaces $S(r, m, F)$, while, as we shall soon see, isospaces $\hat{S}(\hat{r}, \hat{M}, \hat{F})$ reconstruct the exact symmetries of $S(r, g, F)$.*

The implications of the latter property alone are far reaching because *all symmetries believed to be broken in the 20-th century can be proved to remain exact* *on suitable isospaces over isofields.* In different terms, the “breakings of space-time and internal symmetries” studies through the 20-th century are a direct manifestation of the adaptation of new physical events to a rather limited, pre-existent mathematics because, if the underlying mathematics is suitably lifted, all believed breakings cease to exist, as already proved in Vol. II of this series [7] and updated in this volume.

Santilli's isodual isospaces for the treatment of *antimatter* are the anti-isomorphic image of isospaces under the isodual map (3.2.15) and can be written

$$\hat{S}^d(\hat{r}^d, \hat{M}^d, \hat{F}^d): \quad \hat{r}^d = -\hat{r}^\dagger, \quad \hat{M}^d = -\hat{M}, \tag{3.2.67a}$$

$$\hat{r}^{\hat{2}d} = \hat{r}^d \hat{\times}^d \hat{M}^d \hat{\times}^d \hat{r}^{t,d}. \tag{3.2.67b}$$

Isodual isospaces were introduced in Vol. I of this series [6] and then treated in various other works (see, e.g., [10,17,18]). As we shall see, they play a crucial role for the treatment of antimatter in interior conditions. The tensorial product of isospaces and their isoduals appears to be significant for basic advances in biology, e.g., to achieve a quantitative mathematical representation of bifurcations and other biological behavior.

As we shall see, all **industrial applications** of hadronic mechanics are based on isospaces to such an extent that the new isogeometries have acquired evident relevance for new patents assuredly without prior art, evidently in view of their novelty.

3.2.4 Isofunctional Analysis and its Isodual

The lifting of fields evidently requires a corresponding lifting of functional analysis into a form known as *Kadeisvili isofunctional analysis* since it was first studied by J. V. Kadeisvili [19,20] in 1992. Additional studies were done by A. K. Aringazin *et al.* [21] in 1995 and other authors.

A detailed study of isofunctional analysis was also provided in monographs [6,7] of 1995. A knowledge of these studies is necessary for any application of hadronic mechanics because all conventional functions and transforms have to be properly lifted for consistent applications, while the use of conventional (or improperly lifted) functions and transforms leads to catastrophic inconsistencies.

In essence, the consistent formulation of isofunctional analysis requires not only the preservation of the original axioms, but also the preservation of the original numerical values when formulated on isospaces over isofields, under which conditions the broadening of conventional formulations emerge in the projection of the isotopic treatment in the original space.

The latter mathematical requirement has deep physical implications, such as the preservation of the speed of light *in vacuum* as the universal invariant on *isospaces over isofield,* with consequential preservation under isotopies of all axioms of special relativity, while locally varying speeds of light within physical media emerge in the *projection* of the isospace in our spacetime, as we shall see in subsequent sections.

The scope of this section is essentially that of providing the guidelines for the updating of Refs. [19,20,16,6,7] along the above requirements to achieve compatibility with the main lines of this presentation.

DEFINITION 3.2.4 [19,20,21, 6,7] Let $f(x)$ be an ordinary (sufficiently smooth) function on a vector space S with local variable x (such as a coordinate) over the reals R. The isotopic image of $f(x)$, called isofunctions, can be constructed via the use of a noncanonical-nonunitary transform

$$U \times f(x) \times U^{\dagger} = f(x) \times \hat{I} \in \hat{F}, \tag{3.2.68}$$

reformulated on isospace $\hat{S}(\hat{x}, \hat{F})$ over the isofield $\hat{F}$

$$f(x) \times \hat{I} = f(\hat{T} \times \hat{x}) \times \hat{I} = \hat{f}(\hat{x}) \in \hat{F}, \tag{3.2.69}$$

with projection in the original space $S(x, F)$

$$f(\hat{T} \times x) \in F. \tag{3.2.70}$$

As one can see, expression (3.2.68) coincides with the definition of isofunction in the quoted references. A feature identified since that time is the re-interpretation in such a way that the function $f(x)$ preserves its numerical value when formulated as $\hat{f}(\hat{x})$ on the isospace $\hat{S}$ over the isofield $\hat{F}$ because the variable $\hat{x}$ is multiplied by $\hat{T}$ while the unit to which such a variable is referred to is multiplied by the *inverse* amount $\hat{I} = 1/\hat{T}$. All numerical differences emerge in the *projection* of $\hat{f}(\hat{x})$ in the original space.

This is essentially the definition of isofunctions that will allow us to preserve the basic axioms of special relativity on isospaces over isofields and actually expand their applicability from motion in empty space to motion within physical media.

For the case of the simple function $f(x) = x$ we have the lifting

$$\hat{x} = U \times x \times U^{\dagger} = x \times (U \times U^{\dagger}) = x \times \hat{I} = \hat{T} \times \hat{x} \times \hat{I} \in \hat{F}, \tag{3.2.71}$$

with the projection in the original space S being simply given in this case by $\hat{T} \times x$.

More instructive is the lifting of the exponentiation into the *isoexponentiation* given by

$$e^x \rightarrow U \times e^x \times U^{\dagger} =$$
$$= U \times (I + x/1! + x \times x/2! + \ldots) \times U^{\dagger} =$$
$$= \hat{I} + \hat{x}\hat{/}\hat{1}! + \hat{x}\hat{\times}\hat{x}\hat{/}\hat{2}! + \ldots) =$$

$$= \hat{e}^{\hat{x}} = (e^{\hat{x}\times\hat{T}}) \times \hat{I} = \hat{I} \times (e^{\hat{T}\times\hat{x}}) \ \in \ \hat{F}, \tag{3.2.72}$$

with projection in the original space S given by

$$\hat{e}^{x} = (e^{x} \times \hat{T}) \times I = I \times (e^{\hat{T}\times x}) \ \in F, \tag{3.2.73}$$

where one should note that the function in isospace is computed over $\hat{F}$ while its projection in the original space is computed in the original field F.

The above lifting is nontrivial because of the appearance of the nonlinear integro-differential quantity $\hat{T}(t, x, \psi, \partial\psi, \ldots)$ in the exponent. As we shall see shortly, this feature permits the first known extension of the linear and local Lie theory to nonlinear and nonlocal formulations.

Let $M(x) = (M_{ij}(x))$ be an N-dimensional matrix with elements $M_{ij}(x)$ on a conventional space $S(x, F)$ with local coordinates x over a conventional field F with unit I. Then, the isotopic image of $M(x)$ or it isomatrix, is defined by

$$\hat{M}(\hat{x}) = (\hat{M}_{ij}(\hat{x})) = M(\hat{T} \times \hat{x}) \times \hat{I}, \ \ \hat{M}_{ij} \in \hat{F}, \tag{3.2.74}$$

Similarly, the *isodeterminant* of $\hat{M}$ is defined by

$$\hat{\mathrm{Det}}\hat{M} = [\mathrm{Det}(\hat{T} \times M)] \times \hat{I} \tag{3.2.75}$$

where Det represents the conventional determinant, with the preservation of the conventional axioms, e.g.,

$$\hat{\mathrm{Det}}(\hat{M}_1 \hat{\times} \hat{M}_2) = \hat{\mathrm{Det}}(\hat{M}_1) \hat{\times} \hat{\mathrm{Det}}(\hat{M}_2); \tag{3.2.76a}$$

$$\hat{\mathrm{Det}}(\hat{M}^{-\hat{I}}) = (\hat{\mathrm{Det}}\hat{M})^{-\hat{I}}, \tag{3.2.76b}$$

Note that, by construction, isomatrices and isodeterminant preserve the original values on isospaces over isofields, although show deviations when the same quantities are observed from the original space, that is, referred to the original unit.

Similarly, the *isotrace* of $\hat{M}$ is defined by[8]

$$\hat{Tr}\hat{M} = [Tr(\hat{T} \times M)] \times \hat{I}, \tag{3.2.77}$$

where Tr is the conventional trace, and it also verifies the conventional axioms, such as

$$\hat{Tr}(\hat{M}_1 \hat{\times} \hat{M}_2) = \hat{Tr}\hat{M}_1 \hat{\times} \hat{Tr}\hat{M}_2, \tag{3.2.78a}$$

[8]The isodeterminant introduced in Ref. [6], Eq. (6.3.19) is the correct form as in Eq. (3.2.77) above. However, the isotrace introduced in Eq. (6.3.20a) of Ref. [6] preserves the axioms of a trace, but not its value, as a consequence of which it is not fully invariant, the correct definition of isotrace being given by Eq. (3.2.77) above.

$$\hat{Tr}(\hat{M}^{-\hat{I}}) = (\hat{Tr}\hat{M})^{-\hat{I}}. \tag{3.2.78b}$$

The *isologarithm* is hereon defined by[9]

$$\hat{\log}_{\hat{e}}\hat{a} = \log_e a \times \hat{I}, \tag{3.2.79}$$

and admit the unique solution

$$\hat{\log}_{\hat{e}}\hat{a} = \log_e(\hat{T} \times a) \times \hat{I}, \tag{3.2.80}$$

under which the conventional axioms are preserved,

$$\hat{e}^{\hat{\log}_{\hat{e}}\hat{a}} = \hat{a}, \tag{3.2.81a}$$

$$\hat{\log}_{\hat{e}}\hat{e} = \hat{I}, \ \ \hat{\log}_{\hat{e}}\hat{I} = 0, \tag{3.2.81b}$$

$$\hat{\log}_{\hat{e}}(\hat{a}\hat{\times}\hat{b}) = \hat{\log}_{\hat{e}}\hat{a} + \hat{\log}_{\hat{e}}\hat{b}, \tag{3.2.81c}$$

$$\hat{\log}_{\hat{e}}(\hat{a}\hat{/}\hat{b}) = \hat{\log}_{\hat{e}}\hat{a} - \hat{\log}_{\hat{e}}\hat{b}, \tag{3.2.81d}$$

$$\hat{\log}_{\hat{e}}(\hat{a}^{-\hat{I}}) = -\hat{\log}_{\hat{e}}\hat{a}, \tag{3.2.81e}$$

$$\hat{b}\hat{\times}\hat{\log}_{\hat{e}}\hat{a} = \hat{\log}_{\hat{e}}(\hat{a}^{\hat{b}}). \tag{3.2.81f}$$

The lifting of trigonometric functions is intriguing and instructive (see Chapter 6 of Ref. [6] and Chapter 5 of Ref. [7] whose results in this case require no upgrading). Let $E(r, \delta, R)$ be a conventional two-dimensional Euclidean space with coordinates $r = (x, y)$ on the reals R and polar representation $x = r \times \cos\theta$ and $y = r \times \sin\theta$, $x^2 + y^2 = r^2 \times (\cos^2\theta + \sin^2\theta) = r^2$. Consider now the *iso-Euclidean space* in two dimension

$$\hat{E}(\hat{r}, \hat{\delta}, \hat{R}) : \ \ \hat{\delta} = Diag.(n_1^{-2}, n_2^{-2}), \ \ \hat{I} = Diag.(n_1^2, n_2^2), \tag{3.2.82a}$$

$$\hat{r}^{\hat{2}} = (x^2/n_1^2 + y^2/n_2^2) \times \hat{I} \ \ \in \ \ \hat{R}. \tag{3.2.82b}$$

Then, the *isopolar coordinates* and related *isotrigonometric functions* on $\hat{E}$ are defined by

$$\hat{x} = \hat{r}\hat{\times}\hat{\cos}\hat{\phi}, \tag{3.2.83a}$$

$$\hat{\cos}\hat{\phi} = n_1 \times \cos(\phi/n_1 \times n_2), \tag{3.2.83b}$$

$$\hat{y} = \hat{r}\hat{\times}\hat{\sin}\hat{\phi}, \tag{3.2.83c}$$

$$\hat{\sin}\hat{\phi} = n_2 \times \sin(\phi/n_1 \times n_2), \tag{3.2.83d}$$

and they preserve the axioms of conventional trigonometric functions, such as,

$$\hat{r}^{\hat{2}} = (x^2/n_1^2 + y^2/n_2^2) \times \hat{I} \ \ = r^2 \times \hat{I} \in \ \ \hat{R}. \tag{3.2.84}$$

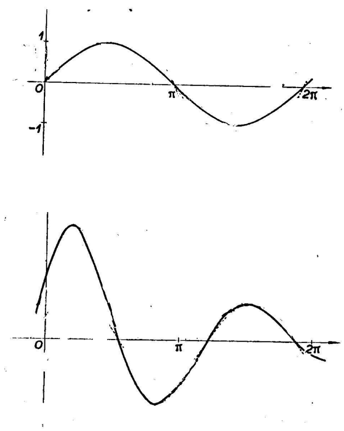

Figure 3.4. A schematic view of the conventional sinus function in Euclidean and iso-Euclidean spaces (top view) and of the projection of a possible example of the isosinus function in the conventional space.

The isotopy of spherical coordinates are treated in detail in Section 5.5 of Ref. [7]. For self-sufficiency of this volume we recall that their definition requires a three-dimensional iso-Euclidean space

$$\hat{E}(\hat{r}, \hat{\delta}, \hat{R}): \quad \hat{\delta} = Diag.(n_1^{-2}, n_2^{-2}, n_3^{-2}), \quad \hat{I} = Diag.(n_1^2, n_2^2, n_3^2), \tag{3.2.85a}$$

$$\hat{r}^{\hat{2}} = (x^2/n_1^2 + y^2/n_2^2 + z^2/n_3^2) \times \hat{I} \quad \in \quad \hat{R}. \tag{3.2.85b}$$

The isotopies of the conventional spherical coordinates in $E(r, \delta, R)$ then yields the following *isospherical coordinates* here presented in the projected form on $\hat{E}(r, \hat{\delta}, R)$

$$x = r \times n_1 \times \sin(\theta/n_3) \times \sin(\phi/n_1 \times n_2), \tag{3.2.86a}$$

$$y = r \times n_2 \times \sin(\theta/n_3) \times \cos(\phi/n_1 \times n_2), \tag{3.2.86b}$$

$$z = r \times n_3 \times \cos(\theta/n_3). \tag{3.2.86c}$$

Via the use of the above general rules, the reader can now construct all needed isofunctions.

The reader should meditate a moment on the isotrigonometric functions. In fact, they provide a *generalization of the Pythagorean theorem to curvilinear triangles.* This is due to the fact that the projection of $\hat{E}(\hat{r}, \hat{\delta}, \hat{R})$ into the original space $E(r, \delta, R)$ characterizes indeed curvilinear triangles, trivially, because the n-characteristic quantities are functions.

[9] Note, again, that a different definition of isologarithm was assumed in Eq. (6.7.5) of Ref. [6].

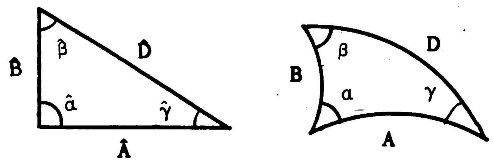

Figure 3.5. An intriguing application of isotrigonometric functions, the generalization of the conventional Pythagorean Theorem (left view) to triangles with curvilinear sides (right view). This is due to the fact that conventional triangles and the Pythagorean theorem are preserved identically on isospaces over isofields, but the projection on conventional Euclidean spaces of straight lines in isospaces over isofields are curves. Therefore in isospace we have expressions such as $\hat{A} = \hat{D}\hat{\times}$isosinus$(\hat{\gamma})$ with projections in the conventional space for curvilinear sides $A = D \times$ isosinus(γ), where A and D are now the lengths of the curvilinear sides.

However, the reader is suggested to verify that the *isotriangle*, that is, the image on $\hat{E}$ of an ordinary triangle on E coincides with the latter because the changes caused by the lifting are compensated by the inverse changes of the unit.

By noting that their value must be isonumbers, the *isointegral* can be defined by (here expressed for the simple case of isounits independent form the integration variable)

$$\hat{\int} d\hat{r} = \hat{I} \times \int \hat{T} \times d(r \times \hat{I}) = \hat{I} \times \int dr, \tag{3.2.87}$$

whose extension to the case of isounits with an explicit functional dependence on the integration variables has a complexity that goes beyond the elementary level of this presentation.

Isointegrals and isoexponentiations then permit the introduction of the following *Fourier-Kadeisvili isotransforms*, first studied in Ref. [19,20] (also represented here to avoid excessive mathematical complexities for the simpler case of isounits without an explicit dependence on the integration variables)[10]

$$\hat{f}(\hat{x}) = (\hat{1}\hat{/}2\hat{\pi})\hat{\times}\hat{\int}_{-\infty}^{+\infty} \hat{g}(\hat{k})\hat{\times}\hat{e}^{\hat{i}\hat{\times}\hat{x}}\hat{d}\hat{k}, \tag{3.2.88a}$$

$$\hat{g}(\hat{x}) = (\hat{1}\hat{/}2\hat{\pi})\hat{\times}\hat{\int}_{-\infty}^{+\infty} \hat{f}(\hat{k})\hat{\times}\hat{e}^{\hat{i}\hat{\times}\hat{x}}\hat{d}\hat{x}, \tag{3.2.88b}$$

with similar liftings for Laplace transforms, etc. Other transforms can be defined accordingly [6].

[10]The reader should be aware that in most applications of hadronic mechanics the isounits can be effectively approximated into constants, thus avoiding the complex mathematics needed for isointegrals and isotransforms with an explicit functional dependence on the integration variables.

We confirm in this way a major feature of isomathematics, the fact that *Hamiltonian quantities preserve not only their axioms, but also their numerical value under isotopic lifting when defined on isospaces over isofields, and all deviations occur in the projection of the lifting into the original space.*

The explicit construction of the *isodual isofunctional analysis* is also instructive and intriguing because they reveal properties that have essentially remained unknown until recently, such as the fact that *the isofourier transforms are isoselfdual* (see also Refs. [6,7]).

3.2.5 Isodifferential Calculus and its Isodual

As indicated in Chapter 1, the delay to complete the construction of hadronic mechanics since its proposal in 1978 [5] was due to difficulties in identifying the origin of the non-invariance of its initial formulation, that is, the lack of prediction of the same numerical values for the same quantities under the same conditions, but at different times, a fundamental invariance property fully verified by quantum mechanics.

These difficulties were related to the lack of a consistent isotopic lifting of the familiar quantum mechanical momentum. More particular, *all* aspects of quantum mechanics could be consistently and easily lifted via a nonunitary transforms, except the eigenvalue equation for the linear momentum, as shown by the following lifting

$$\begin{aligned} p \times \psi(t,r) = -i \times \hbar \times \frac{\partial}{\partial r}\psi(t,r) = K \times \psi(t,r) \rightarrow \\ \rightarrow U \times [p \times \psi(t,r)] = (U \times p \times U^{\dagger}) \times (U \times U^{\dagger})^{-1} \times [U \times \psi(t,r)] = \\ = \hat{p} \times \hat{T} \times \hat{\psi}(\hat{t},\hat{r}) = \hat{p}\hat{\times}\hat{\psi}(\hat{t},\hat{r}) = \\ = -i \times \hbar \times U[\frac{\partial}{\partial r}\psi(t,r)] = K \times U \times \psi(t,r) = \hat{K}\hat{\times}\hat{\psi}(\hat{t},\hat{r}), \end{aligned} \quad (3.2.89)$$

where $\hat{K} = K \times \hat{I}$ is an isonumber.

As one can see, the initial and final parts of the lifting are elementary. The problem rested in the impossibility of achieving a consisting lifting of the intermediate step, that based on the partial derivative.

In the absence of a consistent isotopy of the linear momentum, the early studies of hadronic mechanics lacked consistent formulations of physical quantities depending on the isomomentum, such as the isotopies of angular momentum, kinetic energy, etc.

The origin of the above problem resulted in being where expected the least, in the *ordinary differential calculus,* and this explains the delay in the resolution of the impasse.

The above problem was finally resolved by Santilli in the second edition of Refs. [6,7] of 1995 (see Section 5.4.B of Vol. I and Section 8.4.A of Vol. II) with a mathematical presentation in memoir [10] of 1996. The resulting generalization of the ordinary differential calculus, today known as *Santilli's isodifferential calculus,* plays a fundamental role for these studies beginning with the first known structural generalization of Newton's equations in Newtonian mechanics, and then passing to the correct invariant formulation of all dynamical equations of hadronic mechanics.

For centuries, since its discovery by Newton and Leibnitz in the mid 1600, the ordinary differential calculus had been assumed to be independent from the basic unit and field, and the same assumption was kept in the earlier studies on hadronic mechanics, resulting in the lack of full invariance, inability to formulate physical models and other insufficiencies.

After exhausting all other possibilities, an inspection of the differential calculus soon revealed that, contrary to an erroneous belief kept in mathematics for about four centuries, the ordinary differential calculus is indeed dependent on the basic unit and related field.

In this section we review Santilli's isodifferential calculus in its version needed for applications and verifications of hadronic mechanics. This update is recommendable because of various presentations in which the role of $\hat{I}$ and $\hat{T}$ were interchanged, resulting in possible ambiguities that could cause loss of invariance even under the lifting of the differential calculus.

A main feature is that, *unlike all other aspects of hadronic mechanics, the isotopies of the differential calculus cannot be reached via the use of a noncanonical or nonunitary transform, and have to be built via different, yet compatible methods.*

Let $S(r, m, R)$ an N-dimensional metric or pseudo-metric space with *contravariant* coordinates $R = (r^k)$, metric $m = (m_{ij})$, $i, j, k = 1, 2, \ldots, N$, and conventional unit $I = Diag.(1, 1, \ldots, 1)$ on the reals R. Let $f(r)$ be an ordinary (sufficiently smooth) function on S, let dr^k be the differential in the local coordinates, and let $\partial f(r)/\partial r^k$ be its partial derivative.

As it is well known, the connection between covariant and contravariant coordinates is characterized by the familiar rules

$$r^k = m^{kj} \times r_j, \quad r_i = m_{ik} \times r^k, \tag{3.2.90a}$$

$$m^{ij} = [(m_{qw})^{-1}]^{ij}. \tag{3.2.90b}$$

Let $\hat{S}(\hat{r}, \hat{M}, \hat{R})$ be an isotope of S with N-dimensional isounit $\hat{I} = (\hat{I}^i_j)$, contravariant isocoordinates $\hat{r} = (r^k) \times \hat{I}$ and isometric $\hat{M} = (\hat{M}_{ij}) = (\hat{T}^s_i \times m_{sj}) \times \hat{I}$ on the isoreals $\hat{R}$.

The connection between covariant and contravariant isocoordinates is then given by

$$\hat{r}^k = \hat{M}^{kj} \hat{\times} \hat{r}_j, \quad \hat{r}_i = \hat{M}_{ik} \hat{\times} \hat{r}^k, \tag{3.2.91a}$$

$$\hat{M}^{ij} = [(\hat{M}_{qw})^{-1}]^{ij}. \tag{3.2.91b}$$

Therefore, on grounds of compatibility with the metric and subject to verifications later on geometric grounds, we have the following:

LEMMA 3.2.5 [10]: Whenever the isounit of contravariant coordinates $\hat{r}^k$ on an isospace $\hat{S}(\hat{r}, \hat{M}, \hat{R})$ is given by

$$\hat{I} = (\hat{I}^i_j(t, r, \ldots)) = 1/\hat{T} = (\hat{T}^j_i)^{-1}, \tag{3.2.92}$$

the isounit for the related covariant coordinates $\hat{r}_k$ is given by its inverse

$$\hat{T} = (\hat{T}^i_j(t, r, \ldots)) = 1/\hat{I} = (\hat{I}^i_j)^{-1}, \tag{3.2.93}$$

and viceversa.

The *ordinary differential* of the contravariant isocoordinates is given by $d\hat{r}^k$ with covariant counterpart $d\hat{r}_k$ and they clearly do not constitute an isotopy. The condition for the preservation of the original axioms and value for constant isounits then leads to the following

DEFINITION 3.2.5 [6,7,10]: The isodifferentials of contravariant and covariant coordinates are given respectively by[11]

$$\hat{d}\hat{r}^k = \hat{d}(r^k \times \hat{I}) = \hat{T}^k_i \times d(r^i \times \hat{I}), \tag{3.2.94a}$$

$$\hat{d}\hat{r}_k = \hat{d}(r_k \times \hat{T}) = \hat{I}^i_k \times d(r_i \times \hat{T}). \tag{3.2.94b}$$

LEMMA 3.2.6 [loc. cit.]: For one-dimensional isounits independent from the local variables, isodifferentials coincide with conventional differentials,

$$\hat{d}\hat{r}^k \equiv dr^k, \quad \hat{d}\hat{r}_k \equiv dr_k. \tag{3.2.95}$$

Note that the above property constitutes a *new invariance of the differential calculus.* Its trivial character explains the reason isodifferential calculus escaped detection for centuries. Needless to say, the above triviality is lost for isounit

[11] It should be noted that the role of $\hat{I}$ and $\hat{T}$ in this definition and that of Ref. [10] are inverted. Also, the reader should keep in mind that, since they are assumed to be Hermitian, isounits can always be diagonalized. In fact, diagonal isounits are sufficient for the verifications and applications of hadronic mechanics, while leaving to the interested reader the formulation of hadronic mechanics according to the broader isodifferential calculus of Refs. [6,7,10].

with nontrivial functional dependence from the local variables as it is generally the case for hadronic mechanics.

The *ordinary derivative* of an isofunction of contravariant coordinates is evidently given by

$$\frac{\partial \hat{f}(\hat{r}^k)}{\partial \hat{r}^k} = \lim_{\hat{d}\hat{r}^k \to 0} \frac{\hat{f}(\hat{r}^k + \hat{d}\hat{r}^k) - \hat{f}(\hat{r}^k)}{\hat{d}\hat{r}^k} \tag{3.2.96}$$

with covariant version

$$\frac{\partial \hat{f}(\hat{r}_k)}{\partial \hat{r}_k} = \lim_{\hat{d}\hat{r}_k \to 0} \frac{\hat{f}(\hat{r}_k + \hat{d}\hat{r}_k) - \hat{f}(\hat{r}_k)}{\hat{d}\hat{r}_k}. \tag{3.2.97}$$

It is then simple to reach the following

DEFINITION 3.2.4 [loc. cit.]: The isoderivative of isofunctions on contravariant and covariant isocoordinates are given respectively by

$$\frac{\hat{\partial} \hat{f}(\hat{r}^k)}{\hat{\partial} \hat{r}^k} = \hat{I}^i_k \times \frac{\partial \hat{f}(\hat{r}^i)}{\partial \hat{r}^k}, \tag{3.2.98a}$$

$$\frac{\hat{\partial} \hat{f}(\hat{r}_k)}{\hat{\partial} \hat{r}_k} = \hat{T}^k_i \times \frac{\partial \hat{f}(\hat{r}_i)}{\partial \hat{r}^k}, \tag{3.2.98b}$$

where the isoquotient is tacitly assumed.[12]

A few examples are now in order to illustrate the axiom-preserving character of the isodifferential calculus. Assume that the isounit is not dependent on r. Then, for $\hat{f}(\hat{r}^k) = \hat{r}^k$ we have

$$\frac{\hat{d}\hat{r}^i}{\hat{d}\hat{r}^j} = \hat{\delta}^i_j = \delta^i_j \times \hat{I}. \tag{3.2.99}$$

Similarly we have

$$\frac{\hat{d}(\hat{r}^i)^{\hat{n}}}{\hat{d}\hat{r}^j} = \hat{\delta}^i_j \times \hat{(}r^i)^{\hat{n}-\hat{1}}. \tag{3.2.100}$$

It is instructive for the reader interested in learning Santilli isodifferential calculus to prove that *isoderivatives in different variables "isocommute" on isospace over isofields,*

$$\frac{\hat{\partial}}{\hat{\partial}\hat{r}^i}\frac{\hat{\partial}}{\hat{\partial}\hat{r}^j} = \frac{\hat{\partial}}{\hat{\partial}\hat{r}^j}\frac{\hat{\partial}}{\hat{\partial}\hat{r}^i}, \tag{3.2.101}$$

[12]Note that the isofunction in the numerator contains an additional isounit, $\hat{f} = f \times \hat{I}$, that, however, cancels out with the isounit of the isoquotient, $\hat{/} = / \times \hat{I}$, resulting in expressions (3.2.98). Note also the lack of presence of a *factorized* isounit in the definition of the isodifferentials and isoderivatives, and this explains why the isodifferential calculus cannot be derived via noncanonical or nonunitary transforms.

but their projections on ordinary spaces over ordinary fields do not necessarily "commute".

We are now sufficiently equipped to point out the completion of the construction of hadronic mechanics. First, let us verify the axiom-preserving character of the isoderivative of the isoexponent in a contravariant coordinate for the simple case in which the isounit does not depend on the local variables. In fact, we have the expression

$$\frac{\hat{\partial}}{\hat{\partial}\hat{r}}\hat{e}^{\hat{r}} = \hat{I}\times\frac{\partial}{\partial\hat{r}}[\hat{I}\times e^{\hat{T}\times\hat{r}}] = \hat{I}\times\hat{T}\times[\hat{I}\times e^{\hat{T}\times\hat{r}}] = \hat{e}^{\hat{r}}. \tag{3.2.102}$$

Consider now the *isoplanewave* as a simply isotopy of the conventional planewave solution (again for the case in which the isounit does not depend explicitly on the local coordinates),

$$\hat{e}^{\hat{i}\hat{\times}\hat{r}\hat{\times}\hat{K}} = \hat{I}\times e^{i\times\hat{T}\times K\times\hat{r}}, \tag{3.2.103}$$

for which we have the isoderivatives

$$\frac{\hat{\partial}}{\hat{\partial}\hat{r}}\hat{e}^{\hat{i}\hat{\times}\hat{r}\hat{\times}\hat{K}} = \hat{I}\times\frac{\partial}{\partial\hat{r}}[\hat{I}\times e^{i\times\hat{T}\times K\times\hat{r}}] =$$

$$= -i\times K\times\hat{I}\times e^{i\times\hat{T}\times K\times\hat{r}} = \hat{i}\hat{\times}\hat{K}\hat{\times}\hat{e}^{\hat{i}\hat{\times}\hat{r}\hat{\times}\hat{K}}. \tag{3.2.104}$$

We reach in this way the following fundamental definition of *isomomentum*, first achieved by Santilli in Refs. [6,7] of 1995, that completed the construction of hadronic mechanics (its invariance will be proved later on in Section 3.5).

DEFINITION 3.2.7 [6,7,10]: The isolinear momentum on an iso-Hilbert space over the isofield of isocomplex numbers $\hat{C}$ (see Section 3.5 for details) is characterized by

$$\hat{p}_k\hat{\times}\hat{\psi}(\hat{t},\hat{r}) = -\hat{i}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{r}^k}\hat{\psi}(\hat{t},\hat{r}) = -i\hat{\times}\hat{I}^i_k\times\frac{\partial}{\partial\hat{r}^i}\hat{\psi}(\hat{t},\hat{r}) = \hat{K}\hat{\times}\hat{\psi}(\hat{t},\hat{r}). \tag{3.2.105}$$

Comparing the above formulation with Eq. (3.2.89), and in view of invariance (3.2.95),we reach the following

THEOREM 3.2.4 [6,7,10]: Planck's constant $\hbar$ is the fundamental unit of the differential calculus underlying quantum mechanics, i.e., quantum mechanical eigenvalue equations can be identically reformulated in terms of the isodifferential calculus with basic isounit $\hbar$,

$$p\times\psi(t,r) = -i\times\hbar\times\frac{\partial}{\partial r}\psi(t,r) \equiv -i\times\frac{\hat{\partial}}{\hat{\partial}r}\psi(t,r). \tag{3.2.106}$$

In conclusion, *Santilli's isodifferential calculus establishes that the isounit not only is the algebraic unit of hadronic mechanics, but also replaces Planck's constant with an integro-differential operator $\hat{I}$, as needed to represent contact, nonlinear, nonlocal and nonpotential effects.*

More specifically, Santilli's isodifferential calculus establishes that, while in exterior dynamical systems such as atomic structures, we have the conventional quantization of energy, in interior dynamical systems such as in the structure of hadrons, nuclei and stars, we have a superposition of quantized energy level at atomic distances plus continuous energy exchanges at hadronic distances.

Needless to say, all models of hadronic mechanics will be restricted by the condition

$$\lim_{r\to\infty} \hat{I} \equiv \hbar, \tag{3.2.107}$$

under which hadronic mechanics recovers quantum mechanics uniquely and identically.

DEFINITION 3.2.8 [6,7,17]: The isodual isodifferentials are defined by

$$\hat{d}^d \hat{r}^d = (-\hat{d}^\dagger)(-\hat{r}^\dagger) = \hat{d}\hat{r}, \tag{3.2.108}$$

while isodual isoderivatives are given by

$$\hat{\partial}^d \hat{f}^d(\hat{r}^d)\hat{/}^d \hat{d}^d \hat{r}^d = -\hat{\partial}\hat{f}(\hat{r})\hat{/}\hat{d}\hat{r}. \tag{3.2.109}$$

THEOREM 3.2.5 [6,7,17]: Isodifferentials are isoselfduals.

The latter new invariance constitutes an additional, reason why the isodual theory of antimatter escaped attention during the 20-th century.

3.2.6 Kadeisvili's Isocontinuity and its Isodual

The notion of continuity on an isospace was first studied by Kadeisvili [19] in 1992 and it is today known as *Kadeisvili's isocontinuity.* A review up to 1995 was presented in monographs [6,7]. Rigorous mathematical study of isocontinuity has been done by Tsagas and Sourlas [22–23], R. M. Falcón Ganfornina and J. Núñez Valdés [24–26] and others. For mathematical studies we refer the interested reader to the latter papers. For the limited scope of this volume we shall present the notion of isocontinuity in its most elementary possible form.

Let $\hat{f}(\hat{r}) = f(\hat{T}\times\hat{r})\times\hat{I}$ be an isofunction on an isospace $\hat{S}$ over the isofield $\hat{R}$. The *isomodulus* of said isofunction is defined by [19]

$$\hat{|}\hat{f}(\hat{r})\hat{|} = |f(\hat{T}\times\hat{r})| \times \hat{I}. \tag{3.2.110}$$

DEFINITION 3.2.9 [19,20]: An infinite sequence of isofunctions $\hat{f}_1(\hat{r})$, $\hat{f}_2(\hat{r})$, ... is said to be "strongly isoconvergent" to the isofunction $\hat{f}(\hat{r})$ when

$$\lim_{k\to\infty} \hat{|} \hat{f}_k(\hat{r}) - \hat{f}(\hat{r}) \hat{|} \hat{=} \hat{0}. \tag{3.2.111}$$

while the "iso-Cauchy condition" can be defined by

$$\hat{|} \hat{f}_m(\hat{r}) - \hat{f}_n(\hat{r}) \hat{|} \ < \ \hat{\delta} = \delta \times \hat{I}, \tag{3.2.112}$$

where δ is a sufficiently small real number, and m and n are integers greater than a suitably chosen neighborhood of δ.

The isotopies of other notions of continuity, limits, series, etc. can be easily constructed (see Refs. [6,7] for physical treatments and Refs. [22–26] for mathematical treatments).

Note that *functions that are conventionally continuous are also isocontinuous.* Similarly, *a series that is strongly convergent is also strongly isoconvergent.* However, a series that is strongly isoconvergent is not necessarily strongly convergent. We reach in this way the following important

THEOREM 3.2.6 [6,7]: Under the necessary continuity and regularity conditions, a series that is conventionally divergent can always be turned into a convergent isoform under a suitable selection of the isounit.

This mathematically trivial property has far reaching implications, e.g., the achievement, for the first time in physics, of convergent perturbative series for strong interactions, which perturbative treatments are conventionally divergent (see Section 3.4).

Similarly, the reader may be interested in knowing that, given a function which is not square-integrable in a given interval, there always exists an isotopy which turns the function into a square-integrable form [6,7]. The novelty is due to the fact that the underlying mechanism is not that of a weight function, but that of altering the underlying field.

The *isodual isocontinuity* is a simple isodual image of the preceding notions of continuity and will be hereon assumed.

3.2.7 TSSFN Isotopology and its Isodual

Topology is the ultimate foundation of quantitative sciences because it identifies on rigorous mathematical grounds the limitations of the ensuing description.

Throughout the 20-th century, all quantitative sciences, including particle physics, nuclear physics, astrophysics, superconductivity, chemistry, biology, etc., have been restricted to the use of mathematics based on the conventional *local-differential topology,* with the consequence that the sole admitted representations are those dealing with a finite number of isolated point-like particles.

Since points are dimensionless, they cannot have contact interactions. Therefore, an additional consequence is that the sole possible interactions are those of action-at-a-distance type representable with a potential.

In conclusion, the very assumption of the conventional local-differential topology, such as the conventional topology for the Euclidean space, or the Zeeman topology for the Minkowski space, uniquely and unambiguously restrict the admitted systems to be local, differential and Hamiltonian.

This provided an approximation of systems that proved to be excellent whenever the mutual distances of particles are much greater than their size as it is the case for planetary and atomic systems.

However, the above conditions are the exception and not the rule in nature, because all particles have a well defined extended wavepacket and/or charge distribution of the order of 10^{-13} cm. It is well known in pure and applied mathematics that the representation of the actual shape of particles is impossible with a local-differential topology.

Moreover, once particles are admitted as being extended, there is the emergence of the additional contact, zero-range nonpotential interactions that are nonlocal in the sense of occurring in a finite surface or volume that cannot be consistently reduced to a finite number of isolated points.

Consequently, it is equally know by experts that conventional local-differential topologies cannot represent extended particles at short distances and their nonlocal-nonpotential interactions, as expected in the structure of planets, strongly interacting particles, nuclei, molecules, stars and other interior dynamical systems.

The need to build a new topology, specifically conceived and constructed for hadronic mechanics was suggested since the original proposal [5] of 1978. It was not only until 1995 that the Greek mathematicians Gr. Tsagas and D. S. Sourlas [22,23] proposed the first *isotopology* on scientific record formulated on isospaces over ordinary fields. In 1996, the Italian-American physicist R. M. Santilli [10] extended the formulation to isospaces over isofields. Finally, comprehensive studies on isotopology were conducted by the Spanish Mathematicians R. M. Falcón Ganfornina and J. Núñez Valdés [24,25]. As a result, the new topology is hereon called the *Tsagas-Sourlas-Santilli-Falcón-Núñez isotopology* (or TSSFN Isotopology for short).

The author has no words to emphasize the far reaching implications of the new TSSFN isotopology because, for the first time in the history of science, mathematics can consistently represent the actual extended, generally nonspherical and deformable shape particles, their densities as well as their nonpotential and nonlocal interactions.

As an example, Newton's equations have remained unchanged in Newtonian mechanics since the time of their conception to represent point-particles. No consistent generalization was possible due to the underlying local-differential topology and related differential calculus. As we shall see in the next section, the isodifferential calculus and underlying isotopology will permit the first known structural generalization of Newton's equations in Newtonian mechanics for the representation of extended particles.

New coverings of quantum mechanics, quantum chemistry, special relativity, and other quantitative sciences are then a mere consequence. Perhaps more importantly, the new clean energies and fuels permitted by hadronic mechanics can see their origin precisely in the TSSFN isotopology, as we shall see later on in this chapter.

In their most elementary possible form accessible to experimental physicists, the main lines of the new isotopology can be summarized as follows. Being nowhere singular, Hermitian and positive-definite, N-dimensional isounits can always be diagonalized into the form

$$\hat{I} = Diag.(n_1^2, n_2^2, \ldots, n_N^2), \quad n_k = n_k(t, r, v, \ldots) > 0, \quad k = 1, 2, \ldots, N. \quad (3.2.113)$$

Consider N isoreal isofields $\hat{R}_k(\hat{n}, \hat{+}, \hat{\times})$ each characterized by the isounit $\hat{I}_k = n_k^2$ with (ordered) Cartesian product

$$\hat{R}^N = \hat{R}_1 \times \hat{R}_2 \times \ldots \times \hat{R}_N. \quad (3.2.114)$$

Since each isofield $\hat{R}_k$ is isomorphic to the conventional field of real numbers $R(n, +, \times)$, it is evident that $\hat{R}^N$ is isomorphic to the Cartesian product of N ordinary fields

$$R^N = R \times R \times \ldots \times R. \quad (3.2.115)$$

Let

$$\tau = \{R^N, K_i\} \quad (3.2.116)$$

be the conventional *topology* on R^N (whose knowledge is here assumed for brevity), where K_i represents the subset of R^N defined by

$$K_i = \{P = (a_1, a_2, \ldots, a_N)/n_i < a_1, a_2, \ldots, a_N < m; \quad n_i, m_i, a_i \in R\}. \quad (3.2.117)$$

We therefore have the following:

DEFINITION 3.2.8 [10,22-25]: The isotopology can be defined as the simple lifting on $\hat{R}^N$ of the conventional topology on R^N, and we shall simply write

$$\hat{\tau} = \{\hat{R}^N, \hat{K}_i\}, \quad (3.2.118a)$$

$$\hat{K}_i = \{\hat{P} = (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_N)\hat{/}\hat{n}_i < \hat{a}_1, \hat{a}_2, \ldots, \hat{a}_N < \hat{m}; \ \ \hat{n}_i, \hat{m}_i, \hat{a}_i \in \hat{R}\}. \quad (3.2.118b)$$

As one can see, the above isotopology coincides everywhere with the conventional topology of R^N except at the isounit $\hat{I}$. In particular, $\hat{\tau}$ is everywhere local-differential, except at $\hat{\hat{I}}$ which can incorporate nonlocal integral terms.

It is evident that isotopology can characterize for the first time in scientific history, extended, nonspherical and deformable particles. In fact, for the case of three-dimensions in diagonal representation (3.2.113), we have the characterization of deformable spheroidal ellipsoids with variable semiaxes n_1^2, n_2^2, n_3^2 depending on local quantities, such as energy, density, pressure, etc. For the case of four-dimension the quantity n_4^2 represents, for the first time in scientific record, the density of the particle considered[13].

The reader should be aware that the above formulation of the isotopology is the simplest possible one, being restricted to the description of *one* isolated *isoparticle,* that is, an extended and nonspherical particle on isospace over isofields that, as such, has no interactions.

Consequently, numerous generalizations of the above formulations are possible and actually needed for hadronic mechanics. The first broadening is given by the case of *two* or more isoparticles in which case the basic isounit is given by the Cartesian product of two isounits of type (3.2.113). The second broadening is given by exponential factors incorporating nonlinear integral terms as in the general isounit (3.1.19). In the preceding formulation, these exponential factors have been incorporated in the n's since they are common factors.

A lesser trivial broadening of the above formulation of isotopology is given by *nondiagonal isounits* that are capable of representing nonspheroidal shapes and other complex geometric occurrences (see in Ref. [6], page 213 the case of a nondiagonal isotopy contracting the dimensions from three to one, also reviewed in the next section). The study of the latter more general formulations of isotopology is left to the interested reader.

DEFINITION 3.2.11 [22-25]: An isotopological isospace $\hat{\tau}(\hat{R}^N)$ is the isospace $\hat{R}^N$ equipped with the isotopology $\hat{\tau}$. An isocartesian isomanifold $\hat{M}(\hat{R}^N)$ is the isotopological isospace $\hat{M}(\hat{R}^N)$ equipped with a isovector structure, an isoaffine structure and the mapping

$$\hat{F} : \hat{R}^N \rightarrow \hat{R}^N; \quad \hat{a} \rightarrow \hat{f}(\hat{a}), \ \ \forall \hat{a} \in \hat{R}^N. \quad (3.2.119)$$

[13] The reader is encouraged to inspect any desired textbook in particle physics and verify the complete lack of representation of the density of the particle considered.

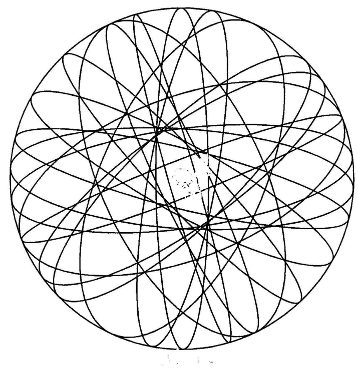

Figure 3.6. A schematic view of the "isosphere", namely, the perfect sphere on isospace over isofield represented by isoinvariant (3.2.121), that is assumed as the geometric representation of hadrons used in this monograph. The actual nonspherical and deformable shape of hadrons is obtained by projecting the isosphere in our Euclidean space, as illustrated in the last identify of Eq. (3.2.122).

An iso-Euclidean isomanifold $\hat{M}(\hat{E}(\hat{r}, \hat{\delta}, \hat{R}))$ *occurs when the* N*-dimensional isospace* $\hat{E}$ *is realized as the Cartesian product (3.2.106) and equipped with isotopology (3.2.118) with basic isounit (3.2.113).*

The *isodual isotopology* and related notions can be easily constructed with the isodual map (3.2.15) and its explicit study is left as an instructive exercise for the interested reader.

3.2.8 Iso-Euclidean Geometry and its Isodual

The isotopies of the Euclidean space and geometry were introduced for the first time by Santilli in Ref. [14] of 1983 as a particular case of the broader isotopies of the Minkowski space and geometry treated in the next section.

The same isotopies were then studied in various works by the same author and a comprehensive treatment was presented in Chapter 5 of Vol. I [6]. These isotopies are today known as the *Euclid-Santilli isospace and isogeometry.* The presentation of Vol. I will not be repeated here for brevity. We merely limit ourselves to outline the main aspects for minimal self-sufficiency of this monograph.

Consider the fundamental isospace for nonrelativistic hadronic mechanics, the three-dimensional *Euclid-Santilli isospace* with contravariant isocoordinates $\hat{r}$, isometric $\hat{\delta}$ over the isoreals $\hat{R} = \hat{R}(\hat{n}, \hat{+}, \hat{\times})$ (see Section 3.3)

$$\hat{E}(\hat{r}, \hat{\delta}, \hat{R}): \quad \hat{r} = (\hat{r}^k) = (\hat{x}, \hat{y}, \hat{z}) = (r^k) \times \hat{I} = (x, y, z) \times \hat{I}, \quad k = 1, 2, 3; \quad (3.2.120a)$$

$$\hat{I} = Diag.(n_1^2, n_2^2, n_3^2) = 1/\hat{T} > 0, \quad n_k = n_k(t, r, v, a, \mu, \tau, \dots) > 0, \qquad (3.2.120b)$$

$$\hat{\Delta} = \hat{\delta} \times \hat{I}; \ \ \hat{\delta} = \hat{T} \times \delta = Diag.(n_1^{-2}, n_2^{-2}, n_3^{-2}), \tag{3.2.120c}$$

with basic isoinvariant on $\hat{E}$

$$\hat{r}^{\hat{2}} = \hat{r}^i \hat{\times} \hat{\Delta}_{ij} \hat{\times} \hat{r}^j = \hat{r}^i \times \hat{\delta}_{ij} \times \hat{r}^j = \hat{r}^i \times (\hat{T}_i^k \times \delta_{kj}) \times \hat{r}^j =$$

$$= \hat{x}^{\hat{2}} + \hat{y}^{\hat{2}} + \hat{z}^{\hat{2}} = \frac{\hat{x}^2}{n_1^2} + \frac{\hat{y}^2}{n_2^2} + \frac{\hat{z}^2}{n_3^2} \ \in \ \hat{R}. \tag{3.2.121}$$

and projection on the conventional Euclidean space

$$r^2 = \frac{x^2}{n_1^2} + \frac{y^2}{n_2^2} + \frac{z^2}{n_3^2} \ \in \ R. \tag{3.2.122}$$

where the scalar functions n_k, besides being sufficiently smooth and positive-definite, have an unrestricted functional dependence on time t, coordinates r, velocities v, acceleration a, density μ, temperature τ, and any needed local variable.

The *Euclid-Santilli isogeometry* is the geometry of the above isospaces. A knowledge of the following main features is essential for an understanding of nonrelativistic hadronic mechanics.

Since the isospaces $\hat{E}$ are all locally isomorphic to the conventional Euclidean space $E(r, \delta, R)$, it is evident that *the Euclid-Santilli isogeometry verifies all axioms of the conventional geometry,* as proved in detail in Section 5.2 of Vol. I [6]. In fact, the conventional and isotopic geometries coincide at the abstract, realization free level to such an extent that they can be expressed with the same abstract symbols, the differences between the conventional and the isotopic geometries emerging only in the selected realizations of said abstract axioms.

Note that, while the Euclidean space and geometry are unique, there exist an infinite family of different yet isomorphic Euclid-Santilli isospaces and isogeometries, evidently characterized by different isometrics in three dimension and signature $(+, +, +)$.

Recall from Section 3.2.3 that the structure of the basic invariant is given by Eq. (3.2.66). Therefore, the *isosphere*, namely, the image on $\hat{E}$ of the perfect sphere on E remains a perfect sphere. However, the projection of the isosphere on the original space E is a spheroidal ellipsoid, as clearly indicated by invariant (3.2.121). Therefore, *the isosphere on isospace over isofields unifies all possible spheroidal ellipsoids on ordinary spaces over ordinary fields.* These features are crucial to understand later on the reconstruction of the *exact* rotational symmetry for *deformed* spheres (see Fig. 3.6).

Since the functional dependence of the isometric is unrestricted except verifying the condition of positive-definiteness, it is easy to see that *the Euclid-Santilli isogeometry unifies all possible three-dimensional geometries with the signature)*

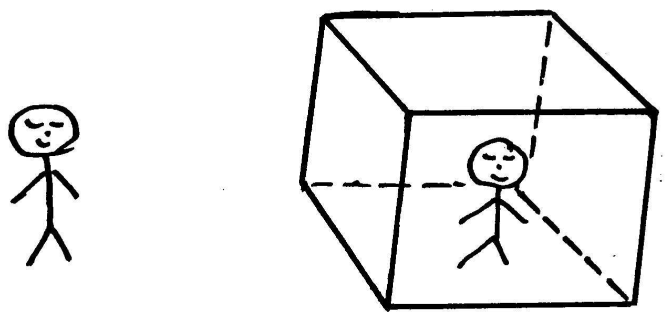

Figure 3.7. A schematic view of the "space isocube", namely, an ordinary cube inspected by two observers, an exterior observer in Euclidean space with basic units of measurements $I = Diag.(1\ \text{cm}, 1\ \text{cm}, 1\ \text{cm})$ and an interior observer on isospace with basic isounits $\hat{I} = Diag.(n_x^2\ \text{cm}, n_y^2\ \text{cm}, n_z^2\ \text{cm})$. It it then evident that, if the exterior observer measures, for instance, the sides of the cube to be $3m$, the interior observers measures different length that can be bigger or smaller than $3m$ depending on whether the isounit is smaller or bigger, respectively, than the original unit. Also, for the case of the Euclidean observer, the units in the three space directions are the same, while the corresponding isounits have different values for different directions. Therefore, the same object appears as a cube of a given size to the external observer, while having a completely different shape and size for the internal observer.

$(+, +, +)$, thus including as particular cases the Riemannian, Finslerian, non-Desarguesian and other geometries. As an example, the Riemannian metric $g_{ij}(r) = g^t$ is a trivial particular case of Santilli's isometric $\hat{\delta}_{ij}(t, r, \ldots)$. This occurrence has profound physical implications that will be pointed out in Section 3.5.

Yet another structural difference between conventional and isotopic geometries is that the former has the same unit for all three reference axes. In fact, the geometric unit $I = Diag.(1, 1, 1)$ is a dimensionless representation of the selected units, for instance, $I = diag.(1\,\text{cm}, 1\,\text{cm}, 1\,\text{cm})$. In the transition to the isospace, the units are different for different axes and we have, for instance, $\hat{I} = Diag.(n_1^2\,\text{cm}, n_2^2\,\text{cm}, n_3^2\,\text{cm})$. It then follows that *shapes detected by our sensory perception are not necessarily absolute, in the sense that they may appear basically different for an isotopic observer* (see Fig. 3.7).

Note that in the conventional space $E(r, \delta, R)$ there are two trivially different trivial units, namely, the unit $I = +1$ of the base field R and the unit $I = Diag.(1, 1, 1)$ of the space, related geometry and symmetries. The isotopies have identified for the first time the fact that *the unit of the space must coincide with the unit of the base field.*

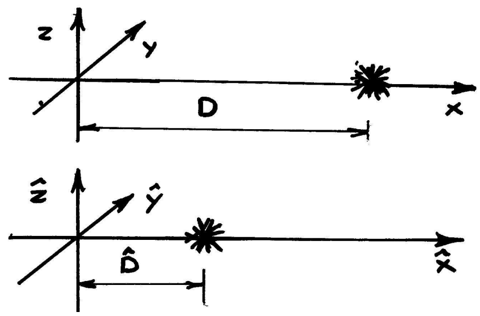

Figure 3.8. A schematic view of the geometric propulsion studied in greater details in Chapter 12, here illustrated via the contraction of distances in the transition from our coordinmates to the isotopiuc ones.

In fact, the isounit of isospace $\hat{E}(\hat{r}, \hat{\delta}, \hat{R})$ must coincide with the isounit of the isofield $\hat{R}$. It is then evident that, at the limit $\hat{I} \rightarrow I = Diag.(1, 1, 1)$ the unit matrix $I = Diag.(1, 1, 1)$ must be the unit of both the Euclidean space and of the basic field. This implies a trivial reformulation of R that is ignored hereon.

Another important notion is that of *isodistance* between two points P_1 and P_2 on $\hat{E}$ that can be defined by the expression

$$\hat{D}^2_{1-2} = (\hat{x}_1 - \hat{x}_2)^2/n_1^2 + (\hat{y}_1 - \hat{y}_2)^2/n_2^2 + (\hat{z}_1 - \hat{z}_2)^2/n_3^2. \qquad (3.2.123)$$

It then follows that *local alterations of the space geometry cause a change in the distance,* an occurrence first identified in Ref. [6] as originating from a lifting of the units, and today known as *isogeometric locomotion* studied in Chapter 13. We are here referring to a new form of non-Newtonian locomotion in which objects can move without the application of a force or, equivalently, without any application of the principle of action and reaction (see Figure 3.8).

Finally, it is important to point out that *the dimensionality of the original Euclidean space is not necessarily preserved under isotopies.* This occurrence constitutes another intriguing epistemological feature because isotopies are axiom-preserving. Therefore, our senses based on the three Eustachian lobes perceive no difference in dimension between a conventional and an isotopic shape.

The epistemological question raised by the isotopies is then whether our perception of space as three-dimensional is real, in the sense of being intrinsic, or it is a mere consequence of our particular sensory perception, with different dimensions occurring for other observers.[14]

[14] As we shall see in Chapter 4, an even deeper epistemological issue emerges from our hyper-isotopies in which the unit is characterized by a *set* of values. In this case, space can be "three-dimensional" yet be "hyper-dimensional", in the sense that each dimension can be multi-valued.

The occurrence was discovered by Santilli in Ref. [6], page 213, via the following isotopic element

$$\hat{\mathbf{T}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad (3.2.124)$$

that is positive definite since $\mathrm{Det}\,\hat{T} = 1$, thus being a fully acceptable isotopic element.

It is easy to see that the isoinvariant of the Euclid-Santilli isospace characterized by the above non-diagonal isotopy is given by

$$\hat{r}^{\hat{2}} = \hat{r}^i \times \hat{T}_i^k \times \delta_{kj} \times \hat{r}^j =$$

$$= \hat{x} \times \hat{z} + \hat{y} \times \hat{z} - \hat{z} \times \hat{y} = \hat{x} \times \hat{x}, \qquad (3.2.125)$$

namely, in this case the *isotopic image of the three-dimensional Euclidean space is one dimensional.*

This occurrence provides another illustration of the fact that, despite their simplicity, the geometric implications of the isotopies are rather deep indeed.

The *isodual Euclid-Santilli isospace* in three dimension can be represented by the expressions

$$\hat{E}^d(\hat{r}^d, \hat{\Delta}^d, \hat{R}^d): \quad \hat{r}^d = (-\hat{x}, -\hat{y}, -\hat{z}); \qquad (3.2.126a)$$

$$\hat{I}^d = Diag.(-n_1^2, -n_2^2, -n_3^2) = -1/\hat{T} > 0, \quad n_k = n_k(t, r, \ldots) > 0,$$

$$\hat{\Delta}^d = \hat{\delta}^d \times \hat{I}, \quad \hat{\delta}^d = \hat{T}^d \times^d \delta^d = Diag.(-n_1^{-2}, -n_2^{-2}, -n_3^{-2}), \qquad (3.2.126b)$$

with isodual isoinvariant on $\hat{R}^d$

$$\hat{r}^{d^{\hat{2}^d}} = \hat{r}^{di} \hat{\times}^d \hat{\Delta}^d_{ij} \hat{\times}^d \hat{r}^{dj} =$$

$$= -\hat{x}^{d^{\hat{2}^d}} - \hat{y}^{d^{\hat{2}^d}} - \hat{z}^{d^{\hat{2}^d}} \quad \in \quad \hat{R}^d. \qquad (3.2.127)$$

and projection on the isodual Euclidean space

$$r^{d^2} = (-x^2/n_1^2 - y^2/n_2^2 - z^2/n_3^2) \times \hat{I} \quad \in \quad R^d. \qquad (3.2.128)$$

A study of the *isodual Euclid-Santilli isogeometry* from Vol. I [6] is essential for a study of antimatter in interior conditions.

3.2.9 Minkowski-Santilli Isogeometry and its Isodual

3.2.9A. Conceptual Foundations. The isotopies of the Minkowski space and geometry are the main mathematical methods of relativistic hadronic mechanics, because they are at the foundations of the Poincaré-Santilli isosymmetry, and related broadening of special relativity for relativistic interior dynamical systems.

The isotopies of the Minkowski space and geometry were first proposed by Santilli in Ref. [14] of 1983 and then studied in numerous papers (see monographs [6,7,14,15] and papers quoted therein) and are today known as *Minkowski-Santilli isospace and isogeometry.*

Due to their fundamental character, the new spaces and geometry were treated in great details in Refs. [6,7], particularly in the second edition of 1995, and that presentation is here assumed as known for brevity.

The primary purpose of this section is to identify the most salient advances occurred since the second edition of Refs. [6,7] with particular reference to the geometric treatment of gravitation.

In essence, the original efforts in the construction of relativistic hadronic mechanics were based on *two different isotopies,* the isotopies of the Minkowskian geometry for nongravitational profiles, and the isotopies of the Riemannian geometry for gravitational aspects. The presentation of Refs. [6,7] was based on this dual approach.

Subsequently, it became known that *the isotopies of the Riemannian geometry could not resolve the catastrophic inconsistencies of gravitation identified in Chapter 1 because they are inherent in the background Riemannian treatment itself, thus persisting under isotopies.*

The resolution of these catastrophic inconsistencies was finally reached by Santilli in Ref. [26] of 1998 via the *unification of the Minkowskian and Riemannian geometries into Minkowski-Santilli isogeometry.* In fact, the isometric of the latter geometry admits, as a particular cases, all possible Riemannian metrics.

Consequently, it became clear that the various methods used for the Riemannian geometry (such as covariant derivative, Christoffel symbols, etc.) are inapplicable to the conventional Minkowski space evidently because flat, but the same methods are fully applicable to the Minkowski-Santilli isogeometry.

The achievement of a geometric unification of the Minkowskian and Riemannian geometries reached in memoir [26] permitted truly momentous advances, such as the geometric unification of the special and general relativities, an axiomatically consistent grand unification of electroweak and gravitational interactions, the first known axiomatically consistent operator form of gravity, and other basic advances reviewed in Section 3.5.

3.2.9B. Minkowski-Santilli Isospaces. We now review in this subsection the foundations of the Minkowski-Santilli isospaces by referring interested readers to volumes [6,7] for details.

DEFINITION 3.2.12 [26]: Consider the conventional Minkowski space

$$M = M(x, \eta, R) : x = (x^\mu) = (r, c_o t), \tag{3.2.129a}$$

$$x^\mu = \eta^{\mu\nu} \times x_\nu, \;\; x_\mu = \eta_{\mu\nu} \times x^\nu, \tag{3.2.129b}$$

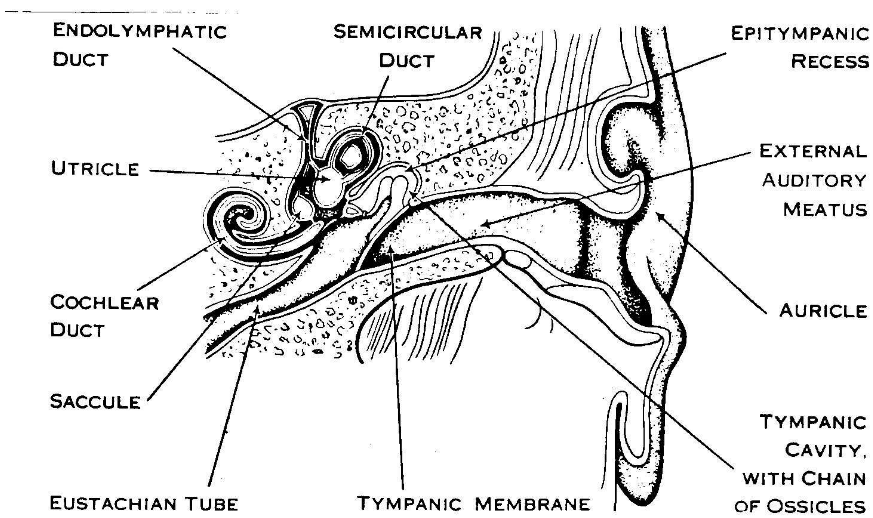

Figure 3.9. A view of the three Eustachian lobes allowing us to perceive three-dimensional shapes. The intriguing epistemological issue raised by the Euclid-Santilli isogeometry is whether living organisms with different senses perceive the same object with different shape and size than ours. As illustrated with the isobox of Figure 3.7, the same object can appear with dramatically different shapes and sizes to a conventional and an isotopic observer, as well as in dimension different than the original ones, as illustrated in the text. Another illustration of the meaning and importance of isotopies is that being axiom-preserving, different shapes, sizes and dimensions on isospaces are rendered compatible with our sensory perception.

where c_o is the speed of light in vacuum, metric

$$\eta = (\eta_{\mu\nu}) Diag.(+1,+1,+1,-1), \quad \eta^{\mu\nu} = [(\eta_{\alpha,\beta})^{-1}]^{\mu\nu}, \tag{3.2.130}$$

basic unit

$$I = Diag.(+1,+1,+1,+1), \tag{3.2.131}$$

and invariant on the reals

$$x^2 = x^\mu \times x_\mu = (x^\mu \times \eta_{\mu\nu} \times x^\nu) \times I \ \in \ R = R(n,+,\times), \tag{3.2.132}$$

$$\mu, \nu, \alpha, \beta = 1,2,3,4.$$

Then, the Minkowski-Santilli isospaces can be defined by isotopies

$$\hat{M} = \hat{M}(\hat{x}, \hat{G}, \hat{R}) : \hat{x} = (\hat{x}^\mu) = (r, c_o t) \times \hat{I}, \tag{3.2.133a}$$

$$\hat{x}^\mu = \hat{G}^{\mu\nu} \hat{\times} \hat{x}_\nu, \quad \hat{x}_\mu = \hat{G}_{\mu\nu} \hat{\times} \hat{x}^\nu, \tag{3.2.133b}$$

with isometric on isospaces over isofields

$$\hat{G} = \hat{\eta} \times \hat{I} = (\hat{T}_\mu^\rho \times \eta_{\rho\nu}) \times \hat{I} =$$

$$= Diag.(\hat{T}_{11}, \hat{T}_{22}, \hat{T}_{33}, \hat{T}_{44}) \times \hat{I} \quad \in \quad \hat{R} = \hat{R}(\hat{n}, \hat{+}, \hat{\times}), \tag{3.2.134a}$$

$$\hat{G}^{\mu\nu} = [(\hat{G}_{\alpha,\beta})^{-1}]^{\mu\nu}, \tag{3.2.134b}$$

and isounit

$$\hat{I} = Diag.(\hat{T}_{11}^{-1}, \hat{T}_{22}^{-1}, \hat{T}_{33}^{-1}, \hat{T}_{44}^{-1}), \tag{3.2.135}$$

where $\hat{T}_{\mu\nu}$ are positive-definite functions of spacetime coordinates x, velocities v, accelerations a, densities μ, temperature τ, wavefunctions, their derivatives and their conjugates and any other needed quantity

$$\hat{T}_{\mu\nu} = \hat{T}_{\mu\nu}(x, v, a, \mu, \tau, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \ldots) \quad > \quad 0 \tag{3.2.136}$$

isoinvariant on isospaces over the isofield of isoreal numbers

$$\hat{x}^{\hat{2}} = \hat{x}^\mu \hat{\times} \hat{x}_\mu = (\hat{x}^\mu \hat{\times} \hat{G}_{\mu\nu} \hat{\times} \hat{x}^\nu) \times I \quad \in \quad \hat{R} = R(\hat{n}, \hat{+}, \hat{\times}) \tag{3.2.137}$$

with projection in our spacetime

$$\hat{M}(x, \hat{\eta}, R) : x = (x^\mu) \times I, \tag{3.2.138a}$$

$$x^\mu = \hat{\eta}^{\mu\nu} \times x_\nu, \ \ x_\mu = \hat{\eta}_{\mu\nu} \times x^\nu, \tag{3.2.138b}$$

metric over the field of real numbers

$$\hat{\eta} = (\hat{\eta}_{\mu\nu}) = (\hat{T}_\mu^\rho \times \eta_{\rho\nu}) = Diag.(\hat{T}_{11}, \hat{T}_{22}, \hat{T}_{33}, \hat{T}_{44}) \in R = R(n, +, \times), \tag{3.2.139a}$$

$$\hat{\eta}^{\mu\nu} = [(\hat{\eta}_{\alpha,\beta})^{-1}]^{\mu\nu}, \tag{3.2.139b}$$

and invariant in our spacetime over the reals

$$x^2 = x^\mu \times x_\nu = x^\mu \times \hat{\eta}_{\mu\nu}(x, v, a, \mu, \tau, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \ldots) \times x^\nu =$$

$$= T_{11} \times x_1^2 + \hat{T}_{22} \times x_2^2 + \hat{T}_{33} \times x_3^2 - \hat{T}_{44} \times x_4^2 \ \in \ R. \tag{3.2.140}$$

Note that all scalars on M must be lifted into *isoscalars* to have meaning for $\hat{M}$, i.e., they must have the structure of the isonumbers $\hat{n} = n \times \hat{I}$. This condition requires the re-definition $x \ \rightarrow \ \hat{x} = x \times \hat{I}$, $\eta_{\mu\nu} \ \rightarrow \ \hat{G}_{\mu\nu} = \hat{\eta}_{\mu\nu} \times \hat{I}$, $x^2 \rightarrow \hat{x}^{\hat{2}}$, etc.

The reader interested in learning in depth the new isogeometry should also study from the preceding sections the different realizations of the isometry whether realized in the original Minkowskian coordinates or in the isocoordinates, since the functional dependence is different in these two cases.

Note however the redundancy in practice for using the forms $\hat{x} = x \times \hat{I}$ and $\hat{G} = \hat{\eta} \times \hat{I}$ because of the identity $\hat{x}^{\hat{2}} = \hat{x}^\mu \hat{\times} \hat{G}_{\mu\nu} \hat{\times} \hat{x}^\nu \equiv (x^\mu \times \hat{\eta}_\nu \times x^\nu) \times \hat{I}$. For simplicity we shall often use the conventional coordinates x and the isometric will be referred to $\hat{\eta} = \hat{T} \times \eta$. The understanding is that the full isotopic formulations are needed for mathematical consistency.

A fundamental property of the infinite family of generalized spaces (3.2.133) is the lifting of the basic unit $I \rightarrow \hat{I}$ while the metric is lifted of the *inverse* amount, $\eta \rightarrow \hat{\eta} = \hat{T} \times \eta$, $\hat{I} = \hat{T}^{-1}$. This implies the preservation of all original axioms, and we have the following:

THEOREM 3.2.7 [26]: All infinitely possible isominkowski spaces $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$ over the isofields $\hat{R}(\hat{n}, \hat{+}, \hat{\times})$ with a common positive-definite isounit $\hat{I}$ preserve all original axioms of the Minkowski space $M(x, \eta, R)$ over the reals $R(n, +, \times)$.

The nontriviality of the lifting is that *the Minkowskian axioms are preserved under an arbitrary functional dependence of the metric $\hat{\eta} = \hat{\eta}(x, v, a, \mu, \tau, \ldots)$* for which the sole x-dependence of the Riemannian metric $g(x)$ is only a simple *particular case*. As a matter of fact, we have the following

THEOREM 3.2.8 [26]: Minkowski-Santilli isospaces are "directly universal" in spacetime, that is, they represent all infinitely possible spacetimes with signature $(+,+,+,-)$ ("universality"), directly with the isometric and without any use of the transformation theory ("direct universality").

Note that all possible "deformations" of the Minkowski space are also particular cases of the above isospaces. However, the former are still referred to the old unit I, thus losing the isomorphic between deformed and Minkowski spaces, while the isotopies preserve the original axioms by construction.

A fundamental physical characteristic of the Minkowski-Santilli isospaces is that *it alters the units of space and time*. Recall that the unit

$$I = Diag.(\{1, 1, 1\}, 1)$$

of the Minkowski space represents in a dimensionless form the units of the three Cartesian axes and time, e.g., $I = (+1\,\mathrm{cm}, +1\,\mathrm{cm}, +1\,\mathrm{cm}, +1\,\mathrm{sec})$. Recall also that the Cartesian space-units are *equal for all axes.*

Consider now the isospaces, and recall that $\hat{I}$ is positive-definite. Consequently, we have the following lifting of the units in which the $\hat{T}_{\mu\mu}$ quantities are reinterpreted as constants

$$I = (+1\,\mathrm{cm}, +1\,\mathrm{cm}, +1\,\mathrm{cm}, +1\,\mathrm{sec}) \rightarrow$$
$$\rightarrow \hat{I} = Diag.(n_1^2, n_2^2, n_3^2, n_4^2) = 1/\hat{T},\ \hat{I}^\mu_\mu = n_\mu^2, n_\mu > 0. \qquad (3.2.141)$$

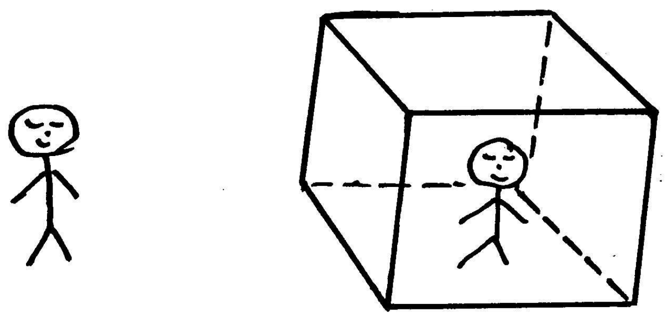

Figure 3.10. A view of the "spacetime isocube" characterized by the "space isocube" of Figure 3.7 now inspected in two spacetimes, the conventional Minkowski spacetime in the exterior and Santilli isospacetime in the interior. In addition to the variations of shape, size and dimensions indicated in Figure 3.7, the same object can be in different times for the two observers, all in a way fully compatible with our sensory perception. Consequently, seeing in a telescope a far away quasar or galaxy it does not mean that astrophysical structure is necessarily in our time, since it could be evolving far away in the future or in the past.

This means that, not only the original units are now lifted into arbitrary positive values, but the *units of different space axes generally have different values.* Jointly, the components of the metric are lifted by the *inverse* amounts n_μ^{-2}. This implies the preservation on $\hat{M}$ over $\hat{R}$ of the original *numerical* values on M over R, including the crucial preservation of the maximal causal speed c_o, as we shall see in Section 3.5.

Note also the necessary condition that *the isospace and isofield have the same isounit* $\hat{I}$. This condition is absent in the conventional Minkowski space where the unit of the space is the unit *matrix* $I = Diag.(1,1,1,1)$, while that of the underlying field is the *number* $I = +1$. Nevertheless, the latter can be trivially reformulated with the common unit matrix I, by achieving in this way the form admitted as a particular case by the covering isospaces

$$M(x, \eta, R): \; x = \{x^\mu \times I\}, \; x^2 = (x^\mu \times \eta_{\mu\nu} \times x^\nu) \times I \in R. \tag{3.2.142}$$

The structure of both the conventional and isotopic invariants is therefore given by Theorem 3.2.66, namely

$$\text{Basic Invariant} = (\text{Length})^2 \times (\text{Unit})^2, \tag{3.2.143}$$

which illustrates more clearly the preservation under the dual lifting $\eta \rightarrow \hat{\eta} = \hat{T} \times \eta$ and $I \rightarrow \hat{I} = 1/\hat{T}$ of the original axioms as well as numerical values.

THEOREM 3.2.9 [6,7,26]: Conventional and isotopic symmetries of spacetime are 11-dimensional.

Proof. In addition to the 10-dimensionality of the Poincaré symmetry, there is an additional 11-th dimensionality characterized by the isotransform

$$\eta \to \hat{\eta} = \eta/n^2, \;\; I \to \hat{I} = n^2 \times I, \tag{3.2.144}$$

where n is a non-null constant. **q.e.d.**

Note the crucial role of Santilli's isonumbers in the above property. This explains why the 11-th dimensionality remained undiscovered throughout the 20-th century.

A significant difference between the conventional space M and its isotopes $\hat{M}$ is that the former admit only *one* formulation, the conventional one, while the latter admit *two* formulations: that on isospace itself (i.e., expressed with respect to the isounit $\hat{I}$) and its *projection* in the original space M (i.e., expressed with respect to the conventional unit I).

Note that the projection of $\hat{M}(\hat{x}, \hat{M}, \hat{R})$ into $M(x, \eta, R)$ is not a conformal map, but an *inverse isotopic map* because it implies the transition from generalized units and fields to conventional units and fields.

The axiomatic motivation for constructing the isotopies of the Minkowskian geometry is that any modification of the Minkowski metric requires the use of *noncanonical transforms* $x \;\to\; x'(x)$,

$$\eta_{\mu\nu} \;\to\; \hat{\eta}_{\mu\nu} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \eta_{\alpha\beta} \frac{\partial x'^{\beta}}{\partial x^{\nu}} \neq \eta_{\mu\nu}, \tag{3.2.145}$$

and this includes the case of the transition from the Minkowskian metric η to the Riemannian metric $g(x)$.

In turn, all noncanonical theories, thus including the Riemannian geometry, do not possess invariant units of space and time, thus having the catastrophic inconsistencies studied in Chapter 1. A primary axiomatic function of the isospace is that of restoring the invariance of the basic units, as established by the Poincaré-Santilli isosymmetry.

This is achieved by embedding all noncanonical content in the generalization of the unit. Invariance for noncanonical structures such as Riemannian metrics is then assured by the fact indicated earlier that, whether conventional or generalized, the unit is the basic invariant of any theory.

Stated in different terms, a primary axiomatic difference between the special and general relativities is that the time evolution of the former is a *canonical transform*, thus implying the majestic mathematical and physical consistency of special relativity recalled in Chapter 1, while the time evolution of the latter is a *noncanonical transform*, thus implying a number of unresolved problematic aspects that have been lingering throughout this century.

The reformulation of the Riemannian geometry in terms of the Minkowskian axioms is the sole possibility known to this author for achieving axiomatic consistency under a nontrivial functional dependence of the metric.

In summary, Minkowski-Santilli isospaces have the following primary applications. First, they are used for a re-interpretation of the Riemannian metrics $g(x)$ for the particular case

$$\hat{\eta} = \hat{\eta}(x) = g(x) \tag{3.2.146}$$

characterizing *exterior gravitational problems in vacuum.* Second, the same isospaces are used for the characterization of *interior gravitational problems* with isometrics of unrestricted functional dependence

$$\hat{\eta} = \hat{\eta}(x, v, a, \mu, \tau, \ldots) = g(x, v, a, \mu, \tau, \ldots) \tag{3.2.147}$$

while preserving the original Minkowskian axioms.

Since the explicit functional dependence is inessential under isotopies, our studies will be generally referred to the interior gravitational problem. Unless otherwise stated, only diagonal realizations of the isounits will be used hereon for simplicity. An example of nondiagonal isounits inherent in a structure proposed by Dirac is indicated in Section 3.5. More general liftings of the Minkowski space of the so-called *genotopic and multivalued-hyperstructural type* will be indicated in Chapter 4.

3.2.9C. Isoderivative, Isoconnection, and Isoflatness. In the preceding subsections we have presented the *Minkowskian* aspects of the new isogeometry. We are now sufficiently equipped to present the novel part of the Minkowski-Santilli isogeometry, its *Riemannian* character as first derived in Ref. [26].

Our study is strictly in local coordinates representing the *fixed* frame of the observer without any un-necessary use of the transformation theory or abstract treatments. Our presentation will be as elementary as possible without reference to advanced topological requirements, such as Kadeisvili's isocontinuity (Section 3.2.6), isomanifolds and related TSSFN isotopology (Section 3.2.7) .

Also, our presentation is made, specifically, for the (3+1)-dimensional isospacetime, with the understanding that the extension to arbitrary dimensions and signatures or signatures different than the conventional one $(+,+,+,-)$ is elementary, and will be left to interested readers.

Let $\hat{M}(\hat{x}, \hat{G}, \hat{R})$ be a Minkowski-Santilli isospace and let $\hat{M}(x, \hat{\eta}, R)$ be its projection in our spacetime as per Definition 3.2.12. To illustrate the transition from isocoordinates $\hat{x}$ to conventional spacetime coordinates x, we shall denote the projection $\hat{M} = \hat{M}(\hat{x}, \hat{\eta}, R)$. This notation emphasizes that the referral of the isospace to the conventional units and field causes the reduction of the isometric from the general form $\hat{G} = \hat{\eta} \times \hat{I}$ to $\hat{\eta} = \hat{T} \times \eta$, where, as now familiar, $\hat{I} = 1/\hat{T}$ and $\eta = Diag.(1, 1, 1, -1)$ is the familiar Minkowskian metric.

According to this notation the Riemannian content of the Minkowski-Santilli isogeometry can be unified in both its isospace formulation properly speaking and its projection in our spacetime. All differences in the interpretations whether occurring in isospace or in our spacetime are then deferred to the selection of the basic unit.

Consider now the infinitesimal version of isoinvariant (3.2.137) permitted by the isodifferential calculus

$$\hat{d}\hat{s}^{\hat{2}} = \hat{d}\hat{x}_\mu \hat{\times} \hat{d}\hat{x}^\mu \in \hat{R}. \tag{3.2.148}$$

The *isonormal coordinates* occur when the isometric $\hat{\eta}$ is reduced to the Minkowski metric η as in conventional Riemannian geometry. Consequently, isonormal coordinates coincide with the conventional normal coordinates, and the Minkowski-Santilli isogeometry verifies the *principle of equivalence* as for the conventional Riemannian geometry.

By using the isodifferential calculus, we now introduce the *isodifferential of a contravariant isovector field* on $\hat{M}$ over $\hat{R}$ [15]

$$\hat{d}\hat{X}^\beta = (\hat{\partial}_\mu \hat{X}^\beta)\hat{\times}\hat{d}\hat{x}^\mu = \hat{I}^\rho_\mu \times (\partial_\rho \hat{X}^\beta)\hat{\times}\hat{T}^\mu_\sigma \times d\hat{x}^\sigma \equiv$$

$$\equiv (\partial_\mu X^\beta) \times d\hat{x}^\mu = (\partial^\rho X^\beta) \times \hat{\eta}_{\rho\sigma} \times d\hat{x}^\sigma, \tag{3.2.149}$$

where the last expression is introduce to recall that the contractions are in isospace. The preceding expression then shows that *isodifferentials of isovector fields coincide at the abstract level with conventional differentials for all isotopies of the class here admitted* (that with $\hat{I} > 0$).

DEFINITION 3.2.13 [26]: The isocovariant isodifferential are defined by

$$\hat{D}\hat{X}^\beta = \hat{d}\hat{X}^\beta + \hat{\Gamma}^\beta_{\alpha\gamma} \hat{\times} \hat{X}^\alpha \hat{\times} \hat{d}\hat{x}^\gamma, \tag{3.2.150}$$

with corresponding isocovariant derivative

$$\hat{X}^\beta_{\hat{|}\mu} = \hat{\partial}_\mu \hat{X}^\beta + \hat{\Gamma}^\beta_{\alpha\mu} \hat{\times} \hat{X}^{\hat{\alpha}}, \tag{3.2.151}$$

where the iso-Christoffel's symbols are given by

$$\hat{\Gamma}^\beta_{\alpha\gamma}(x, v, a, \mu, \tau, \ldots) = \frac{\hat{1}}{\hat{2}} \hat{\times} (\hat{\partial}_\alpha \hat{\eta}_{\beta\gamma} + \hat{\partial}_\gamma \hat{\eta}_{\alpha\beta} - \hat{\partial}_\beta \hat{\eta}_{\alpha\gamma}) \times \hat{I} = \hat{\Gamma}_{\gamma\beta\alpha}, \tag{3.2.152a}$$

$$\hat{\Gamma}^\beta_{\alpha\gamma} = \hat{\eta}^{\beta\rho} \times \hat{\Gamma}_{\alpha\rho\gamma} = \hat{\Gamma}^\beta_{\gamma\alpha}. \tag{3.2.152b}$$

[15] We should note that the role of the isounit and of the isoelement in this presentation and in that of Ref. [26] are interchanged for general compatibility with the various applications and developments.

Note the unrestricted functional dependence of the connection which is notoriously absent in conventional treatments. Note also the abstract identity of the conventional and isotopic connections. Note finally that *local numerical values of the conventional and isotopic connections coincide when computed in their respective spaces.* This is due to the fact that in Eq.s (3.2.152) $\hat{\eta} \equiv g(x)$ for exterior problems, while the value of derivatives ∂_μ and isoderivatives $\hat{\partial}_\mu$ coincide when computed in their respective spaces.

Note however that, when projected in the conventional spacetime, the conventional and isotopic connections are different even in the exterior problem in which $\hat{\eta} = g(x)$,

$$\hat{\Gamma}_{\alpha\beta\gamma} = \frac{1}{2} \times (\hat{I}^\mu_\alpha \times \partial_\mu g_{\beta\gamma} + \hat{I}^\rho_\gamma \times \partial_\rho \hat{\eta}_{\alpha\beta} - \hat{I}^\sigma_\beta \times \partial_\sigma g_{\alpha\gamma}) \times \hat{I} \neq \Gamma_{\alpha\beta\gamma} \times \hat{I}. \quad (3.2.153)$$

The extension to covariant isovector fields and covariant or contravariant isotensor fields is consequential.

Without proof we quote the following important result from Ref. [26]:

LEMMA 3.2.7 (Iso-Ricci Lemma) [26]: Under the assumed conditions, the isocovariant derivatives of all isometrics on Minkowski-Santilli isospaces spaces are identically null,

$$\hat{\eta}_{\alpha\beta\hat{|}\gamma} \equiv 0, \quad \alpha, \beta, \gamma = 1, 2, 3, 4. \quad (3.2.154)$$

The novelty of the isogeometry is then illustrated by the fact that *the Ricci property persists under an arbitrary dependence of the metric, as well as under Minkowskian, rather than Riemannian axioms.*

The *isotorsion* on $\hat{M}$ is defined by

$$\hat{\tau}^\beta_{\alpha\gamma} = \hat{\Gamma}^\beta_{\alpha\gamma} - \hat{\Gamma}^\beta_{\gamma\alpha}, \quad (3.2.155)$$

and coincides again with the conventional torsion at the abstract level, although the two torsions have significant differences in their explicit forms when both projected in our space-time.

DEFINITION 3.2.14 [26]: The Minkowski-Santilli isogeometry is characterized by the following isotensor: the isoflatness isotensor

$$\hat{R}^\beta_{\alpha\gamma\delta} = \hat{\partial}_\delta \hat{\Gamma}^\beta_{\alpha\gamma} - \hat{\partial}_\gamma \hat{\Gamma}^\beta_{\alpha\delta} + \hat{\Gamma}^\beta_{\rho\delta} \hat{\times} \hat{\Gamma}^\rho_{\alpha\gamma} - \hat{\Gamma}^\beta_{\rho\gamma} \hat{\times} \hat{\Gamma}^\rho_{\alpha\delta}; \quad (3.2.156)$$

the iso-Ricci isotensor

$$\hat{R}_{\mu\nu} = \hat{R}^\beta_{\mu\nu\beta}; \quad (3.2.157)$$

the isoflatness isoscalar

$$\hat{R} = \hat{\eta}^{\alpha\beta} \times \hat{R}_{\alpha\beta}; \quad (3.2.158)$$

the iso-Einstein isotensor

$$\hat{G}_{\mu\nu} = \hat{R}_{\mu\nu} - \frac{\hat{1}}{2} \hat{\times} \hat{N}_{\mu\nu} \hat{\times} \hat{R},\ \hat{N}_{\mu\nu} = \hat{\eta}_{\mu\nu} \times \hat{I}; \tag{3.2.159}$$

and the isotopic isoscalar

$$\hat{\Theta} = \hat{N}^{\alpha\beta} \hat{\times} \hat{N}^{\gamma\delta} \hat{\times} (\hat{\Gamma}_{\rho\alpha\delta} \hat{\times} \hat{\Gamma}^{\rho}_{\gamma\beta} - \Gamma_{\rho\alpha\beta} \hat{\times} \hat{\Gamma}^{\rho}_{\gamma\delta}) =$$
$$= \hat{\Gamma}_{\rho\alpha\beta} \hat{\times} \hat{\Gamma}^{\rho}_{\gamma\delta} \hat{\times} (\hat{N}^{\alpha\delta} \hat{\times} \hat{N}^{\gamma\beta} - \hat{N}^{\alpha\beta} \hat{\times} \hat{N}^{\gamma\delta}); \tag{3.2.160}$$

the latter being new for the Minkowski-Santilli isogeometry.

Note the lack of use of the term "isocurvature" and the use instead of the term "isoflatness". This is due to the fact that the prefix "iso-" represents the preservation of the original axioms. The term "isocurvature" would then be inappropriate because the basic axioms of the geometry are flat.

In any case, the main problem underlying the studies herein reported is, as indicated in Chapter 1, that *curvature is the ultimate origin of the catastrophic inconsistencies of general relativity.* Consequently, all geometric efforts are here aimed at the replacement of the notion of curvature with a covering notion resolving the indicated catastrophic inconsistencies.

As we shall see better in Section 3.5, the notion of "isoflatness" does indeed achieve the desired objectives because flatness and its related invariance of gravitation under the Poincaré-Santilli isosymmetry is reconstructed on isospaces over isofields, while the ordinary curvature emerge as a mere projection in our spacetime.

3.2.9D. The Five Identities of the Minkowski-Santilli Isogeometry. By continuing our review of memoir [26], tedious but simple calculations yield the following *five basic identities of the Minkowski-Santilli isogeometry:*

Identity 1: *Antisymmetry of the last two indices of the isoflatness isotensor*

$$\hat{R}^{\beta}_{\alpha\gamma\delta} = -\hat{R}^{\beta}_{\alpha\delta\gamma}; \tag{3.2.161}$$

Identity 2: *Symmetry of the first two indices of the isoflatness isotensor*

$$\hat{R}_{\alpha\beta\gamma\delta} \equiv \hat{R}_{\beta\alpha\gamma\delta}; \tag{3.2.162}$$

Identity 3: *Vanishing of the totally antisymmetric part of the isoflatness isotensor*

$$\hat{R}^{\beta}_{\alpha\gamma\delta} + \hat{R}^{\beta}_{\gamma\delta\alpha} + \hat{R}^{\beta}_{\delta\,\alpha\gamma} \equiv 0; \tag{3.2.163}$$

Figure 3.11. Primary objectives of the Minkowski-Santilli isogeometry are the resolution of the catastrophic inconsistencies of the Riemannian formulation of exterior gravitation (Section 1.4) and a representation of interior gravitation as occurring for the Sun depicted in this figure and any other massive object. These objectives are achieved via the isotopies of the Minkowskian geometry since they are flat in isospace, thus admitting a well defined invariance for all possible gravitation, by adding sources requested by the Freud identity and other reasons, and by unifying exterior and interior gravitational problem in a single formulation in isospace that formally coincides with that for the exterior problem, the interior effects being incorporated in the isounit (see Section 3.5).

Identity 4: *Iso-Bianchi identity*

$$\hat{R}^{\beta}_{\alpha\gamma\delta\hat{|}\rho} + \hat{R}^{\beta}_{\alpha\rho\gamma\hat{|}\delta} + \hat{R}^{\beta}_{\alpha\delta\rho\hat{|}\gamma} \equiv 0; \tag{3.2.164}$$

Identity 5: *Iso-Freud identity*

$$\hat{R}^{\alpha}_{\beta} - \frac{\hat{1}}{2}\hat{\times}\hat{\delta}^{\alpha}_{\beta}\hat{\times}\hat{R} - \frac{\hat{1}}{2}\hat{\times}\hat{\delta}^{\alpha}_{\beta}\hat{\times}\hat{\Theta} = \hat{U}^{\alpha}_{\beta} + \hat{\partial}_{\rho}\hat{V}^{\alpha\rho}_{\beta}, \tag{3.2.165}$$

where $\hat{\Theta}$ is the isotopic isoscalar and

$$\hat{U}^{\alpha}_{\beta} = -\frac{1}{2}\frac{\hat{\partial}\hat{\Theta}}{\hat{\partial}\hat{\eta}^{\alpha\beta}_{\hat{|}\alpha}}\hat{\eta}^{\alpha\beta}_{\hat{|}\beta}, \tag{3.2.166a}$$

$$\hat{V}^{\alpha\rho}_{\beta} = \frac{1}{2}[\hat{\eta}^{\gamma\delta}(\delta^{\alpha}_{\beta}\hat{\Gamma}^{\rho}_{\alpha\delta} - \delta^{\rho}_{\beta}\hat{\Gamma}^{\rho}_{\gamma\delta}) + \tag{3.2.166b}$$

$$+(\delta^{\rho}_{\beta}\hat{\eta}^{\alpha\gamma} - \delta^{\alpha}_{\beta}\hat{\eta}^{\rho\gamma})\hat{\Gamma}^{\delta}_{\gamma\delta} + \hat{\eta}^{\rho\gamma}\hat{\Gamma}^{\alpha}_{\beta\gamma} - \hat{\eta}^{\alpha\gamma}\hat{\Gamma}^{\rho}_{\beta\gamma}], \tag{3.2.166c}$$

Note that the conventional Riemannian geometry is generally thought to possess only *four* identities. In fact, the *fifth* identity (3.2.165) is generally unknown in the contemporary literature in gravitation as the reader is encouraged to verify in the specialized literature in the Riemannian geometry (that is so vast to discourage discriminatory listings).

The latter identity was introduced by Freud [27] in 1939, treated in detail by Pauli in his celebrated book [28] of 1958 and then generally forgotten for a half a century, apparently because of its evident incompatibility between Einstein's conception of exterior gravitation in vacuum as pure curvature without source (see Section 3.4)

$$G^\alpha_\beta = R^\alpha_\beta - \frac{1}{2}\delta^\alpha_\beta R = 0, \tag{3.2.167}$$

and the need for a source term also in exterior gravitation in vacuum mandated by the Freud identity and other reasons

$$R^\alpha_\beta - \frac{1}{2}\delta^\alpha_\beta R - \frac{1}{2}\delta^\alpha_\beta \Theta = U^\alpha_\beta + \hat{\partial}_\rho V^{\alpha\rho}_\beta. \tag{3.2.168}$$

Freud's identity was rediscovered by the author during his accurate study of Pauli's historical book and studied in detail in Refs. [6,7] of 1992. Additional studies of the Freud identity were done by Yilmaz [30]. Following a suggestion by the author, the late mathematician Hanno Rund [29] studied the identity in one of his last papers and proved that:

LEMMA 3.2.8 (Rund's Lemma) [29]: Freud's identity is a bona fide identity for all Riemannian spaces irrespective of dimension and signature.

In this way, Rund confirmed the general need of a source also in vacuum (see Sections 1.4 and 3.5).

Following Ref. [26], in this paper we have presented the isotopies of the Freud identity on Minkowski-Santilli isospaces, as characterized by the isodifferential calculus. Its primary functions for this monograph is to identify the geometric structure of the *interior* gravitational problem. The persistence of the source in vacuum as per the Freud identity, electrodynamics and other needs will then be consequential, thus confirming the inconsistency of Einstein's conception of gravity in vacuum as pure curvature without source.

Note that *all conventional and isotopic identities coincide at the abstract level.*

3.2.9E. Isoparallel Transport and Isogeodesics. An isovector field $\hat{X}^\beta$ on $\hat{M} = \hat{M}(\hat{x}, \hat{M}, \hat{R})$ is said to be transported by *isoparallel* displacement from a point $\hat{m}(\hat{x})$ on a curve $\hat{C}$ on $\hat{M}$ to a neighboring point $\hat{m}'(\hat{x} + \hat{d}\hat{x})$ on $\hat{C}$ if

$$\hat{D}\hat{X}^\beta = \hat{d}\hat{X}^\beta + \hat{\Gamma}^\beta_{\alpha\gamma} \hat{\times} \hat{X}^\alpha \hat{\times} \hat{d}\hat{x}^\gamma \equiv 0, \tag{3.2.169}$$

or in integrated form

$$\hat{X}^{\beta}(\hat{m}') - \hat{X}^{\beta}(m) = \hat{\int}_{\hat{m}}^{\hat{m}'} \frac{\hat{\partial}\hat{X}^{\beta}}{\hat{\partial}\hat{x}^{\alpha}} \frac{\hat{d}\hat{x}^{\alpha}}{\hat{d}\hat{s}} \hat{\times} \hat{d}\hat{s}, \tag{3.2.170}$$

where one should note the isotopic character of the integration. The isotopy of the conventional case then yields the following:

LEMMA 3.2.9 [26]: Necessary and sufficient condition for the existence of an isoparallel transport along a curve $\hat{C}$ on a $(3+1)$-dimensional Minkowski-Santilli isospace is that all the following equations are identically verified along $\hat{C}$

$$\hat{R}^{\beta}_{\alpha\gamma\delta} \hat{\times} \hat{X}^{\alpha} = 0, \quad \alpha, \beta, \gamma, \delta = 1, 2, 3, 4. \tag{3.2.171}$$

Note, again, the abstract identity of the conventional and isotopic parallel transport. However, it is easy to see that the projection of the isoparallel transport in ordinary spacetime is structurally different than the conventional parallel transport.

Consider, as an example, an extended object in gravitational fall in atmosphere (see Figure 3.12). Its trajectory is evidently irregular and depends on the actual shape of the object, as well as its weight. The understanding of the new Minkowski-Santilli isogeometry requires the knowledge of the fact that said trajectory is represented on isospace over isofields as a *straight line,* that is, via the trajectory in the absence of the resistive medium. The actual, irregular trajectory appears only in the projection of said isotrajectory in our spacetime.

If the latter treatment is represented by a rocket, one would note a twisting action as occurring in the reality of motion within physical media, which is evidently absent in the exterior case.

Along similar lines, we say that a smooth isopath $\hat{x}_{\alpha}$ on $\hat{M}$ with isotangent $\hat{v}_{\alpha} = \hat{d}\hat{x}_{\alpha}/\hat{d}\hat{s}$ is an *isogeodesic* when it is solution of the isodifferential equations

$$\frac{\hat{D}\hat{v}^{\beta}}{\hat{D}\hat{s}} = \frac{\hat{d}\hat{v}}{\hat{d}\hat{s}} + \hat{\Gamma}_{\alpha\beta\gamma} \hat{\times} \frac{\hat{d}\hat{x}^{\alpha}}{\hat{d}\hat{s}} \hat{\times} \frac{\hat{d}\hat{x}^{\gamma}}{\hat{d}\hat{s}} = 0. \tag{3.2.172}$$

It is easy to prove the following:

LEMMA 3.2.10 [26]: The isogeodesics of a Minkowski-Santilli isospace $\hat{M}$ are the isocurves verifying the isovariational principle

$$\hat{\delta} \hat{\int} [\hat{G}_{\alpha\beta}(\hat{x}, \hat{v}, \hat{a}, \mu, \tau, \ldots) \hat{\times} \hat{d}\hat{x}^{\alpha} \hat{\times} \hat{d}\hat{x}^{\beta}]^{\hat{1/2}} = 0, \tag{3.2.173}$$

where again isointegration is understood.

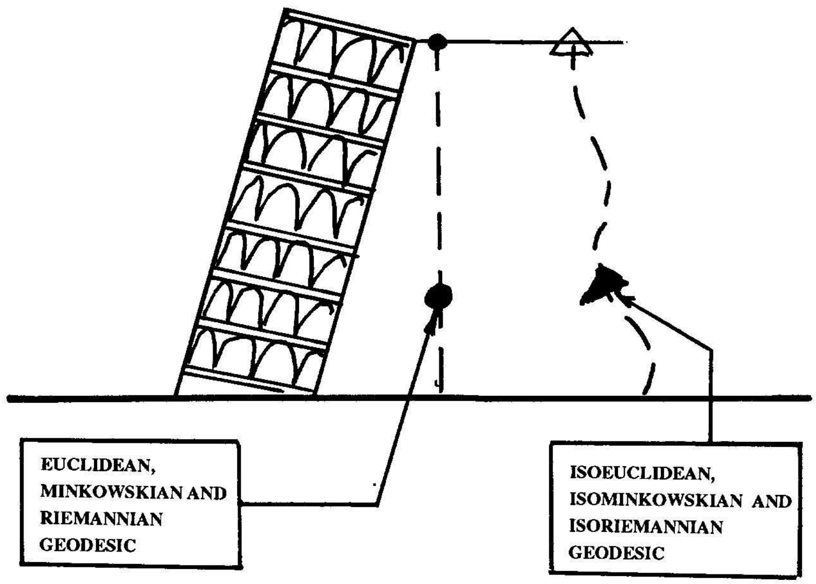

Figure 3.12. A schematic view of two objects released from the Pisa tower. The vertical trajectory represents the approximate geodesic considered by Galileo, used by Einstein and adopted until the end of the 20-th century, namely, the approximation under the lack of resistance due to our atmosphere. The Minkowski-Santilli isogeometry has been built to represent as isogeodesics actual trajectories within physical media.

Finally, we point out the property inherent in the notion of isotopies according to which

COROLLARY 3.2.10A: [26]: Trajectories in an ordinary Riemannian space coincide with the corresponding isogeodesic trajectories in Minkowski-Santilli isospace, but not with the projection of the latter in the original space.

For instance, if a circle is originally a geodesic, its image under isotopy in isospace remains the perfect circle, the *isocircle* (Section 3.2.9), even though its projection in the original space can be an ellipse. The same preservation in isospace occurs for all other curves.

The differences between a geodesic and an isogeodesic therefore emerge only when projecting the latter in the space of the former.

An empirical but conceptually effective rule is that *interior physical media "disappear" under their isogeometrization*, in the sense that actual trajectories under resistive forces due to physical media (which are not geodesics of a Minkowski space) are turned into isogeodesics in isospace having the shape of the geodesics in the absence of resistive forces.

The simplest possible example is given by the iso-Euclidean representation of a straight stick partially immersed in water. In conventional representations the stick penetrating in water with an angle α appears as bended at the point of immersion in water with an angle $\gamma = \alpha + \beta$, where β is the angle of refraction. In iso-Euclidean representation the stick remains straight also in its immersion because the isoangle $\hat{\gamma} = \gamma \times \hat{I}_\gamma$ recovers the original angle α with $\hat{I}_\gamma = \alpha/(\alpha+\beta)$.

The situation is essentially the same for our representation of interior gravitation because the latter is represented in isospace over isofield via field equations (this time necessarily with sources) that formally coincide with conventional equations on a conventional Riemannian spacetime. Being noncanonical, all interior features are invariantly represented via generalized units.

3.2.9F. Isodual Minkowski-Santilli isospaces and isogeometry. The *isodual Minkowski-Santilli isospaces* were introduced for the first time by Santilli in Ref. [8] of 1985 and then studied in various works (see the references of Chapter 1), and can be written

$$\hat{M}^d = \hat{M}^d(\hat{x}^d, \hat{\eta}^d, \hat{R}^d) :$$

$$\hat{x}^d = \{x^{\mu d}\} \times^d \hat{I}^d = \{x^\mu\} \times (-\hat{I}) = \{r^d, c_o^d \times^d t^d\} \times^d \hat{I}^d, \tag{3.2.174a}$$

$$\hat{\eta}^d = -\hat{\eta}. \tag{3.2.174b}$$

The *isodual Minkowski-Santilli isogeometry* is the geometry of isodual isospaces M^d over R^d and was studied for the first time by Santilli in Ref. [26] of 1998.

The physically and mathematically most salient property of the latter geometry is that it is *characterized by negative units of space, time, etc., and negative norms.* Therefore, in addition to a change in the sign of the charge, we also have change of sign of masses, energies, and other quantities normally positive for matter. Similarly, we have the *isodual isospace and isotime coordinates*

$$\hat{x}^d = \hat{x}^d \times^d \hat{I} = -\hat{x}, \;\; \hat{t}^d = t^d \times^d \hat{I}^d = -\hat{t}. \tag{3.2.175}$$

Thus, motion under isoduality is in a time direction *opposite* to the conventional motion. These features are necessary so as to have a classical representation of antimatter in interior conditions whose operator image yields indeed antiparticles (rather than particles with the wrong sign of the charge).

We also have the following important

LEMMA 3.2.12 [17]: Isodualities are independent from spacetime inversions

$$r' = \pi \times r = -r, \;\; t' = \tau \times t = -t. \tag{3.2.176}$$

Proof. Inversions occur within the same original space and keep the unit fixed, while isodualities require a map to a different space, and change the sign of the unit. Therefore, in addition to maps in different spaces, isodualities have numerical value different than the inversions. **q.e.d.**

These are the conceptual roots for the isodual theory of antimatter to predict a *new photon*, the *isodual photon* emitted by antimatter [17]. When applied to the photon, charge conjugation and, more generally, the PCT theorem, do not yield a new photon, as well known. This is not the case under isoduality because all physical characteristics change in sign and numerical value. As a result, *the isodual photon is indistinguishable from the ordinary photon under all interactions except gravitation*. In fact, as indicated in Chapter 1, the isodual photon is predicted to experience antigravity in the field of matter, thus offering, apparently for the first time, a possibility for the future study whether far away galaxies and quasars are made up of matter or of antimatter.

Another important property of isoduality is expressed by the following:

LEMMA 3.2.13 [26]: The intervals of conventional and isotopic Minkowskian spaces are invariant under the joint isodual maps $\hat{I}^d \rightarrow \hat{I}^d$ and $\hat{\eta} \rightarrow \hat{\eta}^d$,

$$\hat{x}^2 = (x^\mu \times \hat{\eta}_{\mu\nu} \times x^\nu) \times \hat{I} \equiv [x^\mu \times (-\hat{\eta}_{\mu\nu}) \times x^\nu] \times (-\hat{I}). \qquad (3.2.177)$$

As a result, *all physical laws applying in conventional Minkowskian geometry for the characterization of matter also apply to its isodual image for the characterization of antimatter.*

Note that, strictly speaking, the intervals are not isoselfdual because

$$\hat{x}^{\hat{2}} = \hat{x}^\mu \hat{\times} \hat{M}_{\mu\nu} \hat{\times} \hat{x}^\mu \ \rightarrow \ \hat{x}^{d\hat{2}d} = \hat{x}^{\mu d} \times^d \hat{M}^d_{\mu\nu} \times^d \hat{x}^{\mu d} = \hat{x}^{d\hat{2}d} = -\hat{x}^{\hat{2}}. \qquad (3.2.178)$$

To outline the *Riemannian* characteristics of the isodual Minkowski-Santilli isogeometry, we consider an *isodual isovector isofield* $\hat{X}^d(\hat{x}^d)$ on $\hat{M}^d$ which is explicitly given by $\hat{X}^d(\hat{x}^d) = -X^t(-x^t \hat{\times} \hat{I}) \hat{\times} \hat{I}$. The *isodual exterior isodifferential* of $\hat{X}^d(\hat{x}^d)$ is given by

$$\hat{D}^d \hat{X}^{\mu d}(\hat{x}^d) = \hat{d}^d \hat{X}^{\mu d}(\hat{x}^d) \hat{+}^d \hat{\Gamma}^{d\,\mu}_{\alpha\beta} \hat{\times}^d \hat{X}^{\alpha d} \hat{\times}^d \hat{d}^d \hat{x}^{\beta d} = \hat{D} \hat{X}^{t\mu}(-\hat{x}^t), \qquad (3.2.179)$$

where the $\hat{\Gamma}^d$'s are the components of the *isodual isoconnection*. The *isodual isocovariant isoderivative* is then given by

$$\hat{X}^{\mu d}(\hat{x}^d)_{\hat{|}d\nu} = \hat{\partial}^d \hat{X}^{\mu d}(\hat{x}^d) \hat{/}^d \hat{\partial}^d \hat{x}^{\nu d} \hat{+} \hat{\Gamma}^{d\,\mu}_{\alpha\nu} \hat{\times}^d \hat{X}^{\alpha d}(\hat{x}^d) = -\hat{X}^{t\mu}(-\hat{x}^t)_{\hat{|}k}. \qquad (3.2.180)$$

The interested reader can then easily derive the remaining notions of the new geometry. It is an instructive exercise for the interested reader to prove the following isodualities:

$$
\begin{array}{cc}
\text{Isodual isounit} & \hat{I} \rightarrow \hat{I}^d = -\hat{I}, \\
\text{Isodual isometric} & \hat{\eta} \rightarrow \hat{\eta}^d = -\eta, \\
\text{Isodual isoconnection coefficients} & \hat{\Gamma}_{\alpha\beta\gamma} \rightarrow \hat{\Gamma}^d_{\alpha\beta\gamma} = \hat{\Gamma}_{\alpha\beta\gamma}, \\
\text{Isoflatness isotensor} & R_{\alpha\beta\gamma\delta} \rightarrow R^d_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\gamma\delta}, \\
\text{Isodual iso-Ricci isotensor} & \hat{R}_{\mu\nu} \rightarrow \hat{R}^d_{\mu\nu} = \hat{R}_{\mu\nu}, \\
\text{Isodual iso-Ricci isoscalar} & \hat{R} \rightarrow \hat{R}^d = \hat{R}, \\
\text{Isodual iso-Freud isoscalar} & \hat{\Theta} \rightarrow \hat{\Theta}^d = -\hat{\Theta}, \\
\text{Isodual Iso-Einstein isotensor} & \hat{G}_{\mu\nu} \rightarrow \hat{G}^d_{\mu\nu} = -\hat{G}_{\mu\nu}, \\
\text{Isodual electromagnetic potentials} & A_\mu \rightarrow A^d_\mu = -A_\mu, \\
\text{Isodual electromagnetic field} & F_{\mu\nu} \rightarrow F^d_{\mu\nu} = -F_{\mu\nu}, \\
\text{Isodual elm energy-mom. isotensor} & T_{\mu\nu} \rightarrow T^d_{\mu\nu} = -T_{\mu\nu}.
\end{array}
\tag{3.2.181}
$$

More detailed isogeometric studies are left to interested readers. Specific applications to gravitational treatments of matter and antimatter are presented in Section 3.5.

3.2.10 Isosymplectic Geometry and its Isodual

As it is well known, the *symplectic geometry* had an important role in the construction of quantum mechanics because it permitted the mathematically rigorous verification, known as *symplectic quantization,* that original quantization procedures, known also as *naive quantization,* were correct.

No broadening of quantum mechanics can be considered mature unless it admits fully equivalent procedures in the map from classical to operator forms known as *isoquantization* also called *hadronization* (rather than quantization).

For this purpose. Santilli [31] presented in 1988 the first known *isotopies of the symplectic geometry,* subsequently studied in various works, with a general presentation available in Vols. I, II of this series (see in particular Chapter 5 of Vol. I [6]). The new geometry is today known as *Santilli's isosymplectic geometry.*

We cannot possibly review here the isosymplectic geometry in detail and have to suggest interested readers to study Refs. [6,7]. Nevertheless, an indication of the basic lines is important for the self-sufficiency of this monograph.

Let us ignore the global (also called abstract) formulation of the symplectic geometry and consider for clarity and simplicity only its realization in a local chart (or coordinates).[16] A *topological manifold* $M(R)$ on the reals R admits the local realization as an Euclidean space $E(r, \delta R)$ with local contravariant coordinates

[16] Again, the literature on the conventional symplectic geometry is so vast to discourage discriminatory quotations.

$r = (r^i),\ i = i, 2, \ldots, N$. The *cotangent bundle* T^*M then becomes the ordinary phase space with local coordinates $(r, p) = (r^i, p_i)$, where p_i represents the tangent vectors (physically the linear momentum). The *canonical one-form* then admits the local realization

$$\theta = p_i \times dr^i. \tag{3.2.182}$$

The *fundamental (canonical) symplectic form* is then given by the exterior derivative of the preceding one form

$$\omega = d\theta = p_i \wedge dr^i, \tag{3.2.183}$$

and one can easily prove that it is closed, namely, that $d\omega \equiv 0$.

Consider now the *isotopological isomanifold* (introduced earlier) $\hat{M}(\hat{R})$ on the isoreals $\hat{R}$ with basic isounit $\hat{I}$. Its realization on local coordinates is given by the Euclid-Santilli isospace $\hat{E}(\hat{r}, \hat{\Delta}, \hat{R})$ with local contravariant isocoordinates $\hat{r} = (r^i) \times \hat{I}$. Then, the *isocotangent isobundle* $\hat{T}^*\hat{M}$ admits as local realization the *isophase isospace* with local coordinates $(\hat{r}^i, \hat{p}_i)$, where $\hat{p}$ is again a tangent isovector. The novelty is given by the fact that the unit of $\hat{p}$ is the *inverse* of that of $\hat{r}$ and we shall write

$$\hat{r} = r \times \hat{I},\ \ \hat{p} = p \times \hat{T},\ \ \hat{I} = 1/\hat{T}. \tag{3.2.184}$$

This property was identified for the first time by Santilli [31] (for a mathematical treatment see also Ref. [10]) because not identifiable in the conventional symplectic geometry due to the use of the trivial unit for which $I^{-1} \equiv I = +1$.

Consequently, we have the isodifferentials

$$\hat{d}\hat{r} = \hat{T} \times d(r \times I),\ \ \hat{d}\hat{p} = \hat{I} \times d(p \times \hat{T}). \tag{3.2.185}$$

The *isocanonical one-isoform* is then given by

$$\hat{\theta} = \hat{p}\hat{\times}\hat{d}\hat{r} = (p \times \hat{T}) \times \hat{I} \times \hat{d}(\hat{r}) = p \times \hat{T} \times d(r \times \hat{I}). \tag{3.2.186}$$

The *fundamental isocanonical two-isoform* is then given by

$$\hat{\omega} = \hat{d}\hat{\theta} = \hat{p}\hat{\wedge}\hat{d}\hat{r} = dp_i \wedge dr^i \equiv \omega, \tag{3.2.187}$$

from which the preservation of closure under isotopy, $\hat{d}\hat{\omega} \equiv \hat{0} = 0$ trivially follows.

LEMMA 3.2.14 [31,10]: The fundamental symplectic and isosymplectic two-forms coincide.

The identity of the fundamental isocanonical and canonical two-forms explains why isosymplectic geometry escaped detection by mathematicians for centuries.

It is evident that, in view of the positive-definiteness of the isounit, *the symplectic and isosymplectic geometries coincide at the global (abstract) realization-free level* to such an extent that there is not even the need of changing formulae in the literature of the symplectic geometry because the isosymplectic geometry can be expressed with the pre-existing formalism and merely subject it to a broader realization.

Despite this simplicity, the physical implications are by far non-trivial. In fact, unlike the conventional two-form, and thanks to the background TSSFN isotopology, the fundamental isocanonical two-form is universal for all possible (sufficiently smooth and regular but otherwise arbitrary) nonlocal and non-Hamiltonian systems. To illustrate this feature, let us consider a *vector field* of the cotangent bundle that must be strictly *local-differential* to avoid catastrophic inconsistencies with the underlying local-differential Euclidean topology, T^*M

$$X(r,p) = A_i(r,p) \times \frac{\partial}{\partial r^i} + B^i(r,p) \times \frac{\partial}{\partial p_i}, \tag{3.2.188}$$

or in unified notations

$$b = (b^\mu) = (r^i, p_j), \quad \mu = 1, 2, \ldots, 2N, \tag{3.2.189}$$

$$X(b) = X_\mu(b) \times \frac{\partial}{\partial b^\mu}, \tag{3.2.190}$$

is said to be a *Hamiltonian vector field* when there exists a function $H(r,p) = H(b)$ on T^*M, called the *Hamiltonian,* verifying the identity

$$A_i \times dr^i + B^i \times dp_i = -dH(r,p) \tag{3.2.191}$$

or in unified notation

$$X \rfloor \omega = dH, \tag{3.2.192}$$

that is

$$\omega_{\mu\nu} \times X^\mu \times db^\nu = -dH, \tag{3.2.193}$$

where the fundamental symplectic form has the components

$$\omega = dp_i \wedge dr^i = \frac{1}{2} \times \omega_{\mu\nu} \times db^\mu \wedge db^\nu, \tag{3.2.194}$$

$$(\omega_{\mu\nu}) = \begin{pmatrix} O_{N\times N} & -I_{N\times N} \\ I_{N\times N} & O_{N\times N} \end{pmatrix}. \tag{3.2.195}$$

Eq. (3.2.192) can hold if and only if

$$\omega_{\mu\nu} \times \frac{db^\nu}{dt} = \frac{\partial H}{\partial b^\mu}, \tag{3.2.196}$$

from which one recovers the familiar truncated Hamilton's equations

$$\frac{dr^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial r^i}. \tag{3.2.197}$$

The main physical limitations is that *the condition for a vector field to be Hamiltonian constitutes a major restrictions because vector fields in the physical reality are generally non-Hamiltonian, besides existing from the limitations of the topology underlying the symplectic geometry.*

As we shall see in Section 3.3, the above restrictions is removed for Santilli isosymplectic geometry that acquire the character of *direct universality,* that is, the capability of representing all sufficiently smooth and regular but otherwise arbitrary vector fields (universality) in the local chart of the experimenter (direct universality).

In fact, expression (3.2.192) is lifted into the form

$$\hat{\omega}_{\mu\nu} \hat{\times} \frac{\hat{d}\hat{b}^\nu}{\hat{d}\hat{t}} = \frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{b}^\mu}, \tag{3.2.198}$$

that, under the assumption for simplicity that $\hat{t} = t$, and by removing common factors, reduces to

$$\frac{dr^i}{dt} = \frac{\hat{\partial} H}{\hat{\partial} p_i} = \hat{T}^i_j(r,p) \times \frac{\partial H}{\partial p_j}; \tag{3.2.199}$$

$$\frac{dp_i}{dt} = -\frac{\hat{\partial} H}{\hat{\partial} r^i} = -\hat{I}^j_i \times \frac{\partial H}{\partial r^j}. \tag{3.2.200}$$

As we shall see better in Section 3.3, direct universality then follows from the number of free functions $\hat{T}^j_i$ as well as the arbitrariness of their functional dependence.

We shall also show that the achievement of a direct isogeometric representation of nonlocal and non-Hamiltonian vector fields representing interior dynamical problems permits their consistent map into an operator form, by therefore reaching hadronic mechanics in a mathematically rigorous, unique and unambiguous way.[17]

The construction of the *isodual isosymplectic geometry* [6] is an instructive exercise for readers interested in serious studies of antimatter in interior dynamical conditions.

[17]Note the crucial role of the isodifferential calculus for the isosymplectic geometry and its implications.

3.2.11 Isolinearity, Isolocality, Isocanonicity and Their Isodualities

In Section 3.1 we pointed out that the primary physical characteristics of particles and antiparticles in interior conditions (such as a neutron in the core of a neutron star) are nonlinear, nonlocality and noncanonicity due to the mutual penetration-overlapping of their wavepackets with those of the surrounding medium.

In the preceding subsections we have identified isotopic means for mapping linear, local and canonical systems into their most general possible nonlinear, nonlocal and noncanonical form. In this section we show how the isotopies permit the reconstruction of linearity, locality and canonicity on isospaces over isofields, called *isolinearity, isolocality and isocanonicity* for the case of particles, with their isodual counterpart for antiparticles.

The understanding of this seemingly impossible task requires the knowledge that conventional methods have only one formulation. By contrast, all isotopic methods have a dual formulation, the first in isospace over isofields, and the second when projected in ordinary spaces over ordinary fields. Deviations from conventional properties can only occur in the latter formulation because in the former all original axiomatic properties are preserved by construction.

Let $S(r, R)$ be a conventional real vector space with local coordinates r over the reals $R = R(n, +, \times)$, and let

$$r' = A(w) \times r, \quad r'^t = r^t \times A^t(w), \quad w \in R. \tag{3.2.201}$$

be a conventional right and left linear, local and canonical transformation on S, where t denotes transpose.

The isotopic lifting $S(r, R) \rightarrow \hat{S}(\hat{r}, \hat{R})$ requires a corresponding necessary isotopy of the transformation theory. In fact, it is instructive for the interested reader to verify that the application of conventional linear transformations to the isospace $\hat{S}(\hat{r}, \hat{R})$ causes the loss of linearity, transitivity and other basic properties.

For these and other reasons, Santilli submitted in the original proposals [4,5] of 1978 (see monographs [6,7] for comprehensive treatments and applications) the isotopy of the transformation theory, called *isotransformation theory,* which is characterized by isotransforms (where we make use of the notion of isofunction of Section 3.2.4)

$$\hat{r}' = \hat{A}(\hat{w}) \hat{\times} \hat{r} = \hat{A}(\hat{w}) \times \hat{T} \times \hat{r} = [A(\hat{T} \times w) \times \hat{I}] \times \hat{T} \times (r \times \hat{I}) =$$

$$= A[\hat{T}(r, \ldots) \times w] \times \hat{r}, \tag{3.2.202a}$$

$$\hat{r}'^{\hat{t}} = \hat{r}^{\hat{t}} \hat{\times} \hat{A}^{\hat{t}} \hat{w} = \hat{r}^{\hat{t}} \times A^t[\hat{T}(r) \times w]. \tag{3.2.202b}$$

The most dominant aspect in the transition from the conventional to the isotopic transforms is that, while the former are linear, local and canonical, the latter are nonlinear in the coordinates as well as other quantities and their derivatives of arbitrary order, nonlocal-integral in all needed quantities, and noncanonical when projected in the original spaces $S(r, R)$. This is due to the unrestricted nature of the functional dependence of the isotopic element $\hat{T} = \hat{T}(r, \ldots)$.

But the conventional and isotopic transforms coincide at the abstract level where we have no distinction between the modular action $A(w) \times r$ and $\hat{A}(\hat{w})\hat{\times}\hat{r}$. Therefore, isotransforms (3.2.202) are *isolinear* when formulated on isospace $\hat{S}$ over the isofield $\hat{R}$, because they verify the conditions

$$\hat{A}\hat{\times}(\hat{n}\hat{\times}\hat{r} + \hat{m}\hat{\times}\hat{p}) = \hat{n}\hat{\times}\hat{A}\hat{\times}\hat{r} + \hat{m}\hat{\times}\hat{A}\hat{\times}\hat{p}, \;\; \hat{r}, \hat{p} \in \hat{S}, \;\; \hat{n}, \hat{m} \in \hat{R}. \tag{3.2.203}$$

Note that conventional transforms are characterized by a *right modular associative action* $A \times r$. Isotransforms are then characterized by the *right isomodular isoassociative action* $\hat{A}\hat{\times}\hat{r}$. Therefore, we do have the preservation of the original axiomatic structure and isotransforms are indeed an isotopy of conventional transforms.

The situation for locality and canonicity follows the same lines [4,5,6,7]. Conventional methods are local in the sense that they are defined at a finite set of isolated points. The isotopic methods are *isolocal* in the sense that they verify the condition of locality in isospaces over isofields. However, their projection on conventional space is nonlocal-integral, because that is the general characteristic of the isotopic element $\hat{T}$, as illustrated, e.g., in Eq. (3.1.202).

Similarly, conventional methods are canonical in the sense that they can be characterized via a first-order canonical action in phase space (or cotangent bundle). The isotopic methods are *isocanonical* in the sense that, as we shall see in Section 3.3, they are derivable from an *isoaction* that is first-order and canonical on isospaces over isofields, although, when projected on ordinary spaces over ordinary fields, such an isoaction is of arbitrary order.

LEMMA 3.2.15 [6,7]: All possible nonlinear, nonlocal and noncanonical transforms on a vector space $S(r, R)$

$$r' = B(w, r, \ldots) \times r, \;\; r \in S, \;\; w \in R, \tag{3.2.204}$$

can always be rewritten in an identical isolinear, isolocal and isocanonical form, that is, there always exists at least one isotopy of the base field, $R \to \hat{R}$, and a corresponding isotopy of the space $S(r, R) \to \hat{S}(\hat{r}, \hat{R})$, such as

$$B(w, r, \ldots) \equiv A(\hat{T} \times w), \tag{3.2.205}$$

under which

$$r' = B(w, r, \ldots) \times r \equiv A(\hat{T} \times w) \times r \equiv \hat{A}(\hat{w})\hat{\times}r, \tag{3.2.206}$$

from which the isolinear form (3.2.202) follows.

COROLLARY 3.2.15A [6,7]: Under sufficient continuity and regularity conditions, all possible ordinary differential equations that are nonlinear in ordinary spaces over ordinary fields can always be turned into an identical form that is isolinear on isospaces over isofields,[18]

$$\dot{r} - E(\dot{r}, w, \ldots) \to \hat{\dot{r}} - A[\hat{T}(\dot{r}, w, \ldots) \times \dot{r} - B[\hat{T}(\dot{r}, w, \ldots] \equiv$$
$$\equiv \hat{\dot{r}} - \hat{A}(\hat{w}) \hat{\times} \hat{\dot{r}} - \hat{B}(w) = 0. \tag{3.2.207}$$

The above properties are at the foundation of the *direct universality* of isotopic methods, that is, their applicability to all possible (sufficiently smooth and regular) nonlinear, nonlocal and noncanonical systems (universality) in the frame of the experimenter (direct universality).

In order to apply isotopic methods to a nonlinear, nonlocal and noncanonical system, one has merely to identity one of its possible isolinear, isolocal and isocanonical identical reformulations in the same system of coordinates. The applicability of the methods studied in this monograph then follows.

The *isodual isotransforms* are given by the image of isotransforms (3.2.202) under isoduality, and, as such, are defined on the isodual isospace $\hat{S}^d(\hat{r}^d, \hat{R}^d)$ over the isodual isofield $\hat{R}^d$ with isodual isounit $\hat{I}^d = 1/\hat{T}^d = -\hat{I}^\dagger$. [6,7] with evident properties

$$\hat{A}^d \hat{\times}^d (\hat{n}^d \hat{\times}^d \hat{r}^d + \hat{m}^d \hat{\times}^d \hat{p}^d) =$$
$$= \hat{n}^d \hat{\times}^d \hat{A}^d \hat{\times}^d \hat{r}^d + \hat{m}^d \hat{\times}^d \hat{A}^d \hat{\times}^d \hat{p}^d, \quad \hat{r}^d, \hat{p}^d \in \hat{S}^d, \quad \hat{n}^d, \hat{m}^d \in \hat{R}^d. \tag{3.2.208}$$

The definition of *isodual isolinearity, isolocality and isocanonicity* then follows.

From now on, we shall use isotransforms for the study of interior dynamical systems of particles and their isodual for interior systems of antiparticles.

3.2.12 Lie-Santilli Isotheory and its Isodual

3.2.12A. Statement of the Problem. As it is well known, Lie's theory has permitted outstanding achievements in various disciplines throughout the 20-th century. Nevertheless, in its current conception and realization, Lie's theory is linear, local-differential and canonical-Hamiltonian.[19]

[18] The author has proposed for over a decade that mathematicians use the property of this Corollary 3.2.15A to identify simpler methods for the solution of nonlinear differential equations, but the request has not been met as yet, to our best knowledge.

[19] The literature on Lie's theory is also vast to discourage discriminatory listings. In any case, its knowledge is a necessary pre-requisite for the understanding of this section.

As such, Lie's theory is exactly valid for exterior dynamical systems, but possesses clear limitations for interior dynamical systems since the latter are nonlinear, nonlocal and noncanonical. This occurrence mandates a suitable revision of Lie's theory such to be exactly valid for interior dynamical systems without approximations.

Independently from that, Lie's theory in its current formulation is solely applicable to matter, evidently because there exists no antiautomorphic version of the conventional Lie's theory as necessary for the correct treatment of antimatter beginning at the classical level, as shown in Chapters 1 and 2.

Another central problem addressed in these studies is the construction of the universal *symmetry* (and not "covariance") of gravitation for matter and, independently, for antimatter, that is, a symmetry for all possible exterior and interior gravitational line elements of matter and, under antiautomorphic image, of antimatter.

Yet another need in physics is the identification of the exact symmetry that can effectively replace broken Lie symmetries, which exact symmetry cannot possibly be a conventional Lie symmetry due to the need of preserving the original dimensions so as to avoid the prediction on nonphysical effects and/or hypothetical new particles.

It is evident that Lie's theory in its current formulation is unable to solve the above identified problems. In a memoir of 1978, Santilli [4] proposed a step-by-step generalization of the conventional Lie theory specifically conceived for nonlinear, nonlocal-integral and nonpotential-noncanonical systems.

The generalized theory was subsequently studied by Santilli in a variety of papers (see monographs [1,2,6,7,14,15] and references quoted therein). The theory was also studied by a number of mathematicians and theoreticians, and it is today called the *Lie-Santilli isotheory* (see, e.g., monographs [32–37] and references quoted therein, as well as specialized papers [38–43]).

A main characteristic of the Lie-Santilli isotheory, that distinguishes it from other possible generalizations, is its isotopic character, that is, the preservation of the original Lie axioms when formulated on isospaces over isofields, despite its nonlinear, nonlocal and noncanonical structure when projected in ordinary spaces. This basic feature is evidently permitted by the reconstruction of linearity, locality and canonicity on isospaces over isofields studied in the preceding section.

To begin, let us recall that Lie's theory is centrally dependent on the basic N-dimensional unit $I = Diag.(1, 1, \ldots, 1)$ of the enveloping algebra. The main idea of the Lie-Santilli isotheory [4] is the reformulation of the entire conventional theory with respect to the most general possible isounit $\hat{I}(x, \dot{x}, \ddot{x}, \ldots)$.

One can therefore see from the very outset the richness and novelty of the isotopic theory since isounits with different topological features (such as Hermiticity, non-Hermiticity, positive-definiteness, negative-definiteness, etc.) characterize different generalized theories.

In this section we outline the rudiments of the Lie-Santilli isotheory properly speaking, that with positive-definite isounits and its isodual with negative-definite isounits. A knowledge of Lie's theory is assumed as a pre-requisite. A true technical knowledge of the Lie-Santilli isotheory can only be acquired from the study of mathematical works such as monographs [2,6,14,36,37].

In inspecting the literature, the reader should be aware that Santilli [4] constructed the isotopies of Lie's theory as a particular case of the broader Lie-admissible theory studied in Chapter 4 occurring for non-Hermitian generalized units, and known as *Lie-Santilli genotheory.* As a matter of fact, a number of aspects of the isotheory can be better identified within the context of the broader genotheory.

The extension to non-Hermitian isounits (that was the main object of the original proposal [4]) requires the exiting of Lie's theory in favor of the covering Lie-admissible theory, and will be studied in Chapter 4.

The isotopies of Lie's theory were proposed by Santilli from first axiomatic principles without the use of any map or transform. It is today known that the isotheory cannot be entirely derived via the use of noncanonical-nonunitary transforms since some of the basic structures (such as the isodifferential calculus) are not entirely derivable via noncanonical-nonunitary transforms.

3.2.12B. Universal Enveloping Isoassociative Algebras. Let ξ be an *associative algebra* over a field $F = F(a, +, \times)$ of characteristic zero with generic elements $A, B, C, \ldots$, trivial associative product $A \times B$ and unit I. The infinitely possible isotopes $\hat{\xi}$ of ξ were first introduced in Ref. [4] under the name of *isoassociative algebras*. In the original proposal $\hat{\xi}$ coincides with ξ as vector spaces but is equipped with Santilli's isoproduct so as to admit the isounit as the correct left and right unit

$$\hat{I}(x, \dot{x}, \ddot{x}, \ldots) = 1/\hat{T} > 0, \tag{3.2.209a}$$

$$\hat{A} \hat{\times} \hat{B} = \hat{A} \times \hat{T} \times \hat{B}, \ \ \hat{A} \hat{\times} (\hat{B} \hat{\times} \hat{C}) = (\hat{A} \hat{\times} \hat{B}) \hat{\times} \hat{C}, \tag{3.2.209b}$$

$$\hat{I} \hat{\times} \hat{A} = \hat{A} \hat{\times} \hat{I} \equiv \hat{A}, \ \ \forall \hat{A} \in \hat{\xi}, \tag{3.2.209c}$$

where $\hat{A}, \hat{B}, \ldots$ denote the original elements $A, B, \ldots$ formulated on isospace over isofields.

Let $\xi = \xi(L)$ be the *universal enveloping associative algebra* of an N-dimensional Lie algebra L with ordered basis X_k, $k = 1, 2, \ldots, N$, and attached antisymmetric algebra isomorphic to the Lie algebras, $[\xi(L)]^- \approx L$ over F, and let the infinite-dimensional basis $I, X_k, X_i \times X_j, i \leq j, \ldots$ of $\xi(L)$ be characterized by the *Poincaré-Birkhoff-Witt theorem.*

A fundamental property submitted in the original proposal [4] (see also [2], pp. 154–163) is the following

THEOREM 3.2.11 (Poincaré-Birkhoff-Witt-Santilli isotheorem): Isocosets of the isounit and the standard, isomonomials

$$\hat{I},\ \ X_k,\ \ \hat{X}_i \hat{\times} \hat{X}_j,\ i \leq j,\ \ \hat{X}_i \hat{\times} \hat{X}_j \hat{\times} \hat{X}_k,\ i \leq j \leq k,\ \ldots, \tag{3.2.210}$$

form a basis of universal enveloping isoassociative algebra $\hat{\xi}(L)$ of a Lie algebra L (also called isoenvelope for short).

The first application of the above infinite-dimensional basis is a rigorous characterization of the isoexponentiation, Eq. (3.2.72), i.e.,

$$\hat{e}^{\hat{i} \hat{\times} \hat{w} \hat{\times} \hat{X}} = \hat{e}^{i \times w \times \hat{X}} =$$

$$= \hat{I} + \hat{i} \hat{\times} \hat{w} \hat{\times} \hat{X} \hat{/} \hat{1}! + (\hat{i} \hat{\times} \hat{w} \hat{\times} \hat{X}) \hat{\times} (\hat{i} \hat{\times} \hat{w} \hat{\times} \hat{X}) \hat{/} \hat{2}! + \ldots =$$

$$= \hat{I} \times (e^{i \times w \times \hat{T} \times \hat{X}}) = (e^{i \times w \times \hat{X} \times \hat{T}}) \times \hat{I},\ \ \hat{i} = i \times \hat{I}, \hat{w} = w \times \hat{I} \in \hat{F}. \tag{3.2.211}$$

The nontriviality of the Lie-Santilli isotheory is illustrated by the emergence of the nonlinear, nonlocal and noncanonical isotopic element $\hat{T}$ directly in the exponent, thus ensuring the desired generalization.

The implications of Theorem 3.2.11 also emerge at the level of isofunctional analysis because all structures defined via the conventional exponentiation must be suitably lifted into a form compatible with Theorem 3.2.11, as illustrated by the *iso-Fourier transforms,* Eq. (3.2.88).

It is today known that the main lines of isoenvelopes can indeed be derived via the use of noncanonical-nonunitary transforms, such as

$$U \times U^{\dagger} \neq I, \tag{3.2.212a}$$

$$I \rightarrow \hat{I} = U \times I \times U^{\dagger}, \tag{3.2.212b}$$

$$X_i \times X_j \rightarrow U \times (X_i \times X_j) \times U^{\dagger} = \hat{X}_i \hat{\times} \hat{X}_j, \tag{3.2.212c}$$

$$X_i \times X_j \times X_k \rightarrow U \times (X_i \times X_j \times X_k) \times U^{\dagger} = \hat{X}_i \hat{\times} \hat{X}_j \hat{\times} \hat{X}_k,\ \ \text{etc.} \tag{3.2.212d}$$

Nevertheless, the uncontrolled use of the above transforms may lead to misrepresentations. In fact, a primary objective of the Lie-Santilli isotheory is that of preserving the original generators and parameters and change instead the associative and Lie products in an axiom-preserving way to accommodate the treatment of nonlinear, nonlocal and noncanonical interactions.

The preservation of the generators is, in particular, necessary for physical consistency because they represent conserved total quantities (such as the total energy, total angular momentum, etc.). These total quantities remain unchanged in the transition from closed Hamiltonian and non-Hamiltonian systems (see Section 3.1.2). Equivalently, the generators of Lie's theory cannot be altered by non-Hamiltonian effects.

This physical requirement can only be achieved by preserving conventional generators X_k and lifting instead their product $X_i \times X_j \rightarrow X_i \hat{\times} X_j = X_i \times \hat{T} \times X_j$, which is the original formulation of the Lie-Santilli isotheory [4] and remain the formulation needed for applications to this day. It is essentially given by the projection of the isotopic formulation on conventional spaces over conventional fields.

3.2.12C. Lie-Santilli Isoalgebras. As it is well known, Lie algebras are the antisymmetric algebras $L \approx [\xi(L)]^-$ attached to the universal enveloping algebras $\xi(L)$. This main characteristic is preserved although enlarged under isotopies (see [4,2] for details). We therefore have the following

DEFINITION 3.2.15 [4]: A finite-dimensional isospace $\hat{L}$ with generic elements $\hat{A}, \hat{B}, \ldots,$ over the isofield $\hat{F}$ with isounit $\hat{I} = 1/\hat{T} > 0$ is called a "Lie-Santilli isoalgebra" over $\hat{F}$ when there is a composition $[\hat{A}\hat{,}\hat{B}]$ in $\hat{L}$, called "isocommutator", that is isolinear as an isovector space and such that all the following axioms are satisfied

$$[\hat{A}\hat{,}\hat{B}] = -[\hat{B}\hat{,}\hat{A}], \tag{3.2.213a}$$

$$[\hat{A}\hat{,}[\hat{B}\hat{,}\hat{C}]] + [\hat{B}\hat{,}[\hat{C}\hat{,}\hat{A}]] + [\hat{C}\hat{,}[\hat{A}\hat{,}\hat{B}]] \equiv 0, \tag{3.2.213b}$$

$$[\hat{A}\hat{\times}\hat{B}\hat{,}\hat{C}] = \hat{A}\hat{\times}[\hat{B}\hat{,}\hat{C}] + [\hat{A}\hat{,}\hat{C}]\hat{\times}\hat{B}, \ \ \forall \hat{A}, \hat{B}, \hat{C} \in \hat{L}. \tag{3.2.213c}$$

The isoalgebras are said to be: *isoreal, isocomplex or isoquaternionic* depending on the assumed isofield and *isoabelian* when $[\hat{A}\hat{,}\hat{B}] \equiv \forall \hat{A}, \hat{B} \in \hat{L}$. A subset $\hat{L}^o$ of $\hat{L}$ is said to be an *isosubalgebra* of $\hat{L}$ when $[\hat{L}^o\hat{,}\hat{L}^o] \subseteq \hat{L}^o$. $\hat{L}^o$ is called an *isoideal* of $\hat{L}$ when $[\hat{L}^o\hat{,}\hat{L}] \subseteq \hat{L}^o$. A *maximal isoideal* verifying the property $[\hat{L}^o\hat{,}\hat{L}^o] = 0$ is called the *isocenter* of $\hat{L}$.

For the isotopies of additional conventional notions, theorems and properties of Lie algebras, one may see monograph [2,6,36,37].

We merely recall the *isotopic generalizations of the celebrated Lie's First, Second and Third Theorems* introduced in the original proposal [4], but which we do not review here for brevity. For instance, the *Lie-Santilli Second Isotheorem* reads

$$[\hat{X}_i\hat{,}\hat{X}_j] = \hat{X}_i\hat{\times}\hat{X}_j - \hat{X}_j\hat{\times}\hat{X}_i = \tag{3.2.214a}$$

$$= \hat{X}_i \times \hat{T}(x, \dot{x}, \ddot{x}, \ldots) \times \hat{X}_j - \hat{X}_j \times \hat{T}(x, \dot{x}, \ddot{x}, \ldots) \times \hat{X}_i = \hat{C}^k_{ij}(x, \dot{x}, \ddot{x}, \ldots)\hat{\times}\hat{X}_k, \tag{3.2.214b}$$

where the C's, called the *structure isofunctions*, generally have an explicit dependence on the underlying isovariable (see the examples later on), and verify certain restrictions from the Isotopic Third Theorem.

It is today known that Lie-Santilli isoalgebras can be reached via a noncanonical-nonunitary transform of conventional Lie algebras. In fact, we have

$$[X_i, X_j] = C_{ij}^k \times X_k \rightarrow$$
$$U \times [X_i, X_j] \times U^\dagger = [\hat{X}_i \hat{,} \hat{X}_j] =$$
$$U \times (C_{ij}^k \times X_k) \times U^\dagger = \hat{C}_{ij}^k(x, \dot{x}, \ddot{x}, \ldots) \hat{\times} \hat{X}_k. \quad (3.2.215)$$

However, again, this type of derivation of the isotheory may be misleading in physical applications due to the need to preserve the original generators unchanged, in accordance with the original formulation [4] of 1978. In this case we shall use the following projection of the isoalgebras on the original space over the original field

$$[X_i \hat{,} X_j] = X_i \times \hat{T} \times X_j - X_j \times \hat{T} \times X_i = C_{ij}^k(x, \dot{x}, \ldots) \times X_k. \quad (3.2.216)$$

It has been proved (see, e.g., [2,4,6] for details) that *Lie-Santilli isoalgebras $\hat{L}$ are isomorphic to the original algebra L.* In other words, the isotopies with $\hat{I} > 0$ cannot characterize any new algebra because all possible Lie algebras are known from Cartan classification. Therefore, Lie-Santilli isoalgebras merely provide new nonlinear, nonlocal and noncanonical realizations of existing algebras. It should be stresses that the above isomorphism is lost for more general liftings as shown in the next chapter.

3.2.12D. Lie-Santilli Isogroups. Under certain integrability conditions hereon assumed, Lie algebras L can be "exponentiated" to their corresponding *Lie transformation groups* G and, vice-versa, Lie transformation groups G admit their corresponding Lie algebra L when computed in the neighborhood of the identity I.

These basic properties are preserved under isotopies although broadened to the most general possible nonlinear, nonlocal and noncanonical transformations groups.

DEFINITION 3.2.16 [4]: A right isomodular Lie-Santilli isotransformation group $\hat{G}$ on an isospace $\hat{S}(\hat{x}, \hat{F})$ over an isofield $\hat{F}$ with common isounit $\hat{I} = 1/\hat{T} > 0$ is a group mapping each element $\hat{x} \in \hat{S}$ into a new element $\hat{x}' \in \hat{S}$ via the isotransformations

$$\hat{x}' = \hat{g}(\hat{w}) \hat{\times} \hat{x}, \;\; \hat{x}, \hat{x}' \in \hat{S}, \;\; \hat{w} \in \hat{F}, \quad (3.2.217)$$

such that:

1) The map $\hat{g}\hat{\times}\hat{S}$ *into* $\hat{S}$ *is isodifferentiable* $\forall \hat{g} \in \hat{G}$*;*
2) $\hat{I}$ *is the left and right unit*

$$\hat{I}\hat{\times}\hat{g} = \hat{g}\hat{\times}\hat{I} \equiv \hat{g}, \ \ \forall \hat{g} \in \hat{G}; \tag{3.2.218}$$

3) the isomodular action is isoassociative, i.e.,

$$\hat{g}_1\hat{\times}(\hat{g}_2\hat{\times}\hat{x}) = (\hat{g}_1\hat{\times}\hat{g}_2)\hat{\times}\hat{x}, \ \ \forall \hat{g}_1, \hat{g}_2 \in \hat{G}; \tag{3.2.219}$$

4) in correspondence with every element $\hat{g}(\hat{w}) \in \hat{G}$ *there is the inverse element* $\hat{g}^{-\hat{I}} = \hat{g}(-\hat{w})$ *such that*

$$\hat{g}(\hat{0}) = \hat{g}(\hat{w})\hat{\times}\hat{g}(-\hat{w}) = \hat{I}; \tag{3.2.220}$$

5) following composition laws are verified

$$\hat{g}(\hat{w})\hat{\times}\hat{g}(\hat{w}') = \hat{g}(\hat{w}')\hat{\times}\hat{g}(\hat{w}) = \hat{g}(\hat{w} + \hat{w}'), \forall \hat{g} \in \hat{G}, \ \hat{w} \in \hat{F}. \tag{3.2.221}$$

The I left isotransformation group is defined accordingly.

The notions of *connected or simply connected transformation groups* carry over to the isogroups in their entirety.

The most direct realization of the (connected) isotransformation groups is that via isoexponentiation,

$$\hat{g}(w) = \prod_k \hat{e}^{\hat{i}\hat{\times}\hat{w}_k\hat{X}_k} = \Big(\prod_k e^{i\times w_k\times X_k\times\hat{T}(x,\dot{x},\ddot{x},\ldots)}\Big) \times \hat{I}, \tag{3.2.222}$$

where the X's and w's are the infinitesimal generators and parameters, respectively, of the original algebra L, with corresponding connected isotransformations

$$\begin{aligned}\hat{x}' = \hat{g}(\hat{w})\hat{\times}\hat{x} &= \Big(\prod_k \hat{e}^{\hat{i}\hat{\times}\hat{w}_k\hat{X}_k}\Big) \times \hat{I} \times \hat{T} \times x \times \hat{I} = \\ &= \Big(\prod_k e^{i\times w_k\times X_k\times\hat{T}(x,\dot{x},\ddot{x},\ldots)}\Big) \times x \times \hat{I}.\end{aligned} \tag{3.2.223}$$

Equations (3.2.223) hold in some open neighborhood N of the isoorigin of $\hat{L}$ and, in this way, characterize some open neighborhood of the isounit of $\hat{G}$. Consequently, under the assumed continuity and connectivity properties, Lie-Santilli isoalgebras can be obtained as infinitesimal versions of finite Lie-Santilli isogroups, as illustrated by the following finite isotransform

$$\begin{aligned}\hat{A}(\hat{w}) &= (\hat{e}^{\hat{i}\hat{\times}\hat{w}\hat{\times}\hat{X}})\hat{\times}\hat{A}(\hat{0})\hat{\times}(\hat{e}^{-\hat{i}\hat{\times}\hat{w}\hat{\times}\hat{X}}) = \\ &= (e^{i\times w\times\hat{X}\times\hat{T}}) \times \hat{A}(\hat{0}) \times (e^{-i\times w\times\hat{T}\times\hat{X}})\end{aligned} \tag{3.2.224}$$

with infinitesimal version in the neighborhood of $\hat{I}$

$$\hat{A}(\hat{d}\hat{w}) = (\hat{I} + \hat{i}\hat{\times}\hat{d}\hat{w}\hat{\times}\hat{X} + \ldots)\hat{\times}\hat{A}(0)\hat{\times}(\hat{I} - \hat{i}\hat{\times}\hat{d}\hat{w}\hat{\times}\hat{X} + \ldots) =$$

$$= \hat{A}(\hat{0}) + \hat{i}\hat{\times}\hat{d}\hat{w}\hat{\times}\hat{X}\hat{\times}\hat{A}(\hat{0}) - \hat{i}\hat{\times}\hat{d}\hat{w}\hat{\times}\hat{A}(\hat{0})\hat{\times}\hat{X}, \tag{3.2.225}$$

that can be written

$$\hat{i}\hat{\times}\frac{\hat{d}\hat{A}(\hat{w})}{\hat{d}\hat{w}} = \hat{A}\hat{\times}\hat{X} - \hat{X}\hat{\times}\hat{A} = [\hat{A}\hat{,}\hat{X}]. \tag{3.2.226}$$

Note the crucial appearance of the isotopic element $\hat{T}(x, \dot{x}, \ddot{x}, \ldots)$ in the exponent of the isogroup. This ensures a structural generalization of Lie's theory of the desired nonlinear, nonlocal and noncanonical form.

Still another important property is that conventional group composition laws admit a consistent isotopic lifting, resulting in the following *Baker-Campbell-Hausdorff-Santilli Isotheorem* [4]

$$(\hat{e}^{\hat{X}_1})\hat{\times}(\hat{e}^{X_2}) = \hat{e}^{\hat{X}_3}, \tag{3.2.227a}$$

$$\hat{X}_3 = \hat{X}_1 + \hat{X}_2 + [\hat{X}_1\hat{,}\hat{X}_2]\hat{/}\hat{2} + [(\hat{X}_1 - \hat{X}_2)\hat{,}[\hat{X}_1\hat{,}\hat{X}_2]]\hat{/}\hat{12} + \ldots \tag{3.2.227b}$$

Let $\hat{G}_1$ and $\hat{G}_2$ be two isogroups with respective isounits $\hat{I}_1$ and $\hat{I}_2$. The *direct isoproduct* $\hat{G}_1\hat{\times}\hat{G}_2$ is the isogroup of all ordered pairs

$$(\hat{g}_1, \hat{g}_2), \quad \hat{g}_1 \in \hat{G}_1, \hat{g}_2 \in \hat{G}_2, \tag{3.2.228}$$

with isomultiplication

$$(\hat{g}_1, \hat{g}_2)\hat{\times}(\hat{g}_1', \hat{g}_2') = (\hat{g}_1\hat{\times}\hat{g}_1', \hat{g}_2\hat{\times}\hat{g}_2'), \tag{3.2.229}$$

total isounit $(\hat{I}_1, \hat{I}_2)$ and inverse $(\hat{g}_1^{-\hat{I}_1}, \hat{g}_2^{-\hat{I}_2})$.

The following particular case is important for the isotopies of inhomogeneous groups. Let $\hat{G}$ be an isogroup and $\hat{G}_{\hat{a}}$ the isogroup of all its inner isoautomorphisms. Let $\hat{G}^o_{\hat{a}}$ be a subgroup of $\hat{G}_{\hat{a}}$, and let $\Lambda(\hat{g})$ be the image of $\hat{g} \in \hat{G}$ under $\hat{G}_{\hat{a}}$. The *semidirect isoproduct* $\hat{G}\hat{\times}\hat{G}^o_{\hat{a}}$ is the isogroup of all ordered pairs

$$(\hat{g}, \hat{\Lambda})\hat{\times}(\hat{g}^o, \hat{\Lambda}^o) = (\hat{g}, \hat{\Lambda}(\hat{g}^o), (\hat{\Lambda}, \hat{\Lambda}^o), \tag{3.2.230}$$

with total isounit given by $\hat{I}_{tot} = \hat{I} \times \hat{I}^o$.

The studies of the isotopies of the remaining aspects of the structure theory of Lie groups is then consequential.

It is hoped that the reader can see from the above elements that the entire conventional Lie theory does indeed admit a consistent and nontrivial lifting into the covering Lie-Santilli formulation.

3.2.12E. Isorepresentations of Lie-Santilli Isoalgebras. Despite considerable research on the Lie-Santilli isotheory over the past 26 years, the study of the *isorepresentations* of the Lie-Santilli isoalgebras remains vastly unknown at this writing (summer 2004), with the sole exception of the *fundamental (or regular) isorepresentations* that were also identified by Santilli in the original proposal [4].

In this monograph we shall primarily use in the applications of hadronic mechanics the fundamental isorepresentations or other isorepresentations reducible to the latter.

Let L be an N-dimensional Lie algebra with N-dimensional unit $I = Diag.(1, 1, \ldots, 1)$. Let R be the fundamental, N-dimensional matrix representation of L. Let $\hat{L}$ be the isotope of L characterized by the N-dimensional isounit $\hat{I} = U \times U^\dagger > 0$. It is then evident that the *fundamental isorepresentation* of $\hat{L}$ is given by

$$\hat{R} = U \times R \times U^\dagger, \quad U \times U^\dagger = \hat{I} \neq I, \hat{I} > 0. \tag{3.2.231}$$

Interested colleagues are encouraged to study the isorepresentation theory because, as we shall see in the next sections, the fundamental notion of hadronic mechanics, that of *isoparticles,* is characterized by an irreducible isorepresentation of the Poincaré-Santilli isosymmetry.

3.2.12F. Isodual Lie-Santilli Isotheory. As indicated Chapters 1 and 2, the contemporary formulation of Lie's theory is one of the most serious obstacles for a consistent *classical* representation of antimatter, because it lacks an appropriate conjugate formulation that, after quantization, is compatible with charge conjugation.[20]

It is easy to verify that the isotheory presented above admits a consistent antiautomorphic image under isoduality, thus permitting the treatment of antimatter under nonlinearity, nonlocality and noncanonicity as occurring in interior conditions, such as for the structure of an antimatter star.

In fact, we have the *isodual universal enveloping isoassociative isoalgebra* $\hat{\xi}^d$ characterized by the *isodual Poincaré-Birkhoff-Witt-Santilli isotheorem* with infinite dimensional basis

$$\hat{I}^d, \;\; X^d_k, \;\; \hat{X}^d_i \hat{\times}^d \hat{X}^d_j, \; i \leq j, \;\; \hat{X}^d_i \hat{\times}^d \hat{X}^d_j \hat{\times}^d \hat{X}^d_k, \; i \leq j \leq k, \; \ldots . \tag{3.2.232}$$

The *isodual Lie-Santilli isoalgebra* $\hat{L}^d \approx (\hat{\xi}^d)^-$ attached to $\hat{\xi}^d$ is characterized by the *isodual Lie-Santilli Second Isotheorem*

$$[\hat{X}^d_i \hat{,} \hat{X}^d_j] = \hat{X}^d_i \hat{\times}^d \hat{X}^d_j - \hat{X}^d_j \hat{\times}^d \hat{X}^d_i = \hat{C^d_{ij}}^k \hat{\times}^d \hat{X}^d_k. \tag{3.2.233}$$

[20]The reader is urged to verify that the classical treatment of antimatter via the so-called *dual Lie algebras* does not achieve antiparticles under quantization, trivially, because of the uniqueness of the quantization channel for both particles and antiparticles.

Under the needed continuity and connectivity property, the *isodual exponentiation* of $\hat{L}^d$ characterizes the *connected isodual Lie-Santilli transformation isogroup*

$$\hat{x}'^d = (\hat{g}^d(\hat{w}^d) = \hat{\prod}_k \hat{e}^{d^{\hat{i}^d \hat{\times}^d \hat{w}_k^d \hat{X}_k^d}}) \hat{\times}^d \hat{x}^d. \tag{3.2.234}$$

Interested readers can then easily derive any additional needed isodual property.

3.2.13 Unification of All Simple Lie Algebras into Lie-Santilli Isoalgebras

The original proposal [4] of 1978 included the *conjecture that all simple Lie algebras of dimension N can be unified into a single Lie-Santilli isoalgebra of the same dimension,* and gave an explicit example. The conjecture was subsequently proved by the late mathematicians Gr. Tsagas [42] in 1996 for all simple Lie algebras of type A, B, C and D. The premature departure of Prof. Tsagas while working at the problem prevented him to complete the proof of the conjecture for the case of all exceptional Lie algebras. As a result, the proof of the indicated conjecture remain incomplete at this writing.

For the unification here considered it is important to eliminate the restriction that the isounits are necessarily positive definite, while preserving all other characteristics, such as nowhere singularity and Hermiticity. As a result, in its simple possible form, the isounit can be diagonalized into the form whose elements can be either positive or negative,

$$\hat{I} = Diag.(\pm n_1^2, \pm n_2^2, \ldots, \pm n_N^2) = 1/\hat{T},\ n_k \in R,\ n_k \neq 0,\ k = 1, 2, \ldots, N. \tag{3.2.235}$$

The example provided in the original proposal [4], subsequently studied in detail in Refs. [8], consisted in the *classification of all possible simple Lie algebra of dimension 3.* In this case, Cartan's classification produces two non-isomorphic Lie algebras, the compact rotational algebra in three dimension $SO(3)$ and the noncompact algebra $SO(2.1)$.

The distinction between compact and noncompact algebras is lost under the class of isotopies here considered. In fact, the *classification of all possible, simple, three-dimensional Lie-Santilli isoalgebras $\hat{L}_3$ for the case of diagonal isounits* is characterized by the isounit itself and can be written

$$\hat{I} = Diag.(+1, +1, +1),\ \ \hat{L}_3 \approx SO(3), \tag{3.2.236a}$$

$$\hat{I} = Diag.(+1, +1, -1),\ \ \hat{L}_3 \approx SO(2.1), \tag{3.2.236b}$$

$$\hat{I} = Diag.(+1, -1, +1),\ \ \hat{L}_3 \approx SO(2.1), \tag{3.2.236c}$$

$$\hat{I} = Diag.(-1, +1, +1),\ \ \hat{L}_3 \approx SOI(2.1), \tag{3.2.236d}$$

$$\hat{I} = Diag.(-1,-1,-1), \quad \hat{L}_3 \approx SO(3)^d, \tag{3.2.236e}$$

$$\hat{I} = Diag.(-1,-1,+1), \quad \hat{L}_3 \approx SO(2.1)^d, \tag{3.2.236f}$$

$$\hat{I} = Diag.(-1,+1,-1), \quad \hat{L}_3 \approx SO(2.1), \tag{3.2.236g}$$

$$\hat{I} = Diag.(+1,-1,-1), \quad \hat{L}_3 \approx SO(2.1)^d, \tag{3.2.236h}$$

$$\hat{I} = Diag.(+n_1^2,+n_2^2,+n_3^2), \quad \hat{L}_3 \approx SO(3), \tag{3.2.236i}$$

$$\hat{I} = Diag.(+n_1^2,+n_2^2,-n_3^2), \quad \hat{L}_3 \approx SO(2.1), \tag{3.2.236j}$$

$$\hat{I} = Diag.(+n_1^2,-n_2^2,+n_3^2), \quad \hat{L}_3 \approx SO(2.1), \tag{3.2.236k}$$

$$\hat{I} = Diag.(-n_1^2,+n_2^2,+n_3^2), \quad \hat{L}_3 \approx SOI(2.1), \tag{3.2.236l}$$

$$\hat{I} = Diag.(-n_1^2,-n_2^2,-n_3^2), \quad \hat{L}_3 \approx SO(3)^d, \tag{3.2.236m}$$

$$\hat{I} = Diag.(-n_1^2,-n_2^2,+n_3^2), \quad \hat{L}_3 \approx SO(2.1)^d, \tag{3.2.236n}$$

$$\hat{I} = Diag.(-n_1^2,+n_2^2,-n_3^2), \quad \hat{L}_3 \approx SO(2.1), \tag{3.2.236o}$$

$$\hat{I} = Diag.(+n_1^2,-n_2^2,-n_3^2), \quad \hat{L}_3 \approx SO(2.1)^d, \tag{3.2.236p}$$

In conclusion, when studying simple algebras from the viewpoint of the covering Lie-Santilli isoalgebras, there exist *only one single isoalgebra in three dimensions,* $\hat{L}_3$ without any distinction between compact and noncompact algebras.

The *realization* of the simple isoalgebra $\hat{L}_3$ with diagonal isounits consists of 21 different Lie-Santilli isoalgebras in three dimension that can be reduced to 4 topologically different Lie algebras, namely $SO(3)$, $SO(2.1)$, $SO(3)^d$ and $SO(2.1)^d$.

All distinctions between these 21 different realizations are lost at the level of abstract Lie-Santilli isoalgebra $\hat{L}_3$.

It should be stressed that, by no means, the 21 realizations (3.2.236) exhaust all possible forms of Lie-Santilli simple isoalgebras in three dimensions because in realizations (3.2.236) we have excluded nondiagonal realizations of the isounit, as well as imposed additional restrictions on the isounit, such as single valuedness and Hermiticity.

Essentially the same results hold for the unification of the Lie Algebras of type A, B, C, and D studied by Tsagas [42].

It is hoped that interested mathematicians can complete the proof of Santilli's conjecture for the remaining exceptional algebras. In considering the problem, mathematicians are suggested to keep in mind that Hermitian and diagonal realizations of the isounit (3.2.135) are expected to be insufficient, thus implying the possible use of *nowhere singular, Hermitian, nondiagonal isounits,* or *nowhere singular, Hermitian, nondiagonal and multivalued isounits,* or *nowhere singular, non-Hermitian, nondiagonal and multivalued isounits.*

3.2.14 The Fundamental Theorem for Isosymmetries and Their Isoduals

The fundamental symmetries of the 20-th century physics deal with point-like abstractions of particles in vacuum under linear, local and potential interactions, and are the *Galilei symmetry* $G(3.1)$ for nonrelativistic treatment or the *Poincaré symmetry* for relativistic formulations.

A central objective of hadronic mechanics is the broadening of these fundamental spacetime symmetries to represent extended, nonspherical and deformable particles under linear and nonlinear, local and nonlocal and potential as well as nonpotential interactions.

In fact, as we shall see, all novel industrial applications of hadronic mechanics are crucially dependent on the admission of the extended character of particles or of their wavepackets in conditions of deep mutual penetration. In turn, the latter conditions imply new effects permitting basically new energies and fuels that are completely absent for conventional spacetime and other symmetries.

Alternatively and equivalently a central problem of hadronic mechanics is the *construction in an explicit form of the symmetries of all possible nonsingular, but otherwise arbitrary deformations of conventional spacetime and internal invariants.*

All these problems and others are resolved by the following important:

THEOREM 3.2.12 [6]: Let G be an N-dimensional Lie symmetry group of a K-dimensional metric or pseudo-metric space $S(x, m, F)$ over a field F,

$$G: \ \ x' = \Lambda(w) \times x, \ \ y' = \Lambda(w) \times y, \ \ x, y \in \hat{S}, \tag{3.2.237a}$$

$$(x' - y')^\dagger \times \Lambda^\dagger \times m \times \Lambda \times (x - y) \equiv (x - y)^\dagger \times m \times (x - y), \tag{3.2.237b}$$

$$\Lambda^\dagger(w) \times m \times \Lambda(w) \equiv m. \tag{3.2.237c}$$

Then, all infinitely possible isotopies $\hat{G}$ of G acting on the isospace $\hat{S}(\hat{x}, \hat{M}, \hat{F})$, $\hat{M} = \hat{m} \times \hat{I} = (\hat{T}_i^k \times m_{kj}) \times \hat{I}$ characterized by the same generators and parameters of G and new isounits $\hat{I} = 1/\hat{T} > 0$ leave invariant the isocomposition on the projection $\hat{S}(x, \hat{m}, F)$ of $\hat{S}(\hat{x}, \hat{M}, \hat{F})$ on the original space $S(x, m, F)$

$$\hat{G}: \ \ x' = \hat{\Lambda}(w) \times x, \ \ y' = \hat{\Lambda}(w) \times y, \ \ x, y \in \hat{S}, \tag{3.2.238a}$$

$$(x' - y')^\dagger \times \hat{\Lambda}^\dagger \times \hat{m} \times \hat{\Lambda} \times (x - y) \equiv (x - y)^\dagger \times \hat{m} \times (x - y), \tag{3.2.238b}$$

$$\hat{\Lambda}^\dagger(\hat{w}) \times \hat{m} \times \hat{\Lambda}(\hat{w}) \equiv \hat{m}. \tag{3.2.238c}$$

Similarly,all infinitely possible isodual isotopies $\hat{G}^d$ of $\hat{G}$ acting on the isodual isospace $\hat{S}^d(\hat{x}^d, \hat{M}^d, \hat{F}^d)$, $\hat{M}^d = (\hat{T}^d \times m^d) \times \hat{I}^d$ characterized by the isodual generators $\hat{X}_k^d$ parameters $\hat{w}^d$ and isodual isounit $\hat{I}^d = 1/\hat{T}^d < 0$ leave invariant the *isodual isocomposition on the projection* $\hat{S}^d(x^d, \hat{m}^d, F^d)$

$$\hat{G}^d: \quad x'^d = \hat{\Lambda}^d \times^d x^d, \quad y'^d = \hat{\Lambda}^d \times^d y^d, \quad x^d, y^d \in \hat{S}^d, \tag{3.2.239a}$$

$$(x'-y')^{\dagger^d} \times^d \hat{\Lambda}^{\dagger^d} \times^d \hat{m}^d \times^d \hat{\Lambda}^d \times^d (x-y)^d \equiv (x-y)^{\dagger^d} \times^d \hat{m}^d \times^d (x-y)^d, \tag{3.2.239b}$$

$$\hat{\Lambda}^{\dagger^d} \times^d \hat{m}^d \times^d \hat{\Lambda}^d \equiv \hat{m}^d. \tag{3.2.239c}$$

Proof. Assume that $N = K$ and the representation Λ is the fundamental one. Recall that metrics, isometrics and isounits are diagonal. Then on $\hat{S}(x, \hat{m}, F)$ we have the identities

$$\hat{I} = U \times U^\dagger \neq I, \quad \hat{T} = (U \times U^\dagger)^{-1}, \tag{3.2.240a}$$

$$U \times (\Lambda \times m \times \Lambda) \times U^\dagger =$$
$$= (U \times \Lambda \times U^\dagger) \times (U^{\dagger^{-1}} \times m \times U^{-1}) \times (U \times \Lambda \times U^\dagger) =$$
$$= \hat{\Lambda} \times (\hat{T} \times m) \times \hat{\Lambda} = \hat{\Lambda} \times \hat{m} \times \hat{\Lambda} = \hat{m}. \tag{3.2.240b}$$

The proof of the remaining cases are equally trivial. **q.e.d.**

Note that the isotopic symmetries and their isoduals can be uniquely and explicitly constructed with the methods summarized in this section via the sole use of the original symmetry and the isounit characterizing the deformation of the original metric m.

Under our assumptions, the isosymmetries can be constructed in the needed, explicit, nonlinear, nonlocal and noncanonical forms. In fact, the existence of the original symmetry transformations plus the condition $\hat{I} > 0$ ensure the convergence of the infinite isoseries of the isoexponentiation, resulting in the needed explicit form, as we shall see in various examples in the next sections.

3.3 CLASSICAL LIE-ISOTOPIC MECHANICS FOR MATTER AND ITS ISODUAL FOR ANTIMATTER

3.3.1 Introduction

One of the reasons for the majestic consistency of quantum mechanics is the existence of axiomatically consistent and invariant classical foundations, given by *classical Lagrangian and Hamiltonian mechanics,* namely, the discipline based on the *truncated analytic equations*

$$\frac{d}{dt}\frac{\partial L(t,r,v)}{\partial v^k_a} - \frac{\partial L(t,r,v)}{\partial r^k_a} = 0, \tag{3.3.1a}$$

$$\frac{dr_a^k}{dt} = \frac{\partial H(t,r,p)}{\partial p_{ak}}, \quad \frac{dp_{ak}}{dt} = -\frac{\partial H(t,r,p)}{\partial r_a^k}, \tag{3.3.1b}$$
$$k = 1, 2, 3; \quad a = 1, 2, 3, \ldots, N,$$

with a unique and unambiguous map into operator forms.

Following the original proposal [5] of 1978 to build hadronic mechanics, this author did not consider the new discipline sufficiently mature for experimental verifications and industrial applications until the new discipline had equally consistent and invariant classical foundations with an equally unique and unambiguous map into operator formulations.

Intriguingly, the *operator* foundations of hadronic mechanics were sufficiently identified in the original proposal [5], as we shall see in the next section. However, the identification of the *classical* counterpart turned out to be a rather complex task that required decades of research.

The objective, fully identified in 1978, was the construction of a covering of classical Lagrangian and Hamiltonian mechanics, namely, a covering of Eqs. (3.3.1), admitting a unique and unambiguous map into the already known Lie-isotopic equations of hadronic mechanics.

The mandatory starting point was the consideration of the *true Lagrange and Hamilton equations,* those with external terms

$$\frac{d}{dt}\frac{\partial L(t,r,v)}{\partial v_a^k} - \frac{\partial L(t,r,v)}{\partial r_a^k} = F_{ak}(t,r,v), \tag{3.3.2a}$$

$$\frac{dr_a^k}{dt} = \frac{\partial H(t,r,p)}{\partial p_{ak}}, \quad \frac{dp_{ak}}{dt} = -\frac{\partial H(t,r,p)}{\partial r_a^k} + F_{ak}(t,r,p), \tag{3.3.2b}$$

since they were conceived, specifically, for the interior dynamical systems treated by hadronic mechanics.

In fact, the legacy of Lagrange and Hamilton is that classical systems *cannot* be entirely represented with one single function today called a Lagrangian or a Hamiltonian used for the representation of forces derivable from a potential, but require additional quantities for the representation of contact nonpotential forced represented precisely by the external terms.

As such, the true Lagrange and Hamilton equations constitute excellent candidates for the classical origin of hadronic mechanics.

3.3.2 Insufficiencies of Analytic Equations with External Terms

It was indicated by Santilli [4] also in 1978 (see the review in Chapter 1 for more details) that the true analytic equations cannot be used for the construction of a consistent covering of conventional analytic equations because the new algebraic brackets of the time evolution of a generic quantity $A(r, p)$ in phase space

$$\frac{dA}{dt} = (A, H, F) = [A, H] + \frac{\partial A}{\partial r^k} \times F_k =$$
$$= \frac{\partial A}{\partial r^k} \times \frac{\partial H}{\partial p_k} - \frac{\partial H}{\partial r^k} \times \frac{\partial A}{\partial p_k} + \frac{\partial A}{\partial r^k} \times F_k, \tag{3.3.3}$$

violate the right distributive and scalar laws, Eqs. (3.2.5) and (3.2.6). Consequently, *the true analytic equations in their original formulation lose "all" possible algebras, let alone all possible Lie algebras.* No axiomatically consistent covering can then be build under these premises.[21]

The above insufficiency essentially established the need of rewriting the true analytic equations into a form admitting a consistent algebra in the brackets of the time evolution laws and, in addition, achieves the same invariance possessed by the truncated analytic equations.

Even though its main lines were fully identified in 1978, the achievement of the new covering mechanics resulted to require a rather long and laborious scientific journey.

This section is intended to outline the final formulation of the classical mechanics underlying hadronic mechanics in order to distinguish it from the numerous attempts that were published with the passing of time.

As a brief guide to the literature, the reader should be aware that the true analytic equations (3.3.2) are generally set for *open nonconservative systems.* These systems require the broader *Lie-admissible branch of hadronic mechanics* that will be studied in the next chapter.

Therefore, the reader should be aware that several advances in Lie-isotopies have been obtained and can be originally identified as particular cases of the broader Lie-admissible theories.

This chapter is dedicated to the study of classical and operator closed-isolated systems verifying conventional total conservation laws while having linear and nonlinear, local and nonlocal as well as potential and nonpotential internal forces.

The verification of conventional total conservation law requires classical brackets that, firstly, verify the right and left distributive and scalar laws (as a condition to characterize an algebra), and, secondly, the brackets are necessarily antisymmetric.

The brackets of conventional Hamiltonian mechanics are Lie. Therefore, a necessary condition to build a true covering of Hamiltonian mechanics is the search of brackets that are of the broader Lie-isotopic type. As a matter of

[21] For additional problematic aspects of the true analytic equations, one may consult Ref. [4] or the review in Chapter 1.

fact, this feature, fully identified in 1978 [4,5], was the very motivation for the construction of the isotopies of the Lie theory reviewed in Section 3.2.12.

In summary, the construction of a covering of the conventional Hamiltonian mechanics as the classical foundations of the Lie-isotopic branch of hadronic mechanics must be restricted to a reformulation of the true analytic equations (3.3.2) in such a way that the underlying brackets are Lie-isotopic, and the resulting mechanics is invariant.

3.3.3 Insufficiencies of Birkhoffian Mechanics

Santilli dedicated the second volume of *Foundations of Theoretical Mechanics* published by Springer-Verlag [2] in 1982 to the construction of a covering of classical Hamiltonian mechanics along the above indicated requirement. The resulting new mechanics was released under the name of *Birkhoffian mechanics* to honor G. D. Birkhoff who first discovered the underlying analytic equations in 1927.[22]

Conventional Hamiltonian mechanics is based on the *canonical action principle*

$$\delta A^o = \delta \int (p_k \times dr^k - H \times dt) = 0, \tag{3.3.4}$$

and, via the use of the unified notation

$$b = (b^\mu) = (r^i, p_j), \tag{3.3.5a}$$

$$R^o = (R^o_\mu) = (p_k, 0), \quad \mu = 1, 2, \ldots, 6, \tag{3.3.5b}$$

can be written

$$\delta A^o = \delta \int (R^o_\mu \times db^\mu - H \times dt) \equiv$$

$$\equiv \delta \int (p_k \times dr^k - H \times dt) = 0. \tag{3.3.6}$$

from which the conventional Hamilton's equations (3.3.1b) acquire the unified form

$$\omega_{\mu\nu} \times \frac{db^\nu}{dt} = \frac{\partial H}{\partial b^\mu}, \tag{3.3.7}$$

where

$$\omega_{\mu\nu} = \frac{\partial R^o_\nu}{\partial b^\mu} - \frac{\partial R^o_\mu}{\partial b^\nu} \tag{3.3.8}$$

is the fundamental (canonical) symplectic tensor (3.2.187).

[22] Interested readers should consult, for brevity, the historical notes of Ref. [2].

The fundamental (conventional Poisson) brackets of the time evolution then acquire the unified form

$$\frac{dA}{dt} = [A, H] = \frac{\partial A}{\partial b^\mu} \times \omega^{\mu\nu} \times \frac{\partial H}{\partial b^\nu}, \tag{3.3.9}$$

where

$$\omega^{\mu\nu} = [(\omega_{\alpha\beta})^{-1}]^{\mu\nu} \tag{3.3.10}$$

is the fundamental (canonical) Lie tensor.

Santilli [2] based the construction of a covering isotopic (that is, axiom-preserving) mechanics on the most general possible *Pfaffian action principle*

$$\delta A = \delta \int (R_\mu \times db^\mu - B \times dt) = 0, \tag{3.3.11}$$

where the $R_\mu(b)$ functions are now arbitrary functions in phase space, e.g., of the type

$$R(b) = (R_\mu) = (E_i(r,p), D^j(r,p)), \tag{3.3.12}$$

verifying certain regularity conditions [2].

It is easy to see that principle (3.3.11) characterizes the following analytic equations[23]

$$\Omega_{\mu\nu} \times \frac{db^\nu}{dt} = \frac{\partial B}{\partial b^\mu}, \tag{3.3.13a}$$

$$\Omega_{\mu\nu} = \frac{\partial R_\nu}{\partial b^\mu} - \frac{\partial R_\mu}{\partial b^\nu} \tag{3.3.13b}$$

is the most general possible symplectic tensor in local coordinates. Eqs. (3.3.12) were called *Birkhoff's equations* because, following a considerable research, they resulted to have been first identified by D. G. Birkhoff in 1927. The function B was called the *Birkhoffian* in order to distinguish it from the conventional Hamiltonian, since the latter represent the total energy, while the former does not.

The fundamental brackets of the time evolution then acquire the unified form

$$\frac{dA}{dt} = \frac{\partial A}{\partial b^\mu} \times \Omega^{\mu\nu} \times \frac{\partial B}{\partial b^\nu}, \tag{3.3.14a}$$

$$\Omega^{\mu\nu} = [(\Omega_{\alpha\beta})^{-1}]^{\mu\nu}. \tag{3.3.14b}$$

The covering nature of Eqs. (3.3.11)–(3.3.14) over the conventional Eqs. (3.3.4)–(3.3.10) is evident. In particular, brackets (3.3.14) are antisymmetric and verify the Lie axioms, although in the generalized Lie-Santilli isotopic form.

[23]The equations are called "analytic" in the sense of being derivable from a variational principle.

Moreover, Birkhoffian mechanics was proved in Ref. [2] to be "directly universal", that is, capable of representing "all" possible (sufficiently smooth and regular) Newtonian systems directly in the "frame of the observer" without any need for the transformation theory.

Therefore, at the time of releasing monograph [2] in 1982, the Birkhoffian mechanics appeared to have all the necessary pre-requisites to be the classical foundation of hadronic mechanics.

Unfortunately, subsequent studies established that *Birkhoffian mechanics cannot be used for consistent physical applications* because it is afflicted by the catastrophic inconsistencies studied in Section 1.4.1, with particular reference to the lack of invariance, namely, the inability to predict the same numbers for the same physical conditions at different times owing to the noncanonical character of the time evolution.

Moreover, canonical action (3.3.4) is independent from the momenta, $A^o = A^o(r)$, while this is not the case for the Pfaffian action (3.3.11) for which we have $A = A(r, p)$. Consequently, any map into an operator form implies "wavefunctions" dependent on both coordinates and momenta, $\psi(r, p)$. Therefore, the operator image of Birkhoffian mechanics is beyond our current knowledge, and its study is deferred to future generations.

The above problems requested the resumption of the search for the consistent classical counterpart of hadronic mechanics from its beginning.

Numerous additional generalized classical mechanics were identified but they still missed the achievement of the crucial invariance (for brevity, see monographs [15,16] of 1991 and the first edition of monograph [6,7] of 1993).

By looking in retrospect, the origin of all the above difficulties resulted to be where one would expect them the least, in the use of the ordinary differential calculus.

Following the discovery in 1995 (see the second edition of monographs [6,7] and Ref. [10]) of the isodifferential calculus, the identification of the final, axiomatically consistent and invariant form of the classical foundations of hadronic mechanics emerged quite rapidly.

3.3.4 Newton-Santilli Isomechanics for Matter and its Isodual for Antimatter

The fundamental character of *Newtonian Mechanics* for all scientific inquiries is due to the preservation at all subsequent levels of treatment (such as Hamiltonian mechanics, Galileo's relativity, special relativity, quantum mechanics, quantum chemistry, quantum field theory, etc.) of its main structural features, such as:

1) The underlying local-differential Euclidean topology;
2) The ordinary differential calculus; and
3) The consequential point-like approximation of particles.

Nevertheless, Newton's equations have well known notable limitations to maintain such a fundamental character for the entirely of scientific knowledge without due generalization for so many centuries.

As indicated in Chapter 1, the point-like approximation is indeed valid for very large mutual distances among particles compared to their size, as occurring for planetary and atomic systems (*exterior dynamical systems*). However, the same approximation is excessive for systems of particles at short mutual distances, as occurring for the structure of planets, hadrons, nuclei and stars (*interior dynamical systems*).

Also, dimensionless particles cannot experience any contact or resistive interactions. Consequently, dissipative or, more generally, nonconservative forces used for centuries in Newtonian mechanics are a mere *approximation* of contact nonpotential nonlocal-integral interactions among extended constituents, the approximation being generally achieved via power series expansion in the velocities.

It should be finally recalled on historical grounds that *Newton had to construct the differential calculus as a pre-requisite for the formulation of his celebrated equations.*

No genuine structural broadening of the disciplines of the 20-th century is possible without a consistent structural generalization of their foundations, Newton's equations in Newtonian mechanics.

Santilli's isomathematics has been constructed to permit *the first axiomatically consistent structural generalization of Newton's equations in Newtonian mechanics since Newton's time, for the representation of extended, nonspherical and deformable particles under linear and nonlinear, local and nonlocal and potential as well as nonpotential interactions as occurring in the physical reality of interior dynamical systems.*

By following Newton's teaching, the author has dedicated primary efforts to the isotopic lifting of the conventional differential calculus, topology and geometries [6,10] as a pre-requisite for the indicated structural generalization of Newton's equations.

To outline the needed isotopies, let us recall that Newtonian mechanics is formulated on a 7-dimensional representation space characterized by the following Kronecker products of Euclidean spaces

$$S_{tot} = E(t, R_t) \times E(r, \delta, R_r) \times E(v, \delta, R_v), \tag{3.3.15}$$

of the one dimensional space $E(t, R_t)$ representing time t, the tree dimensional Euclidean space $E(r, \delta, R_r)$ of the coordinates $r = (r_a^k)$ (where $k = 1, 2, 3$ are the Euclidean axes and $a = 1, 2, \ldots, N$ represents the number of particles), and the velocity space $E(v, \delta, R_v), v = dr/dt$.

It is generally assumed that all variables t, r, and v are defined on the same field of real numbers R. However, the unit of time is the *scalar* $I = +1$, while the unit of the Euclidean space is the *matrix*, and the same happens for the velocities, $I_r = I_v = Diag.(1,1,1)$.

Therefore, on rigorous grounds, the representation space of Newtonian mechanics must be defined on the Kronecker product of the corresponding fields

$$R_{tot} = R_t \times R_r \times R_v \tag{3.3.16}$$

with total unit

$$I_{Tot} = 1_t \times Diag.(1,1,1)_r \times Diag.(1,1,1)_v. \tag{3.3.17}$$

The above total unit can be factorized into the production of seven individual units for time and the two sets of individual Euclidean axes a, y, a with corresponding factorization of the fields

$$I_{tot} = 1_t \times 1_{rx} \times 1_{ry} \times 1_{rz} \times 1_{vx} \times 1_{vy} \times 1_{vz}, \tag{3.3.18a}$$

$$R_{tot} = R_t \times R_{rx} \times R_{ry} \times R_{rz} \times R_{vx} \times R_{vy} \times R_{vz}, \tag{3.3.18b}$$

that constitute the foundations of the conventional *Euclidean topology* here assumed as known.

Via the use of Eqs. (3.1.5), *Newton's equations for closed-non-Hamiltonian systems* can then be written

$$m_a \times a_{ka} = m_a \times \frac{dv_{ka}}{dt} = F_{ka}(t,r,v) = F_{ka}^{SA} + F_{ka}^{NSA}, \tag{3.3.19a}$$

$$\sum_a \mathbf{F}_a^{NSA} = 0, \tag{3.3.19b}$$

$$\sum_a \mathbf{r}_a \bigodot \mathbf{F}_a^{NSA} = 0, \tag{3.3.19c}$$

$$\sum_a \mathbf{r}_a \bigwedge \mathbf{F}_a^{NSA} = 0, \tag{3.3.19d}$$

where SA (NSA) stands for *variational selfadjointness (variational nonselfadjointness),* namely, the verification (violation) of the integrability conditions for the existence of a potential [1], and conditions (3.3.xx), (3.3.xx) and (3.3.xx) assure the verification of conventional total conservation laws.

The isotopies of Newtonian mechanics, today known *Newton-Santilli isomechanics*, were first submitted in the second edition of monograph [5] and in the mathematical treatment [10].

They require the use of: the *isotime* $\hat{t} = t \times \hat{I}_t$ with isounit $\hat{I}_t = 1/\hat{T}_t > 0$ and related isofield $\hat{R}_t$; the *isocoordinates* $\hat{r} = (\hat{r}_a^k) = r \times \hat{I}_r$, with isounit $\hat{I}_r = 1/\hat{T}_r > 0$ and related isofield $\hat{R}_r$; and the *isovelocities* $\hat{v} = (v_{ka}) = v \times \hat{I}_v$ with isounit $\hat{I}_v = 1/\hat{T}_v > 0$ and related isofield $\hat{R}_v$.

The Newton-Santilli isomechanics is then formulated on the 7-dimensional isospace

$$\hat{S}_{tot} = \hat{E}(\hat{t}, \hat{R}_{\hat{t}}) \times \hat{E}(\hat{r}, \hat{\delta}_r, \hat{R}_{\hat{r}}) \times \hat{E}(\hat{v}, \hat{\delta}_v, \hat{R}_{\hat{v}}), \tag{3.3.20}$$

with isometrics

$$\hat{\delta}_r = \hat{T}_r \times \delta = (\hat{T}^k_{ir} \times \delta_{kj}), \quad \hat{\delta}_v = \hat{T}_v \times \delta = (\hat{T}^k_{iv} \times \delta_{kj}), \tag{3.3.21}$$

over the Kronecker product of isofields

$$\hat{R}_{tot} = \hat{R}_t \times \hat{R}_r \times \hat{R}_v, \tag{3.3.22}$$

with total isounit

$$\hat{I}_{tot} = \hat{I}_t \times \hat{I}_r \times \hat{I}_v = \\ = n_t^2 \times Diag.(n_{rx}^2, n_{ry}^2, n_{rz}^2) \times Diag.(n_{vx}^2, n_{vy}^2, n_{vz}^2). \tag{3.3.23}$$

Consequently, the isounit can also be factorized into the product of the following seven distinct isounits, with related product of seven distinct isofields

$$\hat{I}_{tot} = n_t^2 \times n_{rx}^2 \times n_{ry}^2 \times n_{rz}^2 \times n_{vx}^2 \times n_{vy}^2 \times n_{vz}^2, \tag{3.3.24a}$$

$$\hat{R}_{tot} = \hat{R}_t \times \hat{R}_{rx} \times \hat{R}_{ry} \times \hat{R}_{rz} \times \hat{R}_{vx} \times \hat{R}_{vy} \times \hat{R}_{vz}, \tag{3.3.24b}$$

and consequential applicability of the fundamental *Tsagas-Sourlas-Santilli-Falcón-Núñez isotopology* (or TSSFN Isotopology) that allows, for the first time to the author's best knowledge, a consistent representation of extended, nonspherical and deformable shapes of particles in newtonian mechanics, here represented via the semiaxes $n_\alpha^2 = n_\alpha^2(t, r, v, \ldots)$, $\alpha = t, r, v$.

Note that the isospeed is the given by

$$\hat{v} = \frac{\hat{d}\hat{r}}{\hat{d}\hat{t}} = \hat{I}_t \times \frac{d(r \times \hat{I}_r)}{dt} = v \times \hat{I}_t \times \hat{I}_r + r \times \hat{I}_t \times \frac{d\hat{I}_r}{dt} = v \times \hat{I}_v, \tag{3.3.25}$$

thus illustrating that the isounit of the isospeed cannot be the same as that for the isocoordinates, having in particular the value

$$\hat{I}_v = \hat{I}_t \times \hat{I}_r \times \left(1 + \frac{r}{v} \times \frac{1}{\hat{I}_r} \times \frac{d\hat{I}_r}{dt}\right). \tag{3.3.26}$$

The *Newton-Santilli isoequation* [6,10] can be written

$$\hat{m}_a \hat{\times} \frac{\hat{d}\hat{v}_{ka}}{\hat{d}\hat{t}} = -\frac{\hat{\partial}\hat{V}(\hat{r})}{\hat{\partial}\hat{r}^k_a}, \tag{3.3.27}$$

namely, *the equations are conceived in such a way to formally coincide with the conventional equations for selfadjoint forces when formulated on isospace over* *isofields, while all nonpotential forces are represented by the isounits or, equivalently, by the isodifferential calculus.*

Such a conception is the only one known permitting the representation of extended particles with contact interactions that is invariant, thus avoiding the catastrophic inconsistencies of Section 1.4.1 and, in addition, achieves closure, namely, the verification of all conventional total conservation laws.

An inspection of Eqs. (3.3.27) is sufficient to see that *the Newton-Santilli isomechanics reconstructs linearity, locality and canonicity on isospaces over isofields,* as studied in Section 3.2.11. Note that this would not be the case if nonselfadjoint forces appear in the right hand side of Eqs. (3.3.27) as in Eqs. (3.3.2).

Note the truly crucial role of the isodifferential calculus for the above structural generalization of Newtonian mechanics (as well as of the subsequent mechanics), that justifies *a posteriori* its construction.

The verification of conventional total conservation laws is established by a visual inspection of Eqs. (3.3.27) since their symmetry is the *Galileo-Santilli isosymmetry* [14,15] that is isomorphic to the conventional Galilean symmetry, only formulated on isospace over isofields. By recalling that conservation laws are represented by the generators of the underlying symmetry, conventional total conservation laws then follow from the indicated invariance.

When projected in the conventional representation space S_{tot}, Eqs. (3.3.27) can be explicitly written

$$\hat{m}\hat{\times}\frac{\hat{d}\hat{v}}{\hat{d}\hat{t}} = m \times \hat{I}_t \times \frac{d(v \times \hat{I}_v)}{dt} =$$

$$= m \times \frac{dv}{dt} \times \hat{I}_t \times \hat{I}_v + m \times v \times \hat{I}_t \times \frac{d\hat{I}_v}{dt} = -\frac{\hat{\partial}\hat{V}(\hat{r})}{\hat{\partial}\hat{r}} = -\hat{I}_r \times \frac{\partial V}{\partial r}, \qquad (3.3.28)$$

that is

$$m \times \frac{dv}{dt} = -\hat{T}_t \times \hat{T}_v \times \hat{I}_r \times \frac{\partial V}{\partial r} - m \times v \times \hat{T}_v \times \frac{d\hat{I}_v}{dt}. \qquad (3.3.29)$$

The necessary and sufficient conditions for the representation of all possible SA and NSA forces are given by

$$\hat{I}_r = \hat{T}_t \times \hat{T}_r, \qquad (3.3.30a)$$

$$m \times v \times \hat{T}_v \times \frac{d\hat{I}_v}{dt} = F^{NSA}, \qquad (3.3.30b)$$

and they always admit a solution, since they constitute a system of $6n$ *algebraic* (rather than differential) equations in the $6N+1$ unknowns given by $\hat{I}_t$, and the diagonal $3N$-dimensional matrices $\hat{I}_r$ and $\hat{I}_v$.

Note that for $\hat{T}_t = 1$ we recover from a dynamical viewpoint the condition $\hat{I}_r = 1/\hat{I}_v$ obtained in Section 3.2.4 and 3.2.10 on geometric grounds.

As a simple illustration among unlimited possibilities, we have the following equations of motion of an *extended* particle with the ellipsoidal shape experiencing a resistive force $F^{NSA} = -\gamma \times v$ because moving within a physical medium

$$m \times \frac{dv}{dt} = \int d\sigma \, \Gamma(\sigma, \mathbf{r}, p, \dots) \approx -\gamma \times v, \tag{3.3.31a}$$

$$\hat{I}_v = Diag.(n_1^2, n_2^2, n_3^2) \times e^{\gamma \times t/m}, \tag{3.3.31b}$$

where the nonlocal-integral character with respect to a kernel Γ is emphasized. Interested readers can then construct the representation of *any* desired non-Hamiltonian Newtonian system (see also memoir [10] for other examples).

Note the natural appearance in the NSA forces of the velocity dependence, as typical of resistive forces. Note also that the representation of the extended character of particles occurs only in isospace because, when Eqs. (3.3.xx) are projected in the conventional Newtonian space, factorized isounits cancel out and the point characterization of particles is recovered.

Note finally the *direct universality* of the Newton-Santilli isoequations, namely, their capability of representing all infinitely possible Newton's equations in the frame of the observer.

As now familiar earlier, Eqs. (3.3.27) can only describe a system of *particles.* The *isodual Newton-Santilli isoequations* for the treatment of a system of *antiparticles* are given by [6,10]

$$\hat{m}_a^d \hat{\times}^d \frac{\hat{d}^d \hat{v}_{ka}^d}{\hat{d}^d \hat{t}^d} = -\frac{\hat{\partial}^d \hat{V}^d(\hat{r}^d)}{\hat{\partial}^d \hat{r}_a^{kd}}. \tag{3.3.32}$$

The explicit construction of the remaining isodualities of the above isomechanics are instructive for the reader seriously interested in a classical study of antimatter under interior dynamical conditions.

3.3.5 Hamilton-Santilli Isomechanics for Matter and its Isodual for Antimatter

3.3.5A. Isoaction Principle and its Isodual. The isotopies of classical Hamiltonian mechanics were first introduced by Santilli in various works (see monographs [6,7] and references quoted therein), and are today known as the *Hamilton-Santilli isomechanics.*

To identify its representation space, recall that the conventional Hamiltonian mechanics is represented in a 7-dimensional space of time, coordinates and momenta (rather than velocity), the latter characterizing phase space (or cotangent bundle of the symplectic geometry).

Correspondingly, the new isomechanics is formulated in the 7-dimensional isospace of isotime $\hat{t}$, isocoordinates $\hat{r}$ and isomomenta $\hat{p}$

$$\hat{S}_{tot} = \hat{E}(\hat{t}, \hat{R}_{\hat{t}}) \times \hat{E}(\hat{r}, \hat{\delta}_r, \hat{R}_{\hat{r}}) \times \hat{E}(\hat{p}, \hat{\delta}_p, \hat{R}_{\hat{p}}), \tag{3.3.33}$$

with isometrics

$$\hat{\delta}_{\hat{r}} = \hat{T}_{\hat{r}} \times \delta = (\hat{T}^k_{ir} \times \delta_{kj}), \hat{\delta}_{\hat{p}} = \hat{T}_{\hat{p}} \times \delta = (\hat{T}^k_{ip} \times \delta_{kj}), \tag{3.3.34}$$

over the Kronecker product of isofields and related isounits

$$\hat{R}_{tot} = \hat{R}_t \times \hat{R}_r \times \hat{R}_p, \tag{3.3.35a}$$

$$\hat{I}_{tot} = \hat{I}_{\hat{t}} \times \hat{I}_{\hat{r}} \times \hat{I}_{\hat{p}} =$$

$$= n_t^2 \times Diag.(n_{rx}^2, n_{ry}^2, n_{rz}^2) \times Diag.(n_{px}^2, n_{py}^2, n_{pz}^2). \tag{3.3.35b}$$

The following new feature now appears. The *isophasespace,* or, more technically, the *isocotangent bundle* of the isosymplectic geometry in local isochart $(\hat{r}, \hat{p})$ requires that the isounits of the variables $\hat{r}$ and $\hat{p}$ are *inverse* of each others (Section 3.2.3 and 3.2.10)

$$\hat{I}_{\hat{r}} = 1/\hat{T}_{\hat{r}} = \hat{I}_{\hat{p}}^{-1} = \hat{T}_{\hat{p}} > 0. \tag{3.3.36}$$

Consequently, by ignoring hereon for notational simplicity the indices for the N particles, the total isounit of the isophase space can be written

$$\hat{I}_{tot} = \hat{I}_{\hat{t}} \times \hat{I}_{\hat{r}} \times \hat{T}_{\hat{r}} = \hat{I}_{\hat{t}} \times \hat{I}_6, \tag{3.3.37a}$$

$$\hat{I}_6 = (\hat{I}^\nu_\mu) = \hat{I}_{\hat{r}} \times \hat{T}_{\hat{r}}. \tag{3.3.37b}$$

The fundamental *isoaction principle* for the classical treatment of matter in interior conditions can be written in the explicit form in the $\hat{r}$ and $\hat{p}$ isovariables

$$\hat{\delta}\hat{A}^o = \hat{\delta} \int_{t_1}^{t_2} (\hat{p}_k \hat{\times} \hat{d}\hat{r}^k - \hat{H} \hat{\times} \hat{d}\hat{t}) = \hat{\delta} \int_{t_1}^{t_2} [p_k \times \hat{T}_{\hat{r}}^{k_i(t,r,p,\ldots)} \times \hat{d}\hat{r}^i - \hat{H} \times \hat{T}_{\hat{t}} \times d\hat{t}] = 0, \tag{3.3.38}$$

where

$$\hat{H} = \hat{p}^{\hat{2}} \hat{/} \hat{2} \hat{\times} \hat{m} - \hat{V}(\hat{r}), \tag{3.3.39}$$

is the *isohamiltonian* or simple the Hamiltonian because its projection on conventional spaces represents the orginary total energy except an inessential multiplicative factor.

By using the unified notation

$$\hat{b} = (\hat{b}^\mu) = (\hat{r}^i, \hat{p}_j) = (r^i, p_j) \times \hat{I}_6 = b \times \hat{I}_6, \tag{3.3.40}$$

and the isotopic image of the canonical R^o functions, Eqs. (3.3.xx),

$$\hat{R}^o = (\hat{R}^o_\mu) = (\hat{r}, \hat{0}), \tag{3.3.41}$$

the fundamental isoaction principle can be written in unified notation

$$\hat{\delta}\hat{A}^o = \hat{\delta}\int_{t_1}^{t_2}(\hat{p}_k \hat{\times} \hat{d}\hat{r}^k - \hat{H}\hat{\times}\hat{d}\hat{t}) \equiv \hat{\delta}\int_{t_1}^{t_2}(\hat{R}^o_\mu \hat{\times} \hat{d}\hat{b}^\mu - \hat{H}\hat{\times}\hat{d}\hat{t}) =$$

$$= \hat{\delta}\int_{t_1}^{t_2}(R^o_\mu \times \hat{T}^\mu_{6\nu} \times d\hat{b}^\nu - H \times \hat{T}_{\hat{t}} \times d\hat{t}) = 0. \tag{3.3.42}$$

A visual inspection of principle (3.3.38) establishes the *isocanonicity* of Hamilton-Santilli isomechanics (Section 3.2.11), namely, the reconstruction of canonicity on isospaces over isofield that is crucial for the consistency of hadronic mechanics.

In fact, the conventional action principle (3.3.4) and isoprinciple (3.3.38) coincide at the abstract, realization-free level by conception and construction.

The direct universality of classical isomechanics can be seen from the arbitrariness of the integrand of isoaction functional (3.3.38) once projected on conventional spaces over conventional fields.

An important property of the isoaction is that its functional dependence on isospaces over isofields is restricted to that on isocoordinates only, i.e., $\hat{A} = \hat{A}(\hat{r})$. However, when projected on conventional spaces, the functional dependence is arbitrary, i.e., $\hat{A}(\hat{r}) = \hat{A}(r \times \hat{I}) = \hat{A}(t, r, p, \ldots)$. This feature will soon have a crucial role for the operator image of the classical isomechanics.

It should finally be noted that isoprinciple (3.3.38) essentially eliminates the entire field of *Lagrangian and action principles of orders higher than the first,* e.g., $L = L(t, r, \dot{r}, \ddot{r}, \ldots)$ because of these higher order formulations can be easily reduced to the isotopic first-order form (3.3.38).

Recall that the action principle has the important application via the use of the *optimal control theory* of optimizing dynamical systems, However, the latter can have only been Hamiltonian until now due to the lack of a universal action functional for non-Hamiltonian systems (that constitute, by far, the system most significant for optimization). Recall also that the optimal control theory can only be applied for local-differential systems due to the underlying Euclidean topology, thus secluding from the optimization process the most important systems, those of extended, and, therefore, of nonlocal type.

Note that isoaction principle (3.3.38) occurs for all possible non-Hamiltonian as well as nonlocal-integral systems, thanks also to the underlying TSSFN isotopology (Section 3.2.7). We, therefore, have the following important:

THEOREM 3.3.1 [6,10]: Isoaction principle (3.3.38) permits the (first known) optimization of all possible nonpotential/non-Hamiltonian and nonlocal-integral systems.

The *isodual isoaction principle* [10] for the classical treatment of antimatter in interior conditions is given by

$$\hat{\delta}^d \hat{A}^d = \hat{\delta}^d \int_{t_1}^{t_2}{}^d (\hat{p}_k^d \hat{\times}^d \hat{d}^d \hat{r}^{kd} - \hat{H}^d \hat{\times}^d \hat{d}^d \hat{t}^d) =$$

$$= \hat{\delta}^d \int_{t_1}^{t_2}{}^d (\hat{R}_\mu^{od} \hat{\times}^d \hat{d}^d \hat{b}^\mu - \hat{H}^d \hat{\times} \hat{d}^d \hat{t}^d) = 0. \tag{3.3.43}$$

Additional isodual treatments are left to the interested reader.

3.3.5B. Hamilton-Santilli Isoequations and their Isoduals. The discovery of the isodifferential calculus in 1995 permitted Santilli [6,10] the identification of the following classical dynamical equations for the treatment of matter at the foundations of hadronic mechanics, today known as the *Hamilton-Santilli isoequations*. They are easily derived via the isovariational principle and can be written from isoprinciple (3.3.38) in disjoint notation

$$\frac{\hat{d}\hat{r}^k}{\hat{d}\hat{t}} = \frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{p}_k}, \quad \frac{\hat{\partial}\hat{p}_k}{\hat{d}\hat{t}} = -\frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{r}^k}. \tag{3.3.44}$$

The same equations can be written in unified notation from principle (3.3.40)

$$\hat{\omega}_{\mu\nu} \hat{\times} \frac{\hat{d}\hat{b}^\mu}{\hat{d}\hat{t}} = \frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{b}^\mu}, \tag{3.3.45}$$

where

$$\hat{\omega}_{\mu\nu} = \omega_{\mu\nu} \times \hat{I}_6 \tag{3.3.46}$$

is the *isocanonical isosymplectic tensor* that coincides with the conventional canonical symplectic tensor $\omega_{\mu\nu}$ except for the factorization of the isounit (Section 3.2.10).

To verify the latter property from an analytic viewpoint, it is instructive for the reader to verify the following identify under isounits (3.3.37)

$$\hat{\omega}_{\mu\nu} = \frac{\hat{\partial}\hat{R}_\nu^o}{\hat{\partial}\hat{b}^\mu} - \frac{\hat{\partial}\hat{R}_\mu^o}{\hat{\partial}\hat{b}^\nu} = \omega_{\mu\nu} \times \hat{I}_6. \tag{3.3.47}$$

A simple comparison of the above isoanalytic equations with the isotopic and conventional Newton's equations established the following:

THEOREM 3.3.2: Hamilton-Santilli isoequations (3.3.5) are "directly universal" in Newtonian mechanics, that is, capable of representing all possible, conventional or isotopic, hamiltonian and non-Hamiltonian Newtonian systems directly in the fixed coordinates of the experimenter.

It is now important to show that Eqs. (3.3.45) provide an identical reformulation of the true analytic equations (3.3.2). For this purpose, we assume the simple case in which isotime coincide with the conventional time, that is, $\hat{t} = t$, $\hat{I}_t = +1$ and we write isoequations (3.3.45) in the explicit form

$$\begin{gathered}(\omega) \times \begin{pmatrix} dr^k/dt \\ dp_k/dt \end{pmatrix} = \begin{pmatrix} 0_{3\times3} & -I_{3\times3} \\ I_{3\times3} & 0_{3\times3} \end{pmatrix} \times \begin{pmatrix} dr^k/dt \\ dp_k/dt \end{pmatrix} = \\ \begin{pmatrix} -dp_k/dt \\ dr^k/dt \end{pmatrix} = \begin{pmatrix} \hat{\partial}\hat{H}/\hat{\partial}r^k \\ \hat{\partial}\hat{H}/\hat{\partial}p_k \end{pmatrix} = \begin{pmatrix} \hat{I}^i_k \times \partial\hat{H}/\partial r^i \\ \hat{T}^k_i \times \partial\hat{H}/\partial p_i \end{pmatrix}.\end{gathered} \tag{3.3.48}$$

It is easy to see that Eqs. (3.3.xx) *coincide* with the true analytic equations (3.3.2) under the trivial algebraic identification

$$\hat{I}_{\hat{r}} = Diag.[I - F/(\partial H/\partial r)]. \tag{3.3.49}$$

As one can see, the main mechanism of Eqs. (3.3.45) is that of *transforming the external terms $F = F^{NSA}$ into an explicit realization of the isounit $\hat{I}_3$*. As a consequence, *reformulation (3.3.45) constitutes direct evidence on the capability to represent non-Hamiltonian forces and effects with a generalization of the unit of the theory.*

Note in particular that *the external terms are embedded in the isoderivatives.* However, when written down explicitly, Eqs. (3.3.2) and (3.3.45) coincide. Note that $\hat{I}_3$ as in rule (3.3.49) is fully symmetric, thus acceptable as the isounit of isomathematics. Note also that all nonlocal and nonhamiltonian effects are embedded in $\hat{I}$.

The reader should note the extreme simplicity in the construction of a representation of given non-Hamiltonian equations of motion, due to the algebraic character of identifications (3.3.49).

Recall that *Hamilton's equations with external terms are not derivable from a variational principle.* In turn, such an occurrence has precluded the identification of the operator counterpart of Eqs. (3.3.2) throughout the 20-th century.

We now learn that *the identical* reformulation (3.3.45) of Eqs. (3.3.2) becomes fully derivable from a variational principle. In turn, this will soon permit the identification of the unique and unambiguous operator counterpart.

It should be noted that *the Hamilton-Santilli isoequations are generally irreversible* due to the general irreversibility of the external forces,

$$F(t,\ldots) \neq F(-t,\ldots), \quad \text{or} \tag{3.3.50a}$$

$$\hat{I}(t,\ldots) = Diag.[I - F(t,\ldots)/(\partial H/\partial t)] \neq \hat{I}(-t,\ldots). \tag{3.3.50b}$$

In particular, we have irreversibility under the conservation of the total energy (see next chapter for full treatment). This feature is important to achieve compatibility with thermodynamics, e.g., to have credible analytic methods for the representation of the internal increase of the entropy for closed-isolated systems such as Jupiter.

The study of these thermodynamical aspects is left to the interested reader. In this chapter we shall solely consider *reversible closed-isolated systems* that occur for external forces not explicitly dependent on time and verify other restrictions.

An important aspect is that *the Hamilton-Santilli isoequations coincide with the Hamilton equations without external terms at the abstract level.* In fact, all differences between I and $\hat{I}$, $\times$ and $\hat{\times}$, ∂ and $\hat{\partial}$, *etc.*, disappear at the abstract level. This proves the achievement of a central objective of isomechanics, the property that the analytic equations with external terms can indeed be *identically* rewritten in a form equivalent to the analytic equations without external terms, provided, however, that the reformulation occurs via the broader isomathematics.

The *isodual Hamilton-Santilli isoequations* for the classical treatment of antimatter, also identified soon after the discovery of the isodifferential calculus, are given by

$$\hat{\omega}^d_{\mu\nu} \hat{\times}^d \frac{\hat{d}^d \hat{b}^{d\mu}}{\hat{d}^d \hat{t}^d} = \frac{\hat{\partial}^d \hat{H}^d}{\hat{\partial}^d \hat{b}^{d\mu}}, \tag{3.3.51}$$

where

$$\hat{\omega}^d_{\mu\nu} = \omega^d_{\mu\nu} \times \hat{I}_6 \tag{3.3.52}$$

is the *isodual isocanonical isosymplectic tensor.* The derivation of other isodual properties is instructed for the interested reader.

3.3.5C. Classical Lie-Santilli Brackets and their Isoduals. It is important to verify that Eqs. (3.3.44) or (3.3.45) resolve the problematic aspects of external terms indicated in Section 3.3.2 [4]. In fact, the isobrackets of the time evolution of matter are given by

$$\frac{d\hat{A}}{\hat{d}\hat{t}} = [\hat{A}\hat{,}\hat{H}] = \frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{r}^k} \hat{\times} \frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{p}_k} - \frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{r}^k} \hat{\times} \frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{p}_k}, \tag{3.3.53}$$

and they verify the left and right distributive and scalar laws, thus characterizing a consistent algebra. Moreover, that algebra results to be Lie-isotopic, for which reasons the above brackets are known as the *Lie-Santilli isobrackets.*

When explicitly written in our spacetime, brackets (3.3.53) recover the brackets (3.3.3) of the true analytic equations (3.3.2)

$$\frac{dH}{dt} = \frac{\partial H}{\partial r^k} \times \frac{\partial H}{\partial p_k} - \frac{\partial H}{\partial p_k} \times \frac{\partial H}{\partial r^k} + \frac{\partial H}{\partial p_k} \times F^k = \frac{\partial H}{\partial p_k} \times F^k \equiv 0, \qquad (3.3.54)$$

where the last identity holds in view of Eqs. (3.3.49). Therefore, the Hamilton-Jacobi isoequations do indeed constitute a reformulation of the true analytic equations with a consistent Lie-isotopic algebraic brackets, as needed (Section 3.3.3).

Note that, in which of their anti-isomorphic character, isobrackets (3.3.53) represent the conservation of the Hamiltonian,

$$\frac{d\hat{H}}{dt} = [\hat{H}\hat{,}\hat{H}] = \frac{\hat{\partial}\hat{H}}{\hat{\partial}r^k} \hat{\times} \frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{p}_k} - \frac{\hat{\partial}\hat{H}}{\hat{\partial}r^k} \hat{\times} \frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{p}_k} \equiv 0. \qquad (3.3.55)$$

This illustrates the reason for assuming closed-isolated Newtonian systems (3.3.19) at the foundations of this chapter.

Basic isobrackets (3.3.53) can be written in unified notation

$$[\hat{A}\hat{,}\hat{B}] = \frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{b}^\mu} \hat{\times} \hat{\omega}^{\mu\nu} \hat{\times} \frac{\hat{\partial}\hat{B}}{\hat{\partial}\hat{b}^\nu}, \qquad (3.3.56)$$

where $\hat{\omega}_{\mu\nu}$ is the Lie-Santilli isotensor. By using the notation $\hat{\partial}^\mu = \hat{\partial}/\hat{\partial}\hat{b}^\mu$, the isobrackets can be written

$$[\hat{A}\hat{,}\hat{B}] = \hat{\partial}_\mu \hat{A} \times \hat{T}^\mu_\rho \times \omega^{\rho\nu} \hat{\partial}_\nu \hat{\partial}_\nu \hat{B}, \qquad (3.3.57)$$

and, when projected in our spacetime, the isobrackets can be written

$$[A\hat{,}B] = \partial_\mu A \times \omega^{\mu\rho} \times \hat{I}^\nu_\rho \times \partial_\nu B, \qquad (3.3.58)$$

where $\omega^{\mu\nu}$ is the canonical Lie tensor.

The *isodual Lie-Santilli isobrackets* for the characterization of antimatter can be written

$$[\hat{A}^d\hat{,}\hat{B}^d] = \hat{\partial}^d_\mu \hat{A}^d \hat{\times}^d \hat{\omega}^{d\rho\nu} \hat{\partial}^d_\nu \hat{\partial}^d_\nu \hat{B}, \qquad (3.3.59)$$

where $\hat{\omega}^{d\mu\nu}$ is the *isodual Lie-Santilli isotensor*. Other algebraic properties can be easily derived by the interested reader.

3.3.5D. Hamilton-Jacobi-Santilli Isoequations and their isoduals. Another important consequence of isoaction principle (3.3.38) is the characterization of the following *Hamilton-Jacobi-Santilli isoequations* for matter [6,10]

$$\frac{\hat{\partial}\hat{A}^o}{\hat{\partial}\hat{t}} + \hat{H} = 0, \qquad (3.3.60a)$$

$$\frac{\hat{\partial}\hat{A}^o}{\hat{\partial}\hat{r}^k} - \hat{p}_k = 0, \tag{3.3.60b}$$

$$\frac{\hat{\partial}\hat{A}^o}{\hat{\partial}\hat{p}_k} \equiv 0, \tag{3.3.60c}$$

which will soon have basic relevance for isoquantization.

Note the *independence of the isoaction $\hat{A}^o$ from the isomomenta* that will soon be crucial for consistent isoquantization.

The isodual equations for antimatter are then given by

$$\frac{\hat{\partial}^d\hat{A}^{od}}{\hat{\partial}^d\hat{t}^d} + \hat{H}^d = 0, \tag{3.3.61a}$$

$$\frac{\hat{\partial}^d\hat{A}^{od}}{\hat{\partial}^d\hat{r}^{kd}} - \hat{p}^d_k = 0, \tag{3.3.61b}$$

$$\frac{\hat{\partial}^d\hat{A}^{od}}{\hat{\partial}^d\hat{p}^d_k} \equiv 0. \tag{3.3.61c}$$

The latter equations will soon result to be essential for the achievement of a consistent operator image of the classical treatment of antimatter in interior conditions.

3.3.5E. Connection Between Isotopic and Birkhoffian Mechanics. Since the Hamilton-Santilli isoequations are directly universal, they can also represent Birkhoff's equations (3.3.13) in the fixed b-coordinates. In fact, by assuming for simplicity that the isotime is the ordinary time, we can write the identities

$$\frac{db^\mu}{dt} = \Omega^{\mu\mu}(b) \times \frac{\partial H(b)}{\partial b^\nu} \equiv \omega^{\mu\nu} \times \frac{\hat{\partial} H(b)}{\hat{\partial} b^\nu} = \omega^{\mu\rho} \times \hat{I}^\nu_{6\rho} \times \frac{\partial H}{\partial b^\nu}. \tag{3.3.62}$$

Consequently, we reach the following *decomposition of the Birkhoffian tensor*

$$\Omega^{\mu\nu}(b) = \omega^{\mu\rho} \times \hat{I}^\nu_{6\rho}(b). \tag{3.3.63}$$

Consequently, Birkhoff's equations can indeed be identically rewritten in the isotopic form, as expected. In the process, the reformulation provides additional insight in the isounit.

The reformulation also carries intriguing geometric implications since it confirms the *direct universality in symplectic geometry of the canonical two-form,* since a general symplectic two-form can always be identically rewritten in the isocanonical form via decomposition of type (3.3.xx) and then the embedding of the isounit in the isodifferential of the exterior calculus.

As an incidental note, the reader should be aware that the construction of an analytic representation via Birkhoff's equations is rather complex, inasmuch as it requires *the solution of nonlinear partial differential equations* or integral equations [2].

By comparison, the construction of the same analytic equations via Hamilton-Santilli isoequations (3.3.44) or (3.3.45) is truly elementary, and merely requires the identification of the isounit according to *algebraic* rule (3.3.49) for arbitrarily given external forces $F_k(t, r, p)$.

3.3.6 Simple Construction of Classical Isomechanics

The above classical isomechanics can be constructed via a simple method which does not need any advanced mathematics, yet it is sufficient and effective for practical applications.

In fact, *the Hamilton-Santilli isomechanics can be constructed via the systematic application of the following noncanonical transform to all quantities and operations of the conventional Hamiltonian mechanics*

$$U = \begin{pmatrix} \hat{I}_3^{1/2} & 0 \\ 0 & \hat{T}_3^{1/2} \end{pmatrix}, \tag{3.3.64a}$$

$$U \times U^t = \hat{I}_6 \neq I, \tag{3.3.64b}$$

$$\hat{I}_3 = I - \frac{F}{\partial H / \partial p} = I - \frac{F}{p/m}. \tag{3.3.64c}$$

The success of the construction depends on the application of the above noncanonical transform to the *totality* of Hamiltonian mechanics, with no exceptions. We have in this way the lifting of: the 6-dimensional unit of the conventional phase space into the isounit

$$I_6 \rightarrow \hat{I}_6 = U \times I_6 \times U^t; \tag{3.3.65}$$

numbers into the isonumbers,

$$n \rightarrow \hat{n} = U \times n \times U^t = n \times (U \times U^t) = n \times \hat{I}_6; \tag{3.3.66}$$

associative product $A \times B$ among generic quantities A, B into the isoassociative product with the correct expression and property for the isotopic element,

$$A \times B \rightarrow A \hat{\times} B = U \times (A \times B) \times U^t = A' \times \hat{T} \times B', \tag{3.3.67a}$$

$$A' = U \times A \times U^t, \quad B' = U \times B \times U^t, \quad \hat{T} = (U \times U^t)^{-1} = T^t; \tag{3.3.67b}$$

Euclidean into iso-Euclidean spaces (where we use only the space component of the transform)

$$\begin{gathered} x^2 = x^t \times \delta \times x \rightarrow \hat{x}^{\hat{2}} = U \times x^2 \times U^t = \\ = (x^t \times U^t) \times (U^{t-1} \times \delta \times U^{-1}) \times (U \times x) \times (U \times U^t) = \\ = [x'^t \times (\hat{T} \times \delta) \times x'] \times \hat{I}; \end{gathered} \tag{3.3.68}$$

and, finally, we have the following isotopic lifting of Hamilton's into Hamilton-Santilli isoequations (here derived for simplicity for the case in which the transform does not depend explicitly on the local coordinates),

$$\begin{gathered} db/dt - \omega \times \partial H/\partial b = 0 \rightarrow \\ \rightarrow U \times db/dt \times U^t - U \times \omega \times \partial H/\partial b \times U^t = \\ = db/dt \times (U \times U^t) - (U \times \omega \times U^t) \times (U^{t\ -1} \times U^{-1}) \times \\ \times (U \times \partial H/\partial b \times U^t) \times (U \times U^t) = \\ = db/dt \times \hat{I} - \omega \times (\hat{\partial} H/\hat{\partial}\hat{b}) \times \hat{I} = 0, \end{gathered} \tag{3.3.69}$$

where we have used the important property the reader is urged to verify

$$U \times \omega \times U^t \equiv \omega. \tag{3.3.70}$$

As one can see, the seemingly complex isomathematics and isomechanics are reduced to a truly elementary construction. e its universality.

3.3.7 Invariance of Classical Isomechanics

A final requirement is necessary for a physical consistency, and that is, *the invariance of isomechanics under its own time evolution*, as it occurs for conventional Hamiltonian mechanics.

Recall that a transformation $b \rightarrow b'(b)$ is called a *canonical transformation* when all the following identities hold

$$\frac{\partial b^\mu}{\partial b'^\alpha} \times \omega_{\mu\nu} \times \frac{\partial b^\nu}{\partial b'^\beta} = \omega_{\alpha\beta}. \tag{3.3.71}$$

The invariance of Hamiltonian mechanics follows from the property that its time evolution constitutes a canonical transformation, as well known.

The proof of the invariant of isomechanics is elementary. In fact, an isotransformation $\hat{b} \rightarrow \hat{b}'(\hat{b})$ constituted an *isocanonical isotransform* when all the following identities old

$$\frac{\hat{\partial}\hat{b}^\mu}{\hat{\partial}\hat{b}'^\alpha} \hat{\times} \hat{\omega}_{\mu\nu} \hat{\times} \frac{\hat{\partial}\hat{b}^\nu}{\hat{\partial}\hat{b}'^\beta} = \hat{\omega}_{\alpha\beta} = \omega_{\alpha\beta} \times \hat{I}_6. \tag{3.3.72}$$

But the above expression can be written

$$(\hat{I}_6^{\mu_\rho} \times \frac{\partial \hat{b}^\rho}{\partial \hat{b}^\alpha} \times \omega_{\mu\nu} \times \hat{I}_6^{\xi_\nu} \times \frac{\partial \hat{b}^\xi}{\partial \hat{b}'^\beta}) \times \hat{I}_6 = \omega_{\mu\nu} \times \hat{I}_6, \qquad (3.3.73)$$

and they coincide with conditions (3.3.xx) in view of the identities

$$\hat{I}_6^{\mu_\rho} \times \omega_{\mu\nu} \times \hat{I}_6^{\nu_\xi} = \omega_{\rho\xi}. \qquad (3.3.74)$$

Consequently, we have the following important

THEOREM 3.3.3 [6,10]: Following factorization of the isounit, isocanonical transformations are canonical.

The desired invariance of the Hamilton-Santilli isomechanics then follows.

It is an instructive exercise for the reader interested in learning isomechanics to verify that *all* catastrophic mathematical and physical inconsistencies of non-canonical theories pointed out in Chapter 1 (see Section 1.4.1 in particular) are indeed resolved by isomechanics as presented in this section.

3.4 OPERATOR LIE-ISOTOPIC MECHANICS FOR MATTER AND ITS ISODUAL FOR ANTIMATTER

3.4.1 Introduction

We are finally equipped to present the foundations of *the Lie-isotopic branch of nonrelativistic hadronic mechanics* for matter and its isodual for antimatter, more simply referred to as *operator isomechanics,* and its isodual for antimatter referred to as *isodual operator isomechanics.* The new mechanics will then be used in subsequent sections for various developments, experimental verifications and industrial applications.

The extension of the results of this section to *relativistic operator isomechanics* is elementary and will be done in the following sections whenever needed for specific applications. the case of *operator genomechanics* with a Lie-admissible, rather than the Lie-isotopic structure, will be studied in the next chapter.

A knowledge of Section 3.2 is necessary for a technical understanding of operator isomechanics. For the mathematically non-inclined readers, we present in Section 3.4.8 a very elementary construction of operator isomechanics via nonunitary transforms.

Unless otherwise specified, all quantities and operations represented with conventional symbols A, H, $\times$, *etc.*, denote quantities and operations on conventional Hilbert spaces over conventional fields. All quantities and symbols of the type $\hat{A}$, $\hat{H}$, $\hat{\times}$, *etc.*, are instead defined on isohilbert spaces over isofields.

Note the use of the terms "operator" isomechanics, rather than "quantum" isomechanics, because, as indicated in Chapter 1, the notion of quantum is fully established within the arena of its conception, the transition of electrons between different stable orbits of atomic structure (exterior problem), while the assumption of the same quantum structure for the same electrons when in the core of a star (interior problems) is a scientific religion at this writing deprived of solid experimental evidence.

3.4.2 Naive Isoquantization and its Isodual

An effective way to derive the basic dynamical equations of operator isomechanics is that via the isotopies of the conventional map of the classical Hamilton-Jacobi equations into their operator counterpart, known as *naive quantization.*. More rigorous methods, such as the isotopies of symplectic quantization, essentially yields the same operator equations and will not be treated in this section for brevity (see monograph [7] for a presentation).

Recall that the *naive quantization* can be expressed via the following map of the canonical action functional

$$A^o = \int_{t_1}^{t_2} (p_k \times dr^k - H \times dt) \rightarrow -i \times \hbar \times \ln |\psi\rangle, \tag{3.4.1}$$

under which the conventional Hamilton-Jacobi equations are mapped into the Schrödinger equations,

$$-\partial_t A^o = H \rightarrow i \times \hbar \times \partial_t |\psi\rangle = H \times |\psi\rangle, \tag{3.4.2a}$$

$$p_k = \partial_k A^o \rightarrow p_k \times |\psi\rangle = -i \times \hbar \times \partial_k |\psi\rangle, \tag{3.4.2b}$$

where $|\psi\rangle$ is the *wavefunction,* or, more technically, a state in a Hilbert space $\mathcal{H}$.

Isocanonical action (3.3.38) is evidently different than the conventional canonical action, e.g., because it is of higher order derivatives. As such, the above naive quantization does not apply.

In its place we have the following *naive isoquantization* first introduced by Animalu and Santilli [44] of 1990, and here extended to the use of the isodifferential calculus

$$\hat{A}^o = \int_{t_1}^{t_2} (\hat{p}_k \hat{\times} \hat{d}\hat{x}^k - \hat{H} \hat{\times} \hat{d}\hat{t}) \rightarrow -i \times \hat{I} \times \ln |\hat{\psi}\rangle, \tag{3.4.3}$$

where $\hat{i} = i \times \hat{I}$, $|\hat{\psi}\rangle$ is the *ospwavefunction,* or, more precisely, a state of the iso-Hilbert space $\hat{\mathcal{H}}$ outlined in the next section, and we should note that $\hat{i} \hat{\times} \hat{I} \times \ln |\hat{\psi}\rangle = i \times \text{isoln}|\hat{\psi}\rangle$.

The use of Hamilton-Jacobi-Santilli isoequations (3.3.60) yields the following operator equations (here written for the simpler case in which $\hat{T}$ has no dependence on r, but admits a dependence on velocities and higher derivatives)

$$-\hat{\partial}_t \hat{A}^o = \hat{H} \rightarrow i \times \hat{\partial}_t |\hat{\psi}\rangle \ = \hat{H} \times \hat{T} \times |\hat{\psi}\rangle \ = \hat{H} \hat{\times} |\hat{\psi}\rangle, \tag{3.4.4a}$$

$$\hat{p}_k = \hat{\partial}_k \hat{A}^o \rightarrow \hat{p}_k \times \hat{T} \times |\hat{\psi}\rangle \ = \hat{p}_k \hat{\times} |\hat{\psi}\rangle \ = -\hat{i} \hat{\times} \hat{\partial}_k |\hat{\psi}\rangle, \tag{3.4.4b}$$

that constitutes the fundamental equations of operator isomechanics, as we shall see in the next section.

As it is well known, *Planck's constant $\hbar$ is the basic unit of quantum mechanics.* By comparing Eqs. (3.4.xx) and (3.4.xx) it is easy to see that $\hat{I}$ is the basic unit of operator isomechanics. Recall also that the isounits are defined at short distances as in Eqs. (3.1.xxx). We therefore have the following important

POSTULATE 3.4.1 [5]: In the transition from quantum mechanics to operator isomechanics Planck's unit $\hbar$ is replaced by the integrodifferential unit $\hat{I}$ under the condition of recovering the former at sufficiently large mutual distances,

$$\lim_{r \rightarrow \infty} \hat{I} = \hbar = 1. \tag{3.4.5}$$

Consequently, in the conditions of deep mutual penetration of the wavepackets and/or charge distributions of particles as studied by operator isomechanics there is the superposition of quantized and continuous exchanges of energy.

3.4.3 Isohilbert Spaces and their Isoduals

As it is well known, the Hilbert space $\mathcal{H}$ used in quantum mechanics is expressed in terms of states $|\psi\rangle,\ |\phi\rangle, \ldots$, with normalization

$$\langle\psi| \times |\psi\rangle = 1, \tag{3.4.6}$$

and inner product

$$\langle\phi| \times |\psi\rangle = \int dr^3 \phi^\dagger(r) \times \psi(r), \tag{3.4.7}$$

defined over the field of complex numbers $\mathcal{C} = C(c, +, \times)$.

The lifting $C(c, +, \times) \rightarrow \hat{C}(\hat{c}, \hat{+}, \hat{\times})$, requires a compatible lifting of $\mathcal{H}$ into the *isohilbert space* $\hat{\mathcal{H}}$ with *isostates* $|\hat{\psi}\rangle, |\hat{\phi}\rangle, \ldots$, *isoinner product* and *isonormalization*

$$\langle\hat{\psi}| \hat{\times} |\hat{\psi}\rangle \times \hat{I} = \left[\hat{\int} d\hat{r}^3 \ \hat{\psi}^\dagger(\hat{r}) \times \hat{T} \times \hat{\psi}(\hat{r}) \right] \times \hat{I} \in \hat{\mathcal{C}}, \tag{3.4.8a}$$

$$\langle\hat{\psi}| \hat{\times} |\hat{\psi}\rangle = 1, \tag{3.4.8b}$$

first introduced by Myung and Santilli in 1982 [45] (see also monographs [6,7] for a comprehensive study).

It is easy to see that the isoinner product is still inner (because $\hat{T} > 0$). Thus, $\hat{\mathcal{H}}$ is still Hilbert and the lifting $\mathcal{H} \to \hat{\mathcal{H}}$ is an isotopy. Also, it is possible to prove that *iso-Hermiticity coincides with conventional Hermiticity*,

$$\langle\hat{\psi}|\hat{\times}(\hat{H}\hat{\times}|\hat{\psi}\rangle) \equiv (\langle\hat{\psi}|\hat{\times}\hat{H}^{\hat{\dagger}})\hat{\times}|\hat{\psi}\rangle, \tag{3.4.9a}$$

$$\hat{H}^{\hat{\dagger}} \equiv \hat{H}^{\dagger} = \hat{H}. \tag{3.4.9b}$$

As a result, *all quantities that are observable for quantum mechanics remain so for hadronic mechanics.*

For consistency, the conventional eigenvalue equation $H \times |\psi\rangle = E \times |\psi\rangle$ must also be lifted into the *isoeigenvalue form* [7]

$$\hat{H}\hat{\times}|\hat{\psi}\rangle = \hat{H} \times \hat{T} \times |\hat{\psi}\rangle = \hat{E}\hat{\times}|\hat{\psi}\rangle = (E \times \hat{I}) \times \hat{T} \times |\hat{\psi}\rangle = E \times |\hat{\psi}\rangle, \tag{3.4.10}$$

where, as one can see, the final results are ordinary numbers.

Note the *necessity* of the isotopic action $\hat{H}\hat{\times}|\hat{\psi}\rangle$, rather than $\hat{H} \times |\hat{\psi}\rangle$. In fact, only the former admits $\hat{I}$ as the correct unit,

$$\hat{I}\hat{\times}|\hat{\psi}\rangle = \hat{T}^{-1} \times \hat{T} \times |\hat{\psi}\rangle \equiv |\hat{\psi}\rangle. \tag{3.4.11}$$

It is possible to prove that *the isoeigenvalues of isohermitian operators are isoreal*, i.e., they have the structure $\hat{E} = E \times \hat{I},\ E \in R(n, +, \times)$. As a result all real eigenvalues of quantum mechanics remain real for hadronic mechanics.

We also recall the notion of *isounitary operators* as the isooperators $\hat{U}$ on $\hat{\mathcal{H}}$ over $\hat{C}$ satisfying the isolaws

$$\hat{U}\hat{\times}\hat{U}^{\hat{\dagger}} = \hat{U}^{\hat{\dagger}}\hat{\times}\hat{U} = \hat{I}, \tag{3.4.12}$$

where we have used the identity $\hat{U}^{\hat{\dagger}} \equiv \hat{U}^{\dagger}$.

We finally indicate the notion of *isoexpectation value* of an isooperators $\hat{H}$ on $\hat{\mathcal{H}}$ over $\hat{C}$

$$\langle\hat{H}\rangle = \frac{\langle\hat{\psi}|\hat{\times}\hat{H}\hat{\times}|\hat{\psi}\rangle}{\langle\hat{\psi}|\hat{\times}|\hat{\psi}\rangle}. \tag{3.4.131}$$

It is easy to see that *the isoexpectation values of isohermitian operators coincide with the isoeigenvalues*, as in the conventional case.

Note also that *the isoexpectation value of the isounit is the isounit*,

$$\langle\hat{I}\rangle = \hat{I}, \tag{3.4.14}$$

provided, of course, that one uses the isoquotient (otherwise $\langle\hat{I}\rangle = I$).

The isotopies of quantum mechanics studied in the next sections are based on the following novel invariance property of the conventional Hilbert space [xxx], here expressed in term of a non-null scalar n independent from the integration variables,

$$\langle\hat{\phi}| \times |\hat{\psi}\rangle \times I \equiv \langle\hat{\phi}| \times n^{-2} \times |\hat{\psi}\rangle \times (n^2 \times I) = \langle\phi|\hat{\times}|\psi\rangle \times \hat{I}. \tag{3.4.15}$$

Note that new invariances (3.4.15) remained undetected throughout the 20-th century because they required the prior discovery of *new numbers*, those with arbitrary units.

3.4.4 Structure of Operator Isomechanics and its Isodual

The structure of operator isomechanics is essentially given by the following main steps [47]:

1) The description of closed-isolated systems is done via *two* quantities, the Hamiltonian representing all action-at-a-distance potential interactions, plus the isounit representing all nonlinear, nonlocal and non-Hamiltonian effects,

$$H(t, r, p) = p^2/2m + V(r), \tag{3.4.16a}$$

$$\hat{I} = \hat{I}(t, r, p, \psi, \nabla\psi, \ldots). \tag{3.4.16b}$$

The explicit form of the Hamiltonian is that conventionally used in quantum mechanics although written on isospaces over isofields,

$$\hat{H} = \hat{p}\hat{\times}\hat{p}\hat{/}\hat{2}\hat{\times}\hat{m} + \hat{V}(\hat{r}). \tag{3.4.17}$$

A generic expression of the isounit for the representation of two spinning particles with point-like change (such as the electrons) in conditions of deep penetration of their wavepackets (as occurring in chemical valence bonds and many other cases) is given by

$$\hat{I} = \exp\left[\Gamma(\psi, \psi^\dagger) \times \int dv\, \psi_\downarrow^\dagger(r)\psi_\uparrow(r)\right], \tag{3.4.18}$$

where the nonlinearity is expressed by $\Gamma(\psi, \psi^\dagger)$ and the nonlocality is expressed by the volume integral of the deep wave-overlappings $\int dv\, \psi_\downarrow^\dagger(r)\psi_\uparrow(r)$. All isounits will be restricted by the conditions of being positive-definite (thus everywhere invertible) as well as of recovering the trivial unit of quantum mechanics for sufficiently big mutual distances r,

$$\lim_{r\to\infty} \int dv\, \psi_\downarrow^\dagger(r)\psi_\uparrow(r) = 0. \tag{3.4.19}$$

2) The lifting of the multiplicative unit $I > 0 \to \hat{I} = 1/\hat{T} > 0$ requires the reconstruction of the entire formalism of quantum mechanics into such a form to

admit $\hat{I}$ as the correct left and right unit at all levels of study, including numbers and angles, conventional and special functions, differential and integral calculus, metric and Hilbert spaces, algebras and groups, etc., without any exception known to the authors. This reconstruction is "isotopic" in the sense of being axiom-preserving. Particularly important is the preservation of all conventional quantum laws as shown below.

3) The mathematical structure of nonrelativistic hadronic mechanics is characterized by [6]:

3a) The isofield $\hat{C} = \hat{C}(\hat{c}, +, \hat{\times})$ with *isounit* $\hat{I} = 1/\hat{T} > 0$, *isocomplex numbers* and related *isoproduct*

$$\hat{c} = c \times \hat{I} = (n_1 + i \times n_2) \times \hat{I}, \;\; \hat{c}\hat{\times}\hat{d} = (c \times d) \times \hat{I}, \;\; \hat{c}, \hat{d} \in \hat{\mathcal{C}}, \;\; c, d \in C, \quad (3.4.20)$$

the isofield $\hat{R}(\hat{n}, +, \hat{\times})$ of *isoreal numbers* $\hat{n} = n \times \hat{I}$, $n \in R$, being a particular case;

3b) The iso-Hilbert space $\hat{\mathcal{H}}$ with isostates $|\hat{\psi}\rangle, |\hat{\phi}\rangle, \ldots,$ *isoinner product* and *isonormalization*

$$\langle\hat{\phi}|\hat{\times}|\hat{\psi}\rangle \times \hat{I} \;\in \hat{S}, \quad \langle\hat{\psi}|\hat{\times}|\hat{\psi}\rangle = 1, \quad (3.4.21)$$

and related theory of isounitary operators;

3c) The Euclid-Santilli isospace $\hat{E}(\hat{r}, \hat{\delta}, \hat{\mathrm{R}})$ with *isocoordinates, isometric and isoinvariant* respectively given by

$$\hat{r} = \{r^k\} \times \hat{I}, \quad (3.4.22a)$$

$$\hat{\delta} = \hat{T}(t, r, p, \psi, \nabla\psi, \ldots) \times \delta, \quad (3.4.22b)$$

$$\delta = Diag.(1, 1, 1), \quad (3.4.22c)$$

$$\hat{r}^{\hat{2}} = (r^i \times \hat{\delta}_{ij} \times r^j) \times \hat{I} \in \hat{\mathrm{R}}; \quad (3.4.22d)$$

3d) The isodifferential calculus and the isofunctional analysis (see Section 3.2);

3e) The Lie-Santilli isotheory with enveloping isoassociative algebra $\hat{\xi}$ of operators $\hat{A}$, $\hat{B}$, ..., with isounit $\hat{I}$, isoassociative product $\hat{A}\hat{\times}\hat{B} = \hat{A} \times \hat{T} \times \hat{B}$, *Lie-Santilli isoalgebra with brackets and isoexponentiation*

$$[\hat{A}\hat{,}\hat{B}] = \hat{A}\hat{\times}\hat{B} - \hat{B}\hat{\times}\hat{A}, \quad (3.4.23a)$$

$$\hat{U} = \hat{e}^X = (e^{X \times \hat{T}}) \times \hat{I} = \hat{I} \times (e^{\hat{T} \times X}), \quad X = X^\dagger, \quad (3.4.13b)$$

and related isosymmetries characterizing groups of isounitary transforms on $\hat{\mathcal{H}}$ over $\hat{C}$,

$$\hat{U}\hat{\times}\hat{U}^\dagger = \hat{U}^\dagger\hat{\times}\hat{U} = \hat{I}. \quad (3.4.24)$$

As we shall see in Sections 3.4.8 and 3.4.9, the above entire mathematical structure can be achieved in a truly elementary way via nonunitary transforms of quantum formalisms. Their isotopic reformulations then proves the invariance of hadronic mechanics, namely, its capability of predicting the same numbers for the same conditions at different times.

Under the above outlined structure we have the following main features:

I) Hadronic mechanics is a covering of quantum mechanics, because the latter theory is admitted uniquely and unambiguously at the limit when the isounit recovers the conventional unit, $\hat{I} \to I$;

II) Said covering is further characterized by the fact that hadronic mechanics coincides with quantum mechanics everywhere except for (as we shall see, generally small) non-Hamiltonian corrections at short mutual distances of particles caused by deep mutual overlapping of the wavepackets and/or charge distributions of particles;

III) Said covering is finally characterized by the fact that the indicated non-Hamiltonian corrections are restricted to verify all abstract axioms of quantum mechanics, with consequential preservation of is basic laws for closed non-Hamiltonian systems as a whole, as we shall see shortly.

Note that *composite hadronic systems*, such as *hadrons, nuclei, isomolecules, etc.*, are represented via the tensorial product of the above structures. This can be best done via the identification first of the *total isounit, total isofields, total isohilbert spaces, etc.*,

$$\hat{I}_{\rm tot} = \hat{I}_1 \times \hat{I}_2 \times \ldots, \ \hat{C}_{\rm tot} = \hat{C}_1 \times \hat{C}_2 \times \ldots, \ \hat{\mathcal{H}}_{\rm tot} = \hat{\mathcal{H}}_1 \times \hat{\mathcal{H}}_2 \times \ldots . \quad (3.4.25)$$

Note also that some of the units, fields and Hilbert spaces in the above tensorial products can be *conventional*, namely, the composite structure may imply *local-potential long range interactions* (e.g., those of Coulomb type), which require the necessary treatment via *conventional* quantum mechanics, and *nonlocal-nonpotential short range interactions* (e.g., those in deep wave-overlappings), which require the use of operator isomechanics.

3.4.5 Dynamical Equations of Operator Isomechanics and their Isoduals

The formulations of the preceding sections permit the identification of the following fundamental dynamical equations of the Lie-isotopic branch of hadronic mechanics, known under the name of *iso-Heisenberg equations* or *Heisenberg-Santilli isoequetions*that were identified in the original proposal of 1978 to build hadronic mechanics [5], are can be presented in their finite and infinitesimal forms,

$$\hat{A}(\hat{t}) = \hat{U} \hat{\times} \hat{A}(\hat{0}) \hat{\times} \hat{U}^{\dagger} = \{\hat{e}^{\hat{i} \hat{\times} \hat{H} \hat{\times} \hat{t}}\} \hat{\times} \hat{A}(\hat{0}) \hat{\times} \{\hat{e}^{-\hat{i} \hat{\times} \hat{t} \hat{\times} \hat{H}}\}, \quad (3.4.26a)$$

$$\hat{i} \hat{\times} \hat{d}\hat{A} \hat{/} \hat{d}\hat{t} = [\hat{A} \hat{,} \hat{H}] = A \hat{\times} \hat{H} - \hat{H} \hat{\times} \hat{A} = \hat{A} \times \hat{T} \times \hat{H} - \hat{H} \times \hat{T} \times \hat{A}, \quad (3.4.26b)$$

with the corresponding *fundamental hadronic isocommutation rules*

$$[\hat{b}^{\mu} \hat{,} \hat{b}^{\nu}] = \hat{i} \hat{\times} \hat{\omega}^{\mu\nu} = i \times \omega^{\mu\nu} \times \hat{I}_6, \quad \hat{b} = (\hat{r}^k, \hat{p}_k), \quad (3.4.27)$$

with corresponding *iso-Schrödinger equations* for the energy, also known as *Schrödinger-Santilli isoequations* identified by Myung and Santilli [45] and Mignani [48] in 1982 over conventional fields and first formulated in an invariant way by Santilli in monograph [7] of 1995

$$\hat{i}\hat{\times}\hat{\partial}_{\hat{t}}|\hat{\psi}\rangle = \hat{H}\hat{\times}|\hat{\psi}\rangle = \hat{H} \times \hat{T} \times |\hat{\psi}\rangle = \hat{E}\hat{\times}|\hat{\psi}\rangle = E \times |\hat{\psi}\rangle, \tag{3.4.28a}$$

$$|\hat{\psi}(\hat{t})\rangle = \hat{U}\hat{\times}|\hat{\psi}(\hat{0})\rangle = \{\hat{e}^{i\hat{H}\hat{\times}\hat{t}}\}\hat{\times}|\hat{\psi}(\hat{0})\rangle, \tag{3.4.28b}$$

and *isolinear momentum* first identified by Santilli in Ref. [7] of 1995 thanks to the discovery of the isodifferential calculus

$$\hat{p}_k\hat{\times}|\hat{\psi}\rangle = \hat{p}_k \times \hat{T} \times |\hat{\psi}\rangle - \hat{i}\hat{\times}\hat{\partial}_k|\hat{\psi}\rangle = -i \times \hat{I}^i_k \times \partial_i|\hat{\psi}\rangle, \tag{3.4.29}$$

It is evident that the iso-Heisenberg equations in their infinitesimal and exponentiated forms are a realization of the Lie-Santilli isotheory of Section 3.2, which is therefore the algebraic and group theoretical structure of the isotopic branch of hadronic mechanics.

Note that Eqs. (3.4.26) and (3.4.28) automatically bring into focus the general need for a *time isounit* and related characterization of the time isodifferential and isoderivative

$$\hat{I}_t(t, r, \psi, \ldots) = \hat{T}_t > 0, \tag{3.4.30a}$$

$$\hat{d}\hat{t} = \hat{I}_t \times d\hat{t}, \quad \hat{\partial}_{\hat{t}} = \hat{I}_t \times \partial_t. \tag{3.4.30b}$$

Note also that $\omega^{\mu\nu}$ in Eqs. (3.4.xxx) is the *conventional* Lie tensor, namely, the same tensor appearing in the conventional canonical commutation rules, thus confirming the axiom-preserving character of isomechanics.

The limited descriptive capabilities of quantum models should be kept in mind, purely Hamiltonian and, as such, they can only represent systems which are linear, local and potential. By comparison, we can write Eq. (3.4.28a) in its explicit form

$$\begin{aligned} \hat{i}\hat{\times}\hat{\partial}_{\hat{t}}\hat{\psi} = i \times \hat{I}_t \times \partial_{\hat{t}}|\hat{\psi}\rangle = \hat{H}\hat{\times}|\hat{\psi}\rangle = \hat{H} \times \hat{T} \times |\hat{\psi}\rangle = \\ = \{\hat{p}_k \times \hat{p}_k/\hat{2}\hat{\times}\hat{m} + \hat{U}_k(\hat{t}, \hat{r})\hat{\times}\hat{v}^k + \\ +\hat{U}_0(\hat{t}, \hat{r})\} \times \hat{T}(\hat{t}, \hat{r}, \hat{p}, \hat{\psi}, \nabla\psi, \ldots) \times |\hat{\psi}(\hat{t}, \hat{r})\rangle = \\ = \hat{E}\hat{\times}|\hat{\psi}(t, \hat{x})\rangle = E \times |\hat{\psi}(\hat{t}, \hat{x})\rangle, \end{aligned} \tag{3.4.31}$$

thus proving the following

THEOREM 3.4.1 [7]: Hadronic mechanics id "directly universal" for all infinitely possible, sufficiently smooth and regular, closed non-Hamiltonian systems, *namely, it can represent in the fixed coordinates of the experimenter all infinitely possible closed-isolated systems with linear and nonlinear, local and nonlocal, and potential as well as nonpotential internal forces verifying the conservation of the total energy.*

A consistent formulation of the isolinear momentum (3.4.29) escaped identification for two decades, thus delaying the completion of the construction of hadronic mechanics, as well as its practical applications. The consistent and invariant form (3.4.29) with consequential isocanonical commutation rules were first identified by Santilli in the second edition of Vol. II of this series, Ref. [7] of 1995 and memoir [10], following the discovery of the isodifferential calculus.

3.4.6 Preservation of Quantum Physical Laws

As one can see, the fundamental assumption of isoquantization is the lifting of the basic unit of quantum mechanics, Planck's constant $\hbar$, into a matrix $\hat{I}$ with nonlinear, integro-differential elements which also depend on the wavefunction and its derivatives

$$\hbar = I > 0 \rightarrow \hat{I} = \hat{I}(t, r, p, \psi, \hat{\psi}, \ldots) = \hat{I}^\dagger > 0. \tag{3.4.32}$$

It should be indicated that the above generalization is only *internal* in closed non-Hamiltonian because, when measured from the outside, *the isoexpectation values and isoeigenvalues of the isounit recover Planck's constant identically* [46],

$$\langle\hat{I}\rangle = \frac{\langle\hat{\psi}|\hat{\times}\hat{I}\hat{\times}|\hat{\psi}\rangle}{\langle\hat{\psi}|\hat{\times}|\hat{\psi}\rangle} = 1 = \hbar, \tag{3.4.33a}$$

$$\hat{I}\hat{\times}|\hat{\psi}\rangle = \hat{T}^{-1} \times \hat{T} \times |\hat{\psi}\rangle = 1 \times |\hat{\psi}\rangle = |\hat{\psi}\rangle. \tag{3.4.33b}$$

Moreover, the isounit is the fundamental invariant of isomechanics, thus preserving all axioms of the conventional unit $I = \hbar$, e.g.,

$$\hat{I}^{\hat{n}} = \hat{I}\hat{\times}\hat{I}\hat{\times}\ldots\hat{\times}\hat{I} \equiv \hat{I}, \tag{3.4.34a}$$

$$\hat{I}^{\frac{1}{2}} \equiv \hat{I}, \tag{3.4.34b}$$

$$\hat{i}\hat{\times}\hat{d}\hat{I}\hat{/}\hat{d}t = [\hat{I}\hat{,}\hat{H}] = \hat{I}\hat{\times}\hat{H} - \hat{H}\hat{\times}\hat{I} \equiv 0. \tag{3.4.34c}$$

Despite their generalized structure, *Eqs. (3.4.26) and (3.4.28) preserve conventional quantum mechanical laws under nonlinear, nonlocal and nonpotential interactions* [7].

To begin an outline, the preservation of Heisenberg's uncertainties can be easily derived from isocommutation rules (3.4.27):

$$\Delta x^k \times \Delta p_k \geq \frac{1}{2} \times \langle[\hat{x}^k\hat{,}\hat{p}_k]\rangle = \frac{1}{2}. \tag{3.4.35}$$

To see the preservation of Pauli's exclusion principle, recall that the regular (two-dimensional) representation of $SU(2)$ is characterized by the conventional *Pauli matrices* σ_k with familiar commutation rules and eigenvalues on $\mathcal{H}$ over C,

$$[\sigma_i, \sigma_j] = \sigma_i \times \sigma_j - \sigma_j \times \sigma_i = 2 \times i\varepsilon_{ijk} \times \sigma_k, \tag{3.4.36a}$$

$$\sigma^2 \times |\psi\rangle = \sigma_k \times \sigma^k \times |\psi\rangle = 3 \times |\psi\rangle, \tag{3.4.36b}$$

$$\sigma_3 \times |\psi\rangle = \pm 1 \times |\psi\rangle. \tag{3.4.36c}$$

The isotopic branch of hadronic mechanics requires the construction of *nonunitary images of Pauli's matrices* first constructed in Ref. [49] that, for diagonal nonunitary transforms and isounits, can be written (see also Section 3.3.6)

$$\hat{\sigma}_k = U \times \sigma_k \times U^\dagger, \quad U \times U^\dagger = \hat{I} \neq I, \tag{3.4.37a}$$

$$\begin{gathered} U = \begin{pmatrix} i\times n_1 & 0 \\ 0 & i\times n_2 \end{pmatrix}, \quad U^\dagger = \begin{pmatrix} -i\times n_1 & 0 \\ 0 & -i\times n_2 \end{pmatrix}, \\ \hat{I} = \begin{pmatrix} n_1^2 & 0 \\ 0 & n_2^2 \end{pmatrix}, \quad \hat{T} = \begin{pmatrix} n_1^{-2} & 0 \\ 0 & n_2^{-2} \end{pmatrix}, \end{gathered} \tag{3.4.37b}$$

where the n's are well behaved nowhere null functions, resulting in the *regular Pauli-Santilli isomatrices* [49]

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & n_1^2 \\ n_2^2 & 0 \end{pmatrix}, \ \hat{\sigma}_2 = \begin{pmatrix} 0 & -i\times n_1^2 \\ i\times n_2^2 & 0 \end{pmatrix}, \ \hat{\sigma}_3 = \begin{pmatrix} n_1^2 & 0 \\ 0 & n_2^2 \end{pmatrix}. \tag{3.4.38}$$

Another realization is given by *nondiagonal unitary transforms* [*loc. cit.*],

$$\begin{gathered} U = \begin{pmatrix} 0 & n_1 \\ n_2 & 0 \end{pmatrix}, \quad U^\dagger = \begin{pmatrix} 0 & n_2 \\ n_1 & 0 \end{pmatrix}, \\ \hat{I} = \begin{pmatrix} n_1^2 & 0 \\ 0 & n_2^2 \end{pmatrix}, \quad \hat{T} = \begin{pmatrix} n_1^{-2} & 0 \\ 0 & n_2^{-2} \end{pmatrix}, \end{gathered} \tag{3.4.39}$$

with corresponding *regular Pauli-Santilli isomatrices*,

$$\begin{gathered} \hat{\sigma}_1 = \begin{pmatrix} 0 & n_1\times n_2 \\ n_1\times n_2 & 0 \end{pmatrix}, \ \hat{\sigma}_2 = \begin{pmatrix} 0 & -i\times n_1\times n_2 \\ i\times n_1\times n_2 & 0 \end{pmatrix}, \\ \hat{\sigma}_3 = \begin{pmatrix} n_1^2 & 0 \\ 0 & n_2^2 \end{pmatrix}, \end{gathered} \tag{3.4.40}$$

or by more general realizations with Hermitian nondiagonal isounits $\hat{I}$ [15].

All Pauli-Santilli isomatrices of the above regular class verify the following isocommutation rules and isoeigenvalue equations on $\hat{\mathcal{H}}$ over $\hat{\mathcal{C}}$

$$[\hat{\sigma}_i \hat{,} \hat{\sigma}_j] = \hat{\sigma}_i \times \hat{T} \times \hat{\sigma}_j - \hat{\sigma}_j \times \hat{T} \times \hat{\sigma}_i = 2 \times i \times \varepsilon_{ijk} \times \hat{\sigma}_k, \tag{3.4.41a}$$

$$\hat{\sigma}^{\hat{2}} \hat{\times} |\hat{\psi}\rangle = (\hat{\sigma}_1 \hat{\times} \hat{\sigma}_1 + \hat{\sigma}_2 \hat{\times} \hat{\sigma}_2 + \hat{\sigma}_3 \hat{\times} \hat{\sigma}_3) \hat{\times} |\hat{\psi}\rangle = 3 \times |\hat{\psi}\rangle, \qquad (3.4.41b)$$

$$\hat{\sigma}_3 \hat{\times} |\hat{\psi}\rangle = \pm 1 \times |\hat{\psi}\rangle, \qquad (3.4.41c)$$

thus preserving conventional spin 1/2, and establishing the preservation in isochemistry of the Fermi-Dirac statistics and Pauli's exclusion principle.

It should be indicated for completeness that the representation of the isotopic $S\hat{U}(2)$ also admit *irregular isorepresentations*, that no longer preserve conventional values of spin [49]. The latter structures are under study for the characterization of spin under the most extreme conditions, such as for protons and electrons in the core of collapsing stars and, as such, they have no known relevance for isomechanics.

The preservation of the superposition principle under nonlinear interactions occurs because of the reconstruction of linearity on isospace over isofields, thus regaining the applicability of the theory to composite systems.

Recall in this latter respect that conventionally nonlinear models,

$$H(t, x, p, \psi, \ldots) \times |\psi\rangle = E \times |\psi\rangle, \qquad (3.4.42)$$

violate the superposition principle and have other shortcomings (see Section 1.5). As such, they cannot be applied to the study of composite systems such as molecules. All these models can be *identically* reformulated in terms of the isotopic techniques via the embedding of all nonlinear terms in the isotopic element,

$$H(t, x, p, \psi, \ldots) \times |\psi\rangle \equiv H_0(t, x, p) \times \hat{T}(\psi, \ldots) \times |\psi\rangle = E \times |\psi\rangle, \qquad (3.4.43)$$

by regaining the full validity of the superposition principle in isospaces over isofields with consequential applicability to composite systems.

The preservation of causality follows from the one-dimensional isounitary group structure of the time evolution (3.4.28) (which is isomorphic to the conventional one); the preservation of probability laws follows from the preservation of the axioms of the unit and its invariant decomposition as indicated earlier; the preservation of other quantum laws then follows.

The same results can be also seen from the fact that operator isomechanics coincides at the abstract level with quantum mechanics by conception and construction. As a result, hadronic and quantum versions are *different realizations of the same abstract axioms and physical laws*.

Note that the preservation of conventional quantum laws under nonlinear, nonlocal and nonpotential interactions is crucially dependent on the capability of isomathematics to reconstruct linearity, locality and canonicity-unitarity on isospaces over isofields.

The preservation of conventional physical laws by the isotopic branch of hadronic mechanics was first identified by Santilli in report [47]. It should be indicated that the same quantum laws *are not* generally preserved by the broader genomechanics, evidently because the latter must represent by assumption *non*-conservation laws and other *departures* from conventional quantum settings.

With the understanding that the theory does not receive the classical determinism, it is evident that isomechanics provides a variety of "completions" of quantum mechanics according to the celebrated E-P-R argument [50], such as:

1) Isomechanics "completes" quantum mechanics via the addition of nonpotential-nonhamiltonian interactions represented by nonunitary transforms.

2) Isomechanics "completes" quantum mechanics via the broadest possible (non-oriented) realization of the associative product into the isoassociative form.

3) Isomechanics "completes" quantum mechanics in its classical image.

In fact, as proved by well known procedures based on *Bell's inequalities*, quantum mechanics does not admit direct classical images on a number of counts. On the contrary, as studied in details in Refs. [51], the nonunitary images of Bell's inequalities permit indeed direct and meaningful classical limits which do not exist for the conventional formulations.

Similarly, it is evident that isomechanics constitutes a specific and concrete realization of "hidden variables" [52] λ which are explicitly realized by the isotopic element, $\lambda = \hat{T}$, and actually turned into an operator hidden variables. The "hidden" character of the realization is expressed by the fact that hidden variables are embedded in the unit and product of the theory.

In fact, we can write the iso-Schrödinger equation $\hat{H} \hat{\times} |\hat{\psi}\rangle = \hat{H} \times \lambda \times |\hat{\psi}\rangle = E \times |\hat{\psi}\rangle$, $\lambda = \hat{T}$. As a result, the "variable" λ (now generalized into the operator $\hat{T}$) is "hidden" in the modular associative product of the Hamiltonian $\hat{H}$ and the state $|\hat{\psi}\rangle$.

Alternatively, we can say that hadronic mechanics provides an explicit and concrete realization of hidden variables because all distinctions between $\hat{H} \hat{\times} |\hat{\psi}\rangle$ and $H \times |\psi\rangle$ cease to exist at the abstract realization-free level.

For studies on the above and related issues, we refer the interested reader to Refs. [51] and quoted literature.

3.4.7 Isoperturbation Theory and its Isodual

We are now sufficiently equipped to illustrate the computational advantages in the use of isotopies.

THEOREM 3.4.2 [7]: Under sufficient continuity conditions, all perturbative and other series that are conventionally divergent (weakly convergent) can be turned into convergent (strongly convergent) forms via the use of isotopies with sufficiently small isotopic element (sufficiently large isounit),

$$|\hat{T}| \ll 1, \quad |\hat{I}| \gg 1. \tag{3.4.44}$$

The emerging perturbation theory was first studied by Jannussis and Mignani [53], and then studied in more detail in monograph [7] under the name of *isoperturbation theory*.

Consider a Hermitian operator on $\mathcal{H}$ over C of the type

$$H(k) = H_0 + k \times V, \quad H_0 \times |\psi\rangle = E_0 \times |\psi\rangle, \tag{3.4.45a}$$

$$H(k) \times |\psi(k)\rangle = E(k) \times |\psi(k)\rangle, \quad k \gg 1. \tag{3.4.45b}$$

Assume that H_0 has a nondegenerate discrete spectrum. Then, conventional perturbative series are *divergent*, as well known. In fact, the eigenvalue $E(k)$ of $H(k)$ up to second order is given by

$$\begin{gathered} E(k) = E_0 + k \times E_1 + k^2 \times E_2 = \\ = E_0 + k \times \langle\psi| \times V \times |\psi\rangle + k^2 \times \sum_{p \neq n} \frac{|\langle\psi_p| \times V \times |\psi_n\rangle|^2}{E_{0n} - E_{0p}}. \end{gathered} \tag{3.4.46}$$

But under isotopies we have

$$H(k) = H_0 + k \times V, \quad H_0 \times \hat{T} \times |\tilde{\psi}\rangle = \tilde{E}_0 \times |\tilde{\psi}\rangle, \quad \tilde{E}_0 \neq E_0, \tag{3.4.47a}$$

$$H(k) \times \hat{T} \times |\hat{\psi}(k)\rangle = \tilde{E}(k) \times |\hat{\psi}(k)\rangle, \quad \tilde{E} \neq E, \quad k > 1. \tag{3.4.47b}$$

A simple lifting of the conventional perturbation expansion then yields

$$\tilde{E}(k) = \tilde{E}_0 + k \times \tilde{E}_1 + k^2 \times \tilde{E}_2 + \hat{O}(k^2) =$$

$$= \tilde{E}_0 + k \times \langle\tilde{\psi}| \times \hat{T} \times V \times \hat{T} \times |\tilde{\psi}\rangle + \tag{3.4.48a}$$

$$+k^2 \times \sum_{p \neq n} \frac{|\langle\hat{\psi}_p| \times \hat{T} \times V \times \hat{T} \times |\hat{\psi}_n\rangle|^{\hat{2}}}{\tilde{E}_{0n} - \tilde{E}_{0p}}, \tag{3.4.48b}$$

whose convergence can be evidently reached via a suitable selection of the isotopic element, e.g., such that $|\hat{T}| \ll k$.

As an example, for a positive-definite constant $\hat{T} \ll k^{-1}$, expression (3.4.46) becomes

$$\begin{gathered} \tilde{E}(k) = \tilde{E}_0 + k \times \hat{T}^2 \times \langle\hat{\psi}| \times V \times |\psi *\rangle + k^2 \times T^5 \times \\ \times \sum_{p \neq n} \frac{|\langle\psi_p| \times V \times |\psi_n\rangle|^2}{\tilde{E}_{0n} - \tilde{E}_{0p}}. \end{gathered} \tag{3.4.49}$$

This shows that the original divergent coefficients $1, k, k^2, \ldots$ are now turned into the manifestly convergent coefficients $1, k \times T^2, k^2 \times T^5, \ldots$, with $k > 1$ and $\hat{T} \ll 1/k$, thus ensuring isoconvergence for a suitable selection of $\hat{T}$ for each given k and V.

A more effective reconstruction of convergence can be seen in the algebraic approach. At this introductory stage, we consider a divergent canonical series,

$$A(k) = A(0) + k \times [A, H]/1! + +k^2 \times [[A, H], H]/2! + \ldots \to \infty, \quad k > 1, \quad (3.4.50)$$

where $[A, H] = A \times H - H \times A$ is the familiar Lie product, and the operators A and H are Hermitian and sufficiently bounded. Then, under the isotopic lifting the preceding series becomes [7]

$$\hat{A}(k) = \hat{A}(0) + k \times [A \hat{,} H]/1! + k^2 \times [[A \hat{,} H] \hat{,} H]/2! + \cdots \leq |N| < \infty, \quad (3.4.51a)$$

$$[A \hat{,} H] = A \times \hat{T} \times H - H \times \hat{T} \times A, \quad (3.4.51b)$$

which holds, e.g., for the case $T = \varepsilon \times k^{-1}$, where ε is a sufficiently small positive-definite constant.

In summary, the studies on the construction of hadronic mechanics have indicated that the apparent origin of divergences (or slow convergence) in quantum mechanics and chemistry is their lack of representation of nonlinear, nonlocal, and nonpotential effects because when the latter are represented via the isounit, full convergence (much faster convergence) can be obtained.

As we shall see, all known applications of hadronic mechanics verify the crucial condition $|\hat{I}| \gg 1$, $|\hat{T}| \ll 1$, by permitting convergence of perturbative series. For instance, in the case of chemical bonds, hadronic chemistry allows computations at least one thousand times faster than those of quantum chemistry, with evident advantages, e.g., a drastic reduction of computer time (see Chapter 9). Essentially the same results are expected for hadronic mechanics and hadronic superconductivity.

The reader should meditate a moment on the evident possibility that *hadronic mechanics offers realistic possibilities of constructing a convergent perturbative theory for strong interactions.* As a matter of fact, the divergencies that have afflicted strong interactions through the 20-th century originates precisely from the excessive approximation of hadrons as points, with the consequential sole potential interactions and related divergencies.

In fact, whenever hadrons are represented as they actually are in reality, extended and hyperdense particles, with consequential potential as well as nonpotential interactions, all divergencies are removed by the isounit.

3.4.8 Simple Construction of Operator Isomechanics and its Isodual

Despite their *mathematical equivalence*, it should be indicated that quantum and hadronic mechanics are *physically inequivalent*, or, alternatively, hadronic mechanics is outside the classes of equivalence of quantum mechanics because the former is a *nonunitary image* of the latter.

As we shall see in the next chapters, the above property provides means for the explicit construction of the new model of isomechanics bonds from the conventional model. The main requirement is that of identifying the *nonhamiltonian* effects one desires to represent, which as such, are necessarily *nonunitary*. The resulting nonunitary transform is then assumed as the fundamental space isounit of the new isomechanics [46]

$$U \times U^\dagger = \hat{I} \neq I, \tag{3.4.52}$$

under which transform we have the liftings of: the quantum unit into the isounit,

$$I \rightarrow \hat{I} = U \times I \times U^\dagger; \tag{3.4.53}$$

numbers into isonumbers,

$$a \rightarrow \hat{a} = U \times a \times U^\dagger = a \times (U \times U^\dagger) = a \times \hat{I}; \quad a = n, c; \tag{3.4.54}$$

associative products $A \times B$ into the isoassociative form with the correct isotopic element,

$$A \times B \rightarrow \hat{A} \hat{\times} \hat{B} = \hat{A} \times \hat{T} \times \hat{B}, \tag{3.4.55a}$$

$$\hat{A} = U \times A \times U^\dagger, \ \hat{B} = U \times B \times U^\dagger, \ \hat{T} = (U \times U^\dagger)^{-1} = T^\dagger; \tag{3.4.55b}$$

Schrödinger's equation into the isoschrödinger's equations

$$\begin{aligned} H \times |\psi\rangle = E \times |\psi\rangle \rightarrow U(H \times |\psi\rangle) = \\ = (U \times H \times U^\dagger) \times (U \times U^\dagger)^{-1} \times (U \times |\psi\rangle) = \\ = \hat{H} \times \hat{T} \times |\hat{\psi}\rangle = \hat{H} \hat{\times} |\hat{\psi}\rangle; \end{aligned} \tag{3.4.56}$$

Heisenberg's equations into their isoheisenberg generalization

$$\begin{aligned} i \times dA/dt - A \times H - H \times A = 0 \rightarrow \\ \rightarrow U \times (i \times dA/dt) \times U^\dagger - U(A \times H - H \times A) \times U^\dagger = \\ = \hat{i} \hat{\times} d\hat{A}/dt - \hat{A} \hat{\times} \hat{H} - \hat{H} \hat{\times} \hat{A} = 0; \end{aligned} \tag{3.4.57}$$

the Hilbert product into its isoinner form

$$\begin{aligned} \langle\psi| \times |\psi\rangle \rightarrow U \times \langle\psi| \times |\psi\rangle \times U^\dagger = \\ = (\langle\psi| \times U^\dagger) \times (U \times U)^{-1} \times (U \times |\psi\rangle) \times (U \times U)^{-1} = \langle\hat{\psi}| \hat{\times} |\hat{\psi}\rangle \times \hat{I}; \end{aligned} \tag{3.4.58}$$

canonical power series expansions into their isotopic form

$$\begin{aligned} A(k) = A(0) + k \times [A, H] + k^2 \times [[A, H,], H] + \ldots \rightarrow U \times A(k) \times U^\dagger = \\ = U \times \Big[A(0) + k \times [A, H] + k^2 \times [[A, H], H] + \ldots\Big] \times U^\dagger = \\ = \hat{A}(\hat{k}) = \hat{A}(0) + \hat{k} \hat{\times} [\hat{A} \hat{,} \hat{H}] + \hat{k}^{\hat{2}} \hat{\times} [[\hat{A} \hat{,} \hat{H}] \hat{,} \hat{H}] + \ldots, \\ k > 1, \quad |\hat{T}| \ll 1; \end{aligned} \tag{3.4.59}$$

Schrödinger's perturbation expansion into its isotopic covering (where the usual summation over states $p \neq n$ is assumed)

$$\begin{aligned}
&E(k) = E(0) + k \times \langle\psi| \times V \times |\psi\rangle + k^2 \frac{|\langle\psi|\times V\times|\psi\rangle|^2}{E_{0n} - E_{0p}} + \ldots \rightarrow \\
&\rightarrow U\times E(k)\times U^\dagger = U\times\Big[E(0) + k\times\langle\psi|\times V\times|\psi\rangle + \ldots\Big]\times U^\dagger = \\
&\quad = \hat{E}(\hat{k}) = \hat{E}(0) + \hat{k}\hat{\times}\langle\hat{\psi}| \times \hat{T} \times \hat{V} \times \hat{T} \times |\hat{\psi}\rangle + \ldots, \\
&\qquad k > 1, \quad |\hat{T}| \ll 1;
\end{aligned} \tag{3.4.60}$$

etc. All remaining aspects of operator isomechanics can then be derived accordingly, including the isoexponent, isologarithm, isodeterminant, isotrace, isospecial functions and transforms, *etc.* The isodual isomechanics can then be constructed via the now famil,kiar isodual map.

Note that the above construction via a nonunitary transform is the correct operator image of the derivability of the classical isohamiltonian mechanics from the conventional form via noncanonical transforms (Section 3.2.12).

The construction of hadronic mechanics via nonunitary transforms of quantum mechanics was first identified by Santilli in the original proposal [5e], and then worked out in subsequent contributions (see [12] for the latest presentation).

3.4.9 Invariance of Operator Isomechanics and of its Isodual

It is important to see that, in a way fully parallel to the classical case (Section 3.3.7), operator isomechanics is indeed invariant under the most general possible nonlinear, nonlocal and nonhamiltonian-nonunitary transforms, provided that, again, the invariance is treated via the isomathematics. In fact, any given nonunitary transform $U \times U^\dagger \neq I$ can always be decomposed into the form [12]

$$U = \hat{U} \times \hat{T}^{1/2},$$

under which nonunitary transforms on $\mathcal{H}$ over C are identically reformulated as isounitary transforms on the isohilbert space $\hat{\mathcal{H}}$ over the isofield $\hat{}$,

$$U \times U^\dagger \equiv \hat{U}\hat{\times}\hat{U}^\dagger = \hat{U}^\dagger\hat{\times}\hat{U} = \hat{I}. \tag{3.4.61}$$

The form-invariance of operator isomechanics under isounitary transforms then follows,

$$\hat{I} \rightarrow \hat{I}' = \hat{U}\hat{\times}\hat{I}\hat{\times}\hat{U}^\dagger \equiv \hat{I},\ \hat{A}\hat{\times}\hat{B} \rightarrow \hat{U}\hat{\times}(\hat{A}\hat{\times}\hat{B})\hat{\times}\hat{U}^\dagger = \hat{A}'\hat{\times}\hat{B}',\ \textit{etc.}, \tag{3.4.62a}$$

$$\begin{aligned}
&\hat{H}\hat{\times}|\hat{\psi}\rangle = \hat{E}\hat{\times}|\hat{\psi}\rangle \rightarrow \hat{U} \times \hat{H}\hat{\times}|\hat{\psi}\rangle = \\
&= (\hat{U} \times \hat{H} \times \hat{U}^\dagger)\hat{\times}(\hat{U}\hat{\times}|\hat{\psi}\rangle) = \hat{H}'\hat{\times}|\hat{\psi}'\rangle = \\
&= \hat{U}\hat{\times}\hat{E}\hat{\times}|\hat{\psi}\rangle = \hat{E}\hat{\times}\hat{U}\hat{\times}|\hat{\psi}\rangle = \hat{E}\hat{\times}|\hat{\psi}'\rangle,
\end{aligned} \tag{3.4.62b}$$

where one should note the preservation of the *numerical values* of the isounit, isoproducts and isoeigenvalues, as necessary for consistent applications. The invariance of isodual isomechanics then follows rather trivially.

Note that the invariance in quantum mechanics holds only for transformations $U \times U^{\dagger} = I$ *with fixed* I. Similarly, the invariance of isomechanics holds only for all nonunitary transforms such $\hat{U} \hat{\times} \hat{U}^{\hat{\dagger}} = \hat{I}$ *with fixed* $\hat{I}$, and *not* for a transform $\hat{W} \hat{\times} \hat{W}^{\dagger} = \hat{I}' \neq \hat{I}$ because the change of the isounit $\hat{I}$ implies the transition to a *different physical system*.

The form-invariance of hadronic mechanics under isounitary transforms was first studied by Santilli in memoir [46].

3.5 SANTILLI ISORELATIVITY AND ITS ISODUAL

3.5.1 Limitations of Special and General Relativities

Special and general relativities are generally presented in contemporary academia as providing final descriptions of all infinitely possible conditions existing in the universe.

The scientific reality is basically different than the above academic posture. In Section 1.1 and Chapter 2, we have shown that special and general relativities *cannot* provide a consistent classical description of antiparticles because they admit no distinction between neutral matter and antimatter and, when used for charged antiparticles, they lead to inconsistent quantum images consisting of particles (rather than charge conjugated antiparticles) with the wrong sign of the charge. Hence, *the entire antimatter content of the universe cannot be credibly treated via special and/or general relativity.*[24]

A widespread academic posture, studiously conceived for adapting nature to preferred doctrines, is the belief that the university can be effectively reduced to point-particles solely under action-at-a-distance, potential interactions. This posture is dictated by the facts that: the mathematics underlying special and general relativities, beginning with their local-differential topology, can only represent (dimensionless) point-like particles; special and genera;l relativity are notoriously incompatible with the deformation theory (that is activated whenever extended particles are admitted); and said relativities are strictly Lagrangian or Hamiltonian, thus being only able to represent potential interactions.

However, in Section 1.3 and in this chapter, we have established the "No Reduction Theorems," according to which a macroscopic extended system in nonconservative conditions (such as a satellite during re-entry in our atmosphere) cannot

[24]Particularly political is the academic posture that "antigravity does not exist because not predicted by Einstein's gravitation," when such a gravitational theory has no means for a credible representation of antimatter. As we shall see in Chapter 14, Volume II, when a credible quantitative representation of antimatter is included, antigravity (defined as gravitational repulsion) between matter and antimatter is unavoidable.

be consistently reduced to a finite number of point-particles all under potential forces and, vice versa, a finite number of quantum (that is, point-like) particles all under potential interactions cannot consistently recover a macroscopic nonconservative system. Hence, *all macroscopic systems under nonconservative forces, thus including all classical interior problems, cannot be consistently treated with special or general relativity.*[25]

Another posture in academia, also intended for adapting nature to a preferred doctrine, is that irreversibility is a macroscopic event that "disappears" (sic) when systems are reduced to their elementary constituents. This widespread academic belief is necessary because special and general relativities are *structurally reversible,* namely, their mathematical and physical axioms, as well as all known Hamiltonians are invariant under time reversal. This posture is complemented with manipulations of scientific evidence, such as the presentation of the probability of the synthesis of two nuclei into a third one, $n_1 + n_2 \rightarrow n_3$ while studiously suppressing the time reversal event that is simply unavoidable for a reversible theory, namely, the finite probability of the *spontaneous* decomposition $n_3 \rightarrow n_1 n_2$ following the synthesis. The latter probability is suppressed evidently because it would prove the inconsistency of the assumed basic doctrine. [26]

Unfortunately for mankind, the above academic postures are also used for all energy releasing processes despite the fact that they are irreversible. The vast majority of the research on energies releasing processes such as the "cold" and "hot" fusions, and the use of the vast majority of public fund,s are restricted to verify quantum mechanics and special relativity under the knowledge by experts that reversible theories cannot be exactly valid for irreversible processes/ In any case, the "No reduction theorems" prevent the consistent reduction of an irreversible macroscopic event to an ideal ensemble of point-like abstractions of

[25]Another political posture in academia is the treatment of the entire universe, thus including interior problems of stars, quasars and black-hole, with Einstein gravitation when it is well known that such a doctrine is purely "external," namely, can only represent point-like masses moving in vacuum in the gravitational field of a massive body. One can then understand the political backing needed for the credibility, e.g., of studies on black holes derived via a purely exterior theory.

[26]Serious physicists should not even redo the calculations for the probability of the spontaneous decay following the synthesis, because it is unavoidable under the assumption of the same Hilbert space for all initial and final nuclei and Heisenberg's uncertainty principle. In fact these assumptions imply that the nucleus n_1 or n_2 has a finite [probability of being outside of n_3 due to the coherence of the interior and interior Hilbert spaces. At this point, numerous additional manipulations of science are attempted to salvage preferred doctrines when inapplicable, rather than admitting their inapplicability and seeking covering theories. One of these manipulations is based on the "argument" that n_3 is extended, when extended sizes cannot be represented by quantum mechanics. Other manipulations are not worth reporting here. *The only scientific case of a rigorously proved, identically null probability of spontaneous disintegrations of a stable nucleus following its synthesis occurs when the initial and final Hilbert spaces are incoherent. This mandates the use of the conventional Hilbert space (quantum mechanics) for the initial states and the use of an incoherent iso-Hilbert space (hadronic mechanics) for the final state. This is the only possibility known to this author following half a century of studies of the problem.*

particles all in reversible conditions. Hence, *special and general relativities are inapplicable for any and all irreversible processes existing in the universe.*[27]

When restricting the arena applicability to those of the original conception (propagation of point particles and electromagnetic waves in vacuum), special relativity remains afflicted by still unresolved basic problems, such as the possibility that the relativity verifying one-way experiments on the propagation of light could be Galilean, rather than Lorentzian; the known incompatibility of special relativity with space conceived as a universal medium; and other unsettled aspects. Independently from that, we have shown in Section 1.4 that general relativity has no case of unequivocal applicability for numerous reasons, such as: curvature cannot possibly represent the free fall of a body along a straight radial line; the "bending of light" is due to Newtonian gravitation (and if curvature is assumed one gets double the bending experimentally measured); gravitation is a noncanonical theory, thus suffering of the Theorems of Catastrophic Inconsistencies of Section 1.5; etc.

In summary, on serious scientific grounds, and contrary to vastly popular political beliefs, special and general relativities have no uncontested arena of exact valid.

Far from pretending final knowledge, in this section we primarily claim the scientific honesty to have identified the above open problems and initiated quantitative studies for their resolution. Our position in regard to special relativity is pragmatic, in the sense that, under the conditions limpidly identified by Einstein, such as particles in accelerators, etc., special relativity works well. Additionally, special relativity has a majestic axiomatic structure emphasized various times by the author.

Hence, we shall assume special relativity at the foundation of this section and seek its *isotopic liftings,* namely, the most general possible formulations verifying at the abstract level the original axioms conceived by Lorentz, Poincaré, Einstein, Minkowski, Weyl and other founders. The first, and perhaps basic understanding of this section is the knowledge that *special relativity and isorelativity coincide at the abstract, realization-free level,* to such an extent that we could use the same formulae and identify the special or isotopic relativity via different meanings of the same symbols. Alternatively, to honor the memory of the founders, it is necessary to identify the widest possible applicability of their axioms before abandoning them for broader vistas.

An additional, century-old, unresolved issue is the incompatibility of special relativity with the absolute reference frame at rest with the universal substratum

[27] To honor the memory of Albert Einstein and other founders of our knowledge, it should be stressed that the use of the word "violation" would be nonscientific, since quantum mechanics and special relativity were not conceived for irreversible processes. Said disciplines have been applied to irreversible processes by Einstein's *followers* seeking money, prestige and power via the abuse of Einstein's name.

(also called *ether*) that appears to be needed for the very characterization of all visible events in the Universe [54,55[. This latter aspect is fundamental for the studies of Volume II and are treated there to avoid unnecessary repetitions.

In regard to general relativity, our position is rather rigid: no research on general relativity can be considered scientifically serious unless the nine theorems s of catastrophic inconsistencies of Ref. [75] are disproved, not in academic corridors, but in refereed technical publications. Since this task appears to be hopeless, we assume the position that general relativity is catastrophically inconsistent and seek an alternative formulation.

As we shall see, *when the memory of the founders is honored in the above sense, the broadest possible realization of their axioms include gravitation and there is no need for general relativity as a separate theory.* Thus, another basic understanding of this section is the knowledge that we shall seek a *unification of special and general relativity into one single formulation based on the axioms of special relativity, known as Santilli isorelativity.* Needless to say, such a unification required several decades of research since it required the construction of the needed new mathematics, the achievement of the unification of the Minkowskian and Riemannian geometries, and the achievement of a universal invariance for all possible spacetime line elements prior to addressing the unification itself.

A further aspect important for the understanding of this section is that, *by no means isorelativity should be believed to be the final relativity of the universe because it is structurally reversible due to the Hermiticity of the isounit and isotopic element.*[28]

This creates the need for a yet broader relativity studied in the next chapte, and known under the name of *Santilli genorelativity,* this time, based on *genotopic liftings* of special relativity or isorelativity, namely, broadening requiring a necessary departure from the abstract axioms of special relativity into a form that is *structurally irreversible,* in the sense of possessing mathematical and physical axioms that are irreversible under all possible reversible Lagrangians or Hamiltonians.

The resolution of the above indicated problems for antimatter is achieved by the isodual image of the studies of this section.

3.5.2 Minkowski-Santilli Isospaces and their Isoduals

As studied in Section 1.2, the "universal constancy of the speed of light" is a philosophical abstraction, particularly when proffered by experts without the

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[28]As we shall see in the next chapter, despite its Hermiticity, the isounit can depend on time in such a way that $\hat{I}(t,...) = \hat{I}^\dagger(t,...) \neq \hat{I}(-t,...)$. In this case isotopic theories represent systems verifying total conservation laws when isolated (because of the antisymmetry of the Lie-Santilli isobrackets), yet being structurally irreversible in their interior processes, as it is the case for all interior problems when considered isolated from the rest of the universe.

additional crucial words "in vacuum", because the constancy of the speed of light has been solely proved in vacuum while, in general, experimental evidence establishes that *the speed of light is a local variable depending on the characteristics of the medium in which it propagates,* with well known expression

$$c = c_\circ / n, \tag{3.5.1}$$

where the familiar *index of refraction* n is a function of a variety of time t, coordinates r, density μ, temperature τ, frequency ω, etc., $n = n(t, r, \mu, \tau, \omega, \ldots)$.

In particular, the speed of light is generally smaller than that in vacuum when propagating within media of low density, such as atmospheres or liquids,

$$c \ll c_\circ, \quad n \gg 1, \tag{3.5.2}$$

while the speed of light is generally bigger than that in vacuum when propagating within special guides, or within media of very high density, such as the interior of stars and quasars,

$$c \gg c_\circ, \quad n \ll 1. \tag{3.5.3}$$

Academic claims of recovering the speed of light in water via photons scattering among the water molecules are afflicted by numerous inconsistencies studied in Section 1.2, and the same holds for other aspects.

Assuming that via some unknown manipulation special relativity is shown to represent consistently the propagation of light within physical media, such a representation would activate the catastrophic inconsistencies of Theorem 1.5.1.

This is due to the fact that *the transition from the speed of light in vacuum to that within physical media requires a noncanonical or nonunitary transform.*

This point can be best illustrated by using the metric originally proposed by Minkowski, which can be written

$$\eta = Diag.(1, 1, 1, -c_\circ^2). \tag{3.5.4}$$

Then, the transition from $c_\circ$ to $c = c_\circ/n$ in the metric can only be achieved via a noncanonical or nonunitary transform

$$\eta = Diag.(1, 1, 1, -c_\circ^2) \rightarrow \hat{\eta} =$$

$$= Diag.(1, 1, 1, -c_\circ/n^2) = U \times \eta \times U^\dagger, \tag{3.5.5a}$$

$$U \times U^\dagger = Diag.(1,\ 1,\ 1,\ 1/n^2) \neq I. \tag{3.5.5b}$$

An invariant resolution of the limitations of special relativity for closed and reversible systems of extended and deformable particles under Hamiltonian and non-Hamiltonian interactions has been provided by the lifting of special relativity into a new formulation today known as *Santilli isorelativity,* where: the prefix "iso" stands to indicate that relativity principles apply on isospacetime over isofields; and the characterization of "special" or "general" is inapplicable because, as shown below, *isorelativity achieves a geometric unification of special and general relativities.*

Isorelativity was first proposed by R. M. Santilli in Ref. [58] of 1983 via the first invariant formulation of *iso-Minkowskian spaces* and related *iso-Lorentz symmetry.* The studies were then continued in: Ref. [59] of 1985 with the first isotopies of the rotational symmetry; Ref. [49] of 1993 with the first isotopies of the SU(2)-spin symmetry; Ref. [60] of 1993 with the first isotopies of the Poincaré symmetry; Ref. [51] of 1998 with the first isotopies of the SU(2)-isospin symmetries, Bell's inequalities and local realism; and Refs. [61,62] on the first isotopies of the spinorial covering of the Poincaré symmetry.

The studies were then completed with memoir [26] of 1998 presenting a comprehensive formulation of the iso-Minkowskian geometry and its capability to unify the Minkowskian and Riemannian geometries, including its formulation via the mathematics of the Riemannian geometry (such iso-Christoffel's symbols, isocovariant derivatives, etc.). The author then dedicated various monographs to the field through the years.

Numerous independent studies on Santilli isorelativity are available in the literature, one can inspect in this respect Refs. [32–43] and papers quoted therein; Aringazin's proof [63] of the direct universality of the Lorentz-Poincaré-Santilli isosymmetry for all infinitely possible spacetimes with signature $(+, +, +, -)$; Mignani's exact representation [64] of the large difference in cosmological redshifts between quasars and galaxies when physically connected; the exact representation of the anomalous behavior of the meanlives of unstable particles with speed by Cardone et al. [65–66]; the exact representation of the experimental data on the Bose-Einstein correlation by Santilli [67] and Cardone and Mignani [68]; the invariant and exact validity of the iso-Minkowskian geometry within the hyperdense medium in the interior of hadrons by Arestov et al. [69]; the first known exact representation of molecular features by Santilli and Shillady [70,71]; and numerous other contributions.

Evidently we cannot review isorelativity in the necessary details to avoid a prohibitive length. Nevertheless, to achieve minimal self-sufficiency of this presentation, it is important to outline at least its main structural lines (see monograph [55] for detailed studies).

The central notion of isorelativity is the lifting of the basic unit of the Minkowski space and of the Poincaré symmetry, $I = Diag.(1, 1, 1, 1)$, into a 4×4-dimensional, nowhere singular and positive-definite matrix $\hat{I} = \hat{I}_{4\times4}$ with an unrestricted functional dependence on local spacetime coordinates x, speeds v, accelerations a, frequencies ω, wavefunctions ψ, their derivative $\partial\psi$, and/or any other needed variables,

$$I = Diag.(1, 1, 1) \rightarrow \hat{I}(x, v, a, \omega, \psi, \partial\psi, \ldots) =$$

$$= 1/\hat{T}(x, v, \omega, \psi, \partial\psi, \ldots) > 0. \quad (3.5.6)$$

Isorelativity can then be constructed via the method of Section 3.4.6, namely, by assuming that the basic noncanonical or nonunitary transform coincides with the above isounit

$$U \times U^\dagger = \hat{I} = Diag.(g_{11}, g_{22}, g_{33}, g_{44}),$$

$$g_{\mu\mu} = g_{\mu\mu}(x, v, \omega, \psi, \partial\psi, \ldots) > 0, \ \ \mu = 1, 2, 3, 4, \quad (3.5.7)$$

and then subjecting the *totality* of quantities and their operation of special relativity to the above transform.

This construction is, however, selected here only for simplicity in pragmatic applications, since the rigorous approach is the construction of isorelativity from its abstract axioms, a task we have to leave to interested readers for brevity (see the original derivations [7]).

This is due to the fact that the former approach evidently preserves the original eigenvalue spectra and does not allow the identification of anomalous eigenvalues emerging from the second approach, such as those of the $SU(2)$ and $SU(3)$ isosymmetries [51].

Let $M(x, \eta, R)$ be the Minkowski space with local coordinates $x = (x^\mu)$, metric $\eta = Diag.(1, 1, 1, -1)$ and invariant

$$x^2 = (x^\mu \times \eta_{\mu\nu} \times x^\nu) \times I \in R. \quad (3.5.8)$$

The fundamental space of isorelativity is the *Minkowski-Santilli isospace* [58] and related topology [10,22–25], $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$ characterized by the liftings

$$I = Diag.(1, 1, 1, 1) \rightarrow U \times I \times U^\dagger = \hat{I} = 1/\hat{T}, \quad (3.5.9a)$$

$$\eta = Diag.(1, 1, 1, -1) \times I \rightarrow (U^{\dagger -1} \times \eta \times U^{-1}) \times \hat{I} = \hat{\eta} =$$

$$= \hat{T} \times \eta = Diag.(g_{11}, g_{22}, g_{33}, -g_{44}) \times \hat{I}, \quad (3.5.9b)$$

with consequential isotopy of the basic invariant

$$x^2 = (x^\mu \times \eta_{\mu\nu} \times x^\nu) \times I \in R \rightarrow$$

$$\rightarrow U \times x^2 \times U^\dagger = \hat{x}^{\hat{2}} = (\hat{x}^\mu \hat{\times} \hat{m}_{\mu\nu} \times x^\nu) \times I \in R, \quad (3.5.10)$$

whose projection in conventional spacetime can be written

$$\hat{x}^{\hat{2}} = [x^\mu \times \hat{\eta}_{\mu\nu}(x, v, a, \omega, \psi, \partial\psi, \ldots) \times x^\nu] \times \hat{I}. \quad (3.5.11)$$

The nontriviality of the above lifting is illustrated by the following:[29]

[29] Fabio Cardone, Roberto Mignani and Alessio Marrani have uploaded a number of papers in the section hep-th of Cornell University arXiv copying *ad litteram* the results of paper [83], including the use of the same symbols, without any quotation at all of Santilli's preceding vast literature in the field. Educators, colleagues and editors of scientific journals are warned of the existence on ongoing legal proceedings one can inspect in the web site http://www.scientificethics.org/

THEOREM 3.5.1: The Minkowski-Santilli isospaces are directly universal, in the sense of admitting as particular cases all possible spaces with the same signature $(+,+,+,-)$*, such as the Minkowskian, Riemannian, Finslerian and other spaces (universality), directly in terms of the isometric within fixed local variables (direct universality).*

Therefore, the correct formulation of the *Minkowski-Santilli isogeometry* requires the isotopy of all tools of the Riemannian geometry, such as the iso-Christoffel symbols, isocovariant derivative, etc. (see for brevity Ref. [15]).

Despite that, one should keep in mind that, in view of the positive-definiteness of the isounit [34,79], *the Minkowski-Santilli isogeometry coincides at the abstract level with the conventional Minkowski geometry, thus having a null isocurvature* (because of the basic mechanism of deforming the metric η by the amount $\hat{T}(x,\ldots)$ while deforming the basic unit of the inverse amount $\hat{I} = 1/\hat{T}$).

The geometric unification of the Minkowskian and Riemannian geometries achieved by the Minkowski-Santilli isogeometry constitutes the evident geometric foundation for the unification of special and general relativities studied below.

It should be also noted that, following the publication in 1983 of Ref. [58], numerous papers on "deformed Minkowski spaces" have appeared in the physical and mathematical literature (generally without a quotation of their origination in Ref. [58]).

These "deformations" are ignored in these studies because they are formulated via conventional mathematics and, consequently, they all suffer of the catastrophic inconsistencies of Theorem 1.5.1.

By comparison, isospaces are formulated via isomathematics and, therefore, they resolve the inconsistencies of Theorem 1.5.1, as shown in Section 3.5.9. This illustrates again the necessity of lifting the basic unit and related field jointly with all remaining conventional mathematical methods.

3.5.3 Poincaré-Santilli Isosymmetry and its Isodual

Let $P(3.1)$ be the conventional Poincaré symmetry with the well known ten generators $J_{\mu\nu}, P_\mu$ and related commutation rules hereon assumed to be known.

The second basic tool of isorelativity is the *Poincaré-Santilli isosymmetry* $\hat{P}(3.1)$ studied in detail in monograph [55] that can be constructed via the isotheory of Section 3.2, resulting in the isocommutation rules [58,60]

$$[J_{\mu\nu}\hat{,}J_{\alpha\beta}] = i \times (\hat{\eta}_{\nu\alpha} \times J_{\beta\mu} - \hat{\eta}_{\mu\alpha} \times J_{\beta\nu} - \hat{\eta}_{\nu\beta} \times J_{\alpha\mu} + \hat{\eta}_{\mu\beta} \times J_{\alpha\nu}), \quad (3.5.12a)$$

$$[J_{\mu\nu}\hat{,}P_\alpha] = i \times (\hat{\eta}_{\mu\alpha} \times P_\nu - \hat{\eta}_{\nu\alpha} \times P_\mu), \quad (3.5.12b)$$

$$[P_\mu\hat{,}P_\nu] = 0, \quad (3.5.12c)$$

where we have followed the general rule of the Lie-Santilli isotheory according to which isotopies leave observables unchanged (since Hermiticity coincides with iso-Hermiticity) and merely change the *operations* among them.

The *iso-Casimir invariants* of $\hat{P}(3.1)$ are given by

$$P^{\hat{2}} = P_\mu \hat{\times} P^\mu = P^\mu \times \hat{\eta}_{\mu\nu} \times P^\nu = P_k \times g_{kk} \times P_k - p_4 \times g_{44} \times P_4, \qquad (3.5.13a)$$

$$W^{\hat{2}} = W_\mu \hat{\times} W^\mu, \;\; W_\mu = \hat{\epsilon}_{\mu\alpha\beta\rho} \hat{\times} J^{\alpha\beta} \hat{\times} P^\rho, \qquad (3.5.13b)$$

and they are at the foundation of classical and operator *isorelativistic kinematics.*

Since $\hat{I} > 0$, it is easy to prove that *the Poincaré-Santilli isosymmetry is isomorphic to the conventional symmetry.* It then follows that *the isotopies increase dramatically the arena of applicability of the Poincaré symmetry, from the sole Minkowskian spacetime to all infinitely possible spacetimes.*

Next, the reader should be aware that *the Poincaré-Santilli isosymmetry characterizes "isoparticles" (and not particles) via its irreducible isorepresentations.*

A mere inspection of the isounit shows that the Poincaré-Santilli isosymmetry characterizes actual nonspherical and deformable shapes as well as internal densities and the most general possible nonlinear, nonlocal and nonpotential interactions.

Since any interaction implies a renormalization of physical characteristics, it is evident that *the transition from particles to isoparticles, that is, from motion in vacuum to motion within physical media, causes an alteration (called isorenormalization), in general, of all intrinsic characteristics, such as rest energy, magnetic moment, charge, etc.*

As we shall see later on, the said isorenormalization has permitted the first exact numerical representation of nuclear magnetic moments, molecular binding energies and other data whose exact representation resulted to be impossible for nonrelativistic and relativistic quantum mechanics despite all possible corrections conducted over 75 years of attempts.

The explicit form of the *Poincaré-Santilli isotransforms* leaving invariant line element (3.5.11) can be easily constructed via the Lie-Santilli isotheory and are given:

(1) The **isorotations** [11]

$$\hat{O}(3) : \hat{\mathbf{x}}' = \hat{\Re}(\hat{\theta}) \hat{\times} \hat{\mathbf{x}}, \;\; \hat{\theta} = \theta \times \hat{I}_\theta \in \hat{R}_\theta, \qquad (3.5.14)$$

that, for isotransforms in the (1, 2)-isoplane, are given by

$$x^{1'} = x^1 \times \cos[\theta \times (g_{11} \times g_{22})^{1/2}] - x^2 \times g_{22} \times g_{11}^{-1} \times \sin[\theta \times (g_{11} \times g_{22})^{1/2}], \;\; (3.5.15a)$$

$$x^{2'} = x^1 \times g_{11} \times g_{22}^{-1} \times \sin[\theta \times (g_{11} \times g_{22})^{1/2}] + x^2 \times \cos[\theta \times (g_{11} \times g_{22})^{1/2}]. \;\; (3.5.15b)$$

For the general expression in three dimensions interested reader can inspect Ref. [7] for brevity.

Note that, since $\hat{O}(3)$ is isomorphic to $O(3)$, Ref. [59] proved, contrary to a popular belief throughout the 20-th century, that

LEMMA 3.5.1: The rotational symmetry remains exact for all possible signature-preserving $(+,+,+)$ *deformations of the sphere.*

The rotational symmetry was believed to be "broken" for ellipsoidal and other deformations of the sphere merely due to insufficient mathematics for the case considered because, when the appropriate mathematics is used, the rotational symmetry returns to be exact, and the same holds for virtually all "broken" symmetries.

The above reconstruction of the exact rotational symmetry can be geometrically visualized by the fact that *all possible signature-preserving deformations of the sphere are perfect spheres in isospace called isosphere.*

This is due to the fact that ellipsoidal deformations of the semiaxes of the perfect sphere are compensated on isospaces over isofields by the *inverse* deformation of the related unit

$$Radius \quad 1_k \rightarrow 1/n_k^2, \tag{3.5.16a}$$

$$Unit \quad 1_k \rightarrow n_k^2. \tag{3.5.16b}$$

We recover in this way the perfect sphere on isospaces over isofields

$$\hat{r}^{\hat{2}} = \hat{r}_1^2 + \hat{r}_2^2 + \hat{r}_3^2 \tag{3.5.17}$$

with exact $\hat{O}(3)$ symmetry, while its projection on the conventional Euclidean space is the ellipsoid

$$r_1^2/n_1^2 + r_2^2/n_2^2 + r_3^2/n_3^2, \tag{3.5.18}$$

with broken $O(3)$ symmetry.

(2) The **Lorentz-Santilli isotransforms** [26,29]

$$\hat{O}(3.1) : \hat{x}' = \hat{\Lambda}(\hat{v}, \ldots)\hat{\times}\hat{x}, \ \hat{v} = v \times \hat{I}_v \in \hat{R}_v, \tag{3.5.19}$$

that, for isotransforms in the (3,4)-isoplane, can be written

$$x^{1'} = x^1, \tag{3.5.20a}$$

$$x^{2'} = x^2, \tag{3.5.20b}$$

$$x^{3'} = x^3 \times \cosh[v \times (g_{33} \times g_{44})^{1/2}] -$$
$$-x^4 \times g_{44} \times (g_{33} \times g_{44})^{-1/2} \times \sinh[v \times (g_{33} \times g_{44})^{1/2}] =$$

$$= \hat{\gamma} \times (x^3 - \hat{\beta} \times \frac{g_{44}^{1/2}}{g_{33}^{1/2}} \times x^4), \tag{3.5.20c}$$

$$x^{4'} = -x^3 \times g_{33} \times (g_{33} \times g_{44})^{-1/2} \times \sinh[v(g_{33} \times g_{44})^{1/2}] +$$

$$+ x^4 \times \cosh[v \times (g_{33} \times g_{44})^{1/2}] =$$

$$= \hat{\gamma} \times (x^4 - \hat{\beta} \times \frac{g_{33}^{1/2}}{g_{44}^{1/2}} \times x^3), \tag{3.5.20b}$$

where

$$\hat{\beta}^2 = \frac{v_k \times g_{kk} \times v_k}{c_o \times g_{44} \times c_o} \hat{\gamma} = \frac{1}{(1 - \hat{\beta}^2)^{1/2}}. \tag{3.5.21}$$

For the general expression interested readers can inspect Ref. [7].

Contrary to another popular belief throughout the 20-th century, Ref. [58] proved that

LEMMA 3.5.2: The Lorentz symmetry remains exact for all possible signature preserving $(+,+,+,-)$ *deformations of the Minkowski space.*

Again, the symmetry remains exact under the use of the appropriate mathematics.

The above reconstruction of the exact Lorentz symmetry can be geometrically visualized by noting that the light cone

$$x_2^2 + x_3^2 - c_o^2 \times t^2 = 0, \tag{3.5.22}$$

can only be formulated in vacuum, while within physical media we have the *light isocone*

$$\frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2} - \frac{c_o^2 \times t^2}{n^2(\omega, \ldots)} = 0, \tag{3.5.23}$$

that, when formulated on isospaces over isofield, is also a perfect cone, as it is the case for the isosphere. This property then explains how the Lorentz symmetry is reconstructed as exact according to Lemma 3.5.2 or, equivalently, that $\hat{O}(3.1)$ is isomorphic to $O(3.1)$.

(3) The **isotranslations** [29]

$$\hat{\mathcal{T}}(4) : \hat{x}' = \hat{\mathcal{T}}(\hat{a}, \ldots)\hat{]} \times x = \hat{x} + \hat{A}(\hat{a}, x, \ldots), \;\; \hat{a} = a \times \hat{I}_a \in \hat{R}_a, \tag{3.5.24}$$

that can be written

$$x^{\mu'} = x^\mu + A^\mu(a, \ldots), \tag{3.5.25a}$$

$$A^\mu = a^\mu(g_{\mu\mu} + a^\alpha \times [g_{i\mu} \hat{,} P_\alpha]/1! + \ldots), \tag{3.5.25b}$$

and there is no summation on the μ indices.

We reach in this way the following important result:

LEMMA 3.5.3 [55]: Isorelativity permits an axiomatically correct extension of relativity laws to noninertial frames.

In fact, noninertial frames are transformed into frames that are inertial on isospaces over isofields, called *isoinertial,* as established by the fact that isotranslations (3.5.25) are manifestly nonlinear and, therefore, noninertial on conventional spaces while they are isolinear on isospaces, according to a process similar to the reconstruction of locality, linearity and canonicity.

The isoinertial character of the frames can also be seen from the isocommutativity of the linear momenta, Eqs. (3.5.12c), while such a commutativity is generally lost in the projection of Eqs. (3.5.12c) on ordinary spaces over ordinary fields, thus confirming the lifting of conventional noninertial frames into an isoinertial form.

This property illustrates again the origin of the name "isorelativity" to indicate that conventional relativity axioms are solely applicable in isospacetime.

(4) The novel **isotopic transformations** [60]

$$\hat{\mathcal{I}}(1) : \hat{x}' = \hat{w}^{-\hat{1}} \hat{\times} \hat{x} = w^{-1} \times \hat{x}, \;\; \hat{I}' = w^{-2} \times \hat{I}, \tag{3.5.26}$$

where w is a constant,

$$\hat{I} \to \hat{I}' = \hat{w}^{-2} \hat{\times} \hat{I} = w^{-2} \times \hat{I} = 1/\hat{T}', \tag{3.5.27a}$$

$$\hat{x}^{\hat{2}} = (x^\mu \times \hat{\eta}_{\mu\nu} \times x^\nu) \times \hat{I} \equiv \hat{x}'^{\hat{2}} =$$
$$= [x^\mu \times (w^2 \times \hat{\eta}_{\mu\nu}) \times x^\nu] \times (w^2 \times \hat{I}). \tag{3.5.27b}$$

Contrary to another popular belief throughout the 20-th century, we therefore have the following

THEOREM 3.5.2: The Poincaré-Santilli isosymmetry, hereon denoted with

$$\hat{P}(3.1) = \hat{O}(3.1) \hat{\times} \hat{\mathcal{T}}(4) \hat{\times} \hat{\mathcal{I}}(1), \tag{3.5.28}$$

and, therefore, the conventional Poincaré symmetry, are eleven dimensional.

The increase of dimensionality of the fundamental spacetime symmetry as, predictably, far reaching implications, including a basically novel and axiomatically consistent grand unification of electroweak and gravitational interactions studied in Chapter 5.

The simplest possible realization of the above formalism for isorelativistic kinematics can be outlined as follows. The first application of isorelativity is that of providing *an invariant description of locally varying speeds of light propagating within physical media.* For this purpose a realization of isorelativity requires the knowledge of the *density* of the medium in which motion occurs.

The simplest possible realization of the fourth component of the isometric is then given by the function

$$g_{44} = n_4^2(x, \omega, \ldots), \tag{3.5.29}$$

normalized to the value $n_4 = 1$ for the vacuum (note that the density of the medium in which motion occurs *cannot* be described by special relativity). The above representation then follows with invariance under $\hat{P}(3.1)$.

In this case the quantities $n_k,\ k = 1, 2, 3$, represent the *inhomogeneity and anisotropy of the medium considered.* For instance, if the medium is homogeneous and isotropic (such as water), all metric elements coincide, in which case

$$\hat{I} = Diag.(g_{11}, g_{22}, g_{33}, g_{44}) = n_4^2 \times Diag.(1, 1, 1, 1), \tag{3.5.30a}$$

$$\hat{x}^{\hat{2}} = \frac{x^2}{n_4^2} \times n_4^2 \times I \equiv x^2, \tag{3.5.30b}$$

thus confirming that *isotopies are hidden in the Minkowskian axioms,* and this may be a reason why they were not been discovered until recently.

Next, isorelativity has been constructed for the invariant description of systems of extended, nonspherical and deformable particles under Hamiltonian and non-Hamiltonian interactions.

Practical applications then require the knowledge of the actual shape of the particles considered, here assumed for simplicity as being spheroidal ellipsoids with semiaxes n_1^2, n_2^2, n_3^2.

Note that the minimum number of constituents of a closed non-Hamiltonian system is two. In this case we have shapes represented with $n_{\alpha k},\ \alpha = 1, 2, , \ldots, n$.

Specific applications finally require the identification of the nonlocal interactions, e.g., whether occurring on an extended *surface* or *volume.* As an illustration, two spinning particles denoted 1 and 2 in condition of deep mutual penetration and overlapping of their wavepackets (as it is the case for valence bonds), can be described by the following Hamiltonian and total isounit

$$H = \frac{p_1 \times p_1}{2 \times m_1} + \frac{p_2 \times p_2}{2 \times m_2} + V(r), \tag{3.5.31a}$$

$$\hat{I}_{Tot} = Diag.(n_{11}^2, n_{12}^2, n_{13}^2, n_{14}^2) \times Diag.(n_{21}^2, n_{22}^2, n_{23}^2, n_{24}^2) \times$$
$$\times e^{N \times (\hat{\psi}_1/\psi_1 + \hat{\psi}_2/\psi_2) \times \int \hat{\psi}_{1\uparrow}(r)^\dagger \times \hat{\psi}_{2\downarrow}(r) \times dr^3}, \tag{3.5.31b}$$

where N is a positive constant.

The above realization of the isounit has permitted the first known *invariant and numerically exact* representation of the binding energy and other features of the hydrogen, water and other molecules [71,72] (see Chapter 9) for which a historical 2% has been missing for about one century. The above isounit has also been instrumental for a number of additional data on two-body systems whose representation had been impossible with quantum mechanics, such as the origin of the spin 1 of the ground state of the deuteron that, according to quantum axioms, should be zero.

Note in isounit (3.5.31) the nonlinearity in the wave functions, the nonlocal-integral character and the impossibility of representing any of the above features with a Hamiltonian.

From the above examples interested readers can then represent any other closed non-Hamiltonian systems.

3.5.4 Isorelativity and Its Isodual

The third important part of the new isorelativity is given by the following isotopies of conventional relativistic axioms that, for the case of motion along the third axis, can be written [29] as follows [60]:

ISOAXIOM I. The projection in our spacetime of the maximal causal invariant isospeed is given by:

$$V_{Max} = c_\circ \times \frac{g_{44}^{1/2}}{g_{33}^{1/2}} = c_\circ \frac{n_3}{n_4} = c \times n_3. \tag{3.5.32}$$

This isoaxiom resolves the inconsistencies of special relativity recalled earlier for particles and electromagnetic waves propagating within physical media such as water.

In fact, water is homogeneous and isotropic, thus requiring that

$$g_{11} = g_{22} = g_{33} = g_{44} = 1/n^2, \tag{3.5.33}$$

where n is the index of refraction.

In this case the maximal causal speed for a massive particle is $c_\circ$ as experimentally established, e.g., for electrons, while the local speed of electromagnetic waves is $c = c_\circ/n$, as also experimentally established.

Note that such a resolution requires *the abandonment of the speed of light as the maximal causal speed for motion within physical media, and its replacement with the maximal causal speed of particles.*

It happens that in vacuum these two maximal causal speeds coincide. However, even in vacuum the correct maximal causal speed remains that of particles and *not* that of light, as generally believed.

At any rate, physical media are generally opaque to light but not to particles. Therefore, the assumption of the speed of light as the maximal causal speed within media in which light cannot propagate would be evidently vacuous.

It is an instructive exercise for interested readers to prove that

LEMMA 3.5.4: The maximal causal isospeed of particles on isominkowski space over an isofield remains $c_\circ$.

In fact, on isospaces over isofields $c_\circ^2$ is deformed by the index of refraction into the form $c_\circ^2/n_4^2$, but the corresponding unit cm^2/sec^2 is deformed by the inverse amount, $n_4^2 \times cm^2/sec^2$, thus preserving the numerical value $c_\circ^2$ due to the structure of the isoinvariant studied earlier.

The understanding of isorelativity requires the knowledge that, when formulated on the Minkowski-Santilli isospace over the isoreals, Isoaxiom I coincides with the conventional axiom that is, the maximal causal speed returns to be c. The same happens for all remaining isoaxioms.

ISOAXIOM II. The projection in our spacetime of the isorelativistic addition of isospeeds within physical media is given by:

$$v_{Tot} = \frac{v_1 + v_2}{1 + \frac{v_1 \times g_{33} \times v_2}{c_\circ \times g_{44} \times c_\circ}} = \frac{v_1 + v_2}{1 + \frac{v_1 \times n_4^2 \times v_2}{c_\circ \times n_3^2 \times c_\circ}}. \tag{3.5.34}$$

We have again the correct result that *the sum of two maximal causal speeds in water,*

$$V_{max} = c_\circ \times (n_3/n_4), \tag{3.5.35}$$

yields the maximal causal speed in water, as the reader is encouraged to verify.

Note that such a result is impossible for special relativity. Note also that *the "relativistic" sum of two speeds of lights in water, $c = c_\circ/n$, does not* yield the speed of light in water, thus confirming that the speed of light within physical media, assuming that they are transparent to light, is not the fundamental maximal causal speed.

ISOAXIOM III. The projection in our spacetime of the isorelativistic laws of dilation of time $t_\circ$ and contraction of length $\ell_\circ$ and the variation of mass $m_\circ$ with speed are given respectively by:

$$t = \hat{\gamma} \times t_\circ, \tag{3.5.36a}$$

$$\ell = \hat{\gamma}^{-1} \times \ell_\circ, \tag{3.5.36b}$$

$$m = \hat{\gamma} \times m_\circ. \tag{3.5.36c}$$

$$\hat{\beta} = \frac{v_k \times g_{kk}}{c_o \times g_{44}} = \frac{v_k}{V_{Max}}, \quad \hat{\gamma} = \frac{1}{(1 - \hat{\beta}^2)^{1/2}}, \tag{3.5.d}$$

where one should npote that, since the speed is always smaller than the maximal possible speed, $\hat{\gamma}$ cannot assume imaginary values.

Note that in water these values coincide with the relativistic ones as it should be since particles such as the electrons have in water the maximal causal speed $c_\circ$.

Note again the necessity of avoiding the interpretation of the local speed of light as the maximal local causal speed. Note also that the mass diverges at the maximal local causal speed, but *not* at the local speed of light.

ISOAXIOM IV. The projection in our spacetime of the iso-Doppler law is given by the isolaw (here formulated for simplicity for 90° *angle of aberration):*

$$\omega = \hat{\gamma} \times \omega_\circ. \tag{3.5.37}$$

This isorelativistic axioms permits an *exact, numerical and invariant representation* of the large differences in cosmological redshifts between quasars and galaxies when physically connected.

In this case light simply exits the huge quasar chromospheres already redshifted due to the decrease of the speed of light, while the speed of the quasars can remain the *same* as that of the associated galaxy. Note again as this result is impossible for special relativity.

Isoaxiom IV also permits a numerical interpretation of the internal blue- and redshift of quasars due to the dependence of the local speed of light on its frequency.

Finally, Isoaxiom IV predicts that *a component* of the predominance toward the red of sunlight at sunset is of iso-Doppler nature. This prediction is based on the different travel within atmosphere of light at sunset as compared to the zenith (evidently because of the travel within a comparatively denser atmosphere).

By contrast, the popular representation of the apparent redshift of sunlight at sunset is that via the scattering of light among the molecules composing our atmosphere. Had this interpretation be correct, the sky at the zenith should be red, while it is blue.

At any rate, the claim of representation of the apparent redshift via the scattering of light is political because of the impossibility of reaching the needed numerical value of the redshift, as serious scholars are suggested to verify.

ISOAXIOM V. The projection in our spacetime of the isorelativistic law of equivalence of mass and energy is given by:

$$E = m \times V_{Max}^2 = m \times c_\circ^2 \times \frac{g_{44}}{g_{33}} = m \times c_\circ^2 \times \frac{n_3^2}{n_4^2} = c \times n_3 \tag{3.5.38}$$

Note a crucial axiomatic difference between the conventional axiom $E = m \times c_circ^2$ and isoaxiom V. They coincide in vacuum, water and other media transparent to light, but are otherwise structurally different. We should note that, in early references, the conventional axiom $E = m \times c_circ^2$, where $c_\circ$ is the speed of light in vacuum, was lifted into the form $E = m \times c^2$ where c is the local speed of light within physical media. However, the latter form lead to inconsistencies in applications studied in Volume II (e.g., when the medium considered is opaque to light in which case both $c_\circ$ and c are meaningless) and had to be further lifted into Isoaxiom V.

Among various applications, *Isoaxiom V removes any need for the "missing mass" in the universe.* This is due to the fact that all isotopic fits of experimental data agree on values $g_{44} \gg 1$ within the hyperdense media in the interior of hadrons, nuclei and stars [7].

As a result, Isoaxiom V yields a value of the total energy of the universe dramatically bigger than that believed until now under the assumption of the universal validity of the speed of light in vacuum.

For other intriguing applications of Isoaxioms V, e.g., for the rest energy of hadronic constituents, we refer the interested reader to monographs [55,61].

The *isodual isorelativity* for the characterization of antimatter can be easily constructed via the isodual map of Chapter 2, and its explicit study is left to the interested reader for- brevity.

3.5.5 Isorelativistic Hadronic Mechanics and its Isoduals

The isorelativistic extension of relativistic hadronic mechanics is readily permitted by the Poincaré-Santilli isosymmetry. In fact, iso-invariant (3.5.13a) characterizes the following *iso-Gordon equation* on $\hat{\mathcal{H}}$ over $\hat{C}$ [55]

$$\hat{p}_\mu \hat{\times} |\hat{\psi}\rangle = -\hat{i} \hat{\times} \hat{\partial}_\mu |\hat{\psi}\rangle = -i \times \hat{I}^\nu_\mu \times \partial_\nu |\hat{\psi}\rangle, \tag{3.5.39a}$$

$$(\hat{p}_\mu \hat{\times} \hat{p}^\mu + \hat{m}_\circ^2 \hat{\times} \hat{c}^4) \hat{\times} |\hat{\psi}\rangle = (\hat{\eta}^{\alpha\beta} \times \partial_\alpha \times \partial_\beta + m_\circ^2 \times c^4) \times |\hat{\psi}\rangle = 0. \tag{3.5.39b}$$

The linearization of the above second-order equations into the *Dirac-Santilli isoequation* has been first studied in Refs. [60–62] and then by other authors (although generally without the use of isomathematics, thus losing the invariance).

By recalling the structure of Dirac's equation as the Kronecker product of a spin 1/2 massive particle and its antiparticle of Chapter 2, the Dirac-Santilli isoequation is formulated on the total isoselfadjoint isospace and related isosymmetry

$$\hat{M}^{tot} = [\hat{M}^{orb}(\hat{x}, \hat{\eta}, \hat{R}) \times \hat{S}^{spin}(2)] \times$$

$$\times [\hat{M}^{d\,orb}(\hat{x}^d, \hat{\eta}^d, \hat{R}^d) \times \hat{S}^{d\,spin}(2)] = \hat{M}^{d\,tot}, \tag{3.5.40a}$$

$$\hat{S}^{tot} = \hat{P}(3.1) \times \hat{P}^d(3.1) = \hat{S}^{d\,tot}, \tag{3.5.40b}$$

and can be written [29]

$$[\hat{\gamma}^{\mu}\hat{\times}(\hat{p}_{\mu} - \hat{e}\hat{\times}\hat{A}_{\mu}) + \hat{i}\hat{\times}\hat{m}]\hat{\times}|\phi(x)\rangle = 0, \tag{3.5.41a}$$

$$\hat{\gamma}^{\mu} = g^{\mu\mu} \times \gamma^{\mu} \times \hat{I}, \tag{3.5.41b}$$

where the γ's are the conventional Dirac matrices.

Note the appearance of the isometric elements directly in the structure of the isogamma matrices and their presence also when the equation is projected in the conventional spacetime.

The following generators

$$J_{\mu\nu} = (S_k,\, L_{k4}), P_{\mu}, \tag{3.5.42a}$$

$$S_k = (\hat{\epsilon}_{kij}\hat{\times}\hat{\gamma}_i\hat{\times}\hat{\gamma}_j)/2, \;\; L_{k4} = \hat{\gamma}_k\hat{\times}\hat{\gamma}_4/2, \;\; P_{\mu} = \hat{p}_{\mu}, \tag{3.5.42b}$$

characterize the *isospinorial covering of the Poincaré-Santilli isosymmetry.*

The notion of "isoparticle" can be best illustrated with the above realization because it implies that, *in the transition from motion in vacuum (as particles have been solely detected and studied until now) to motion within physical media, particles generally experience the alteration, called "mutation", of all intrinsic characteristics,* as illustrated by the following isoeigenvalues,

$$\hat{S}^{\hat{2}}\hat{\times}|\hat{\psi}\rangle = \frac{g_{11} \times g_{22} + g_{22} \times g_{33} + g_{33} \times g_{11}}{4} \times |\hat{\psi}\rangle, \tag{3.5.43a}$$

$$\hat{S}_3\hat{\times}|\hat{\psi}\rangle = \frac{(g_{11} \times g_{22})^{1/2}}{2} \times |\hat{\psi}\rangle. \tag{3.5.43b}$$

The mutation of spin then characterizes a necessary mutation of the intrinsic magnetic moment given by [29]

$$\tilde{\mu} = \left(\frac{g_{33}}{g_{44}}\right)^{1/2} \times \mu, \tag{3.5.44}$$

where μ is the conventional magnetic moment for the same particle when in vacuum. The mutation of the rest energy and of the remaining characteristics has been identified before via the isoaxioms.

Note that the invariance under isorotations allows the rescaling of the radius of an isosphere. Therefore, for the case of the perfect sphere we can always have $g_{11} = g_{22} = g_{33} = g_{44}$ in which case the magnetic moment is not mutated. These results recover conventional classical knowledge according to which *the alteration of the shape of a charged and spinning body implies the necessary alteration of its magnetic moment.*

The construction of the isodual isorelativistic hadronic mechanics is left to the interested reader by keeping in mind that the iso-Dirac equation is isoselfdual as the conventional equation.

To properly understand the above results, one should keep in mind that *the mutation of the intrinsic characteristics of particles is solely referred to the constituents of a hadronic bound state under conditions of mutual penetration of their wave packets (such as one hadronic constituent) under the condition of recovering conventional characteristics for the hadronic bound state as a whole (the hadron considered),* much along Newtonian subsidiary constrains on non-Hamiltonian forces, Eqs. (3.1.6).

It should be also stressed that *the above indicated mutations violate the unitary condition when formulated on conventional Hilbert spaces, with consequential catastrophic inconsistencies, Theorem 1.5.2.*

As an illustration, the violation of causality and probability law has been established for all eigenvalues of the angular momentum M different than the quantum spectrum

$$M^2 \times |\psi\rangle = \ell(\ell + 1) \times |\psi\rangle, \quad \ell = 0, 1, 2, 3, \ldots \ . \tag{3.5.45}$$

As a matter of fact, these inconsistencies are the very reason why the mutations of internal characteristics of particles for bound states at short distances could not be admitted within the framework of quantum mechanics.

By comparison, hadronic mechanics has been constructed to recover unitarity on iso-Hilbert spaces over isofields, thus permitting an invariant description of internal mutations of the characteristics of the constituents of hadronic bound states, while recovering conventional features for states as a whole.

Far from being mere mathematical curiosities, the above mutations permit basically new structure models of hadrons, nuclei and stars, with consequential, new clean energies and fuels (see Chapters 11, 12).

These new advances are prohibited by quantum mechanics precisely because of the preservation of the *intrinsic* characteristics of the constituents in the transition from bound states at large mutual distance, for which no mutation is possible, to the bound state of the same constituents in condition of mutual penetration, in which case mutations have to be admitted in order to avoid the replacement of a scientific process with unsubstantiated personal beliefs one way or the other (see Chapter 12 for details).

3.5.6 Isogravitation and its Isodual

As indicated in Section 1.4, there is no doubt that the classical and operator formulations of gravitation on a curved space have been the most controversial theory of the 20-th century because of an ever increasing plethora of problematic aspects remained vastly ignored. By contrast, as also reviewed in Section 1.4, special relativity in vacuum has a majestic axiomatic consistence in its *invariance* under the Poincaré symmetry.

Recent studies have shown that the formulation of gravitation on a curved space or, equivalently, the formulation of gravitation based on "covariance", is necessarily noncanonical at the classical level and nonunitary at the operator level, thus suffering of all catastrophic inconsistencies of Theorems 1.4.1 and 1.4.2.

These catastrophic inconsistencies can only be resolved via a new conception of gravity based on a *universal invariance,* rather than covariance.

Additional studies have identified profound axiomatic incompatibilities between gravitation on a curved space and electroweak interactions. These incompatibilities have resulted to be responsible for the lack of achievement of an axiomatically consistent grand unification since Einstein's times (see Chapter 14).

No knowledge of isotopies can be claimed without a knowledge that isorelativity has been constructed to resolve at least some of the controversies on gravitation. The fundamental requirement is *the abandonment of the formulation of gravity via curvature on a Riemannian space and its formulation instead on an iso-Minkowskian space* via the following steps characterizing *exterior isogravitation in vacuum,* first presented in Refs. [73,74]:

I) Factorization of any given Riemannian metric representing exterior gravitation $g^{ext}(x)$ into a nowhere singular and positive-definite 4×4-matrix $\hat{T}(x)$ times the Minkowski metric η,

$$g^{ext}(x) = \hat{T}^{ext}_{grav}(x) \times \eta; \tag{3.5.47}$$

II) Assumption of the inverse of $\hat{T}_{grav}$ as the fundamental unit of the theory,

$$\hat{I}^{ext}_{grav}(x) = 1/\hat{T}^{ext}_{grav}(x); \tag{3.5.48}$$

III) Submission of the totality of the Minkowski space and relative symmetries to the noncanonical/nonunitary transform

$$U(x) \times I^{\dagger}(x) = \hat{I}^{ext}_{grav}. \tag{3.5.49}$$

The above procedure yields the isominkowskian spaces and related geometry $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$, resulting in a new conception of gravitation, exterior isogravity, with the following main features [26]:

i) Isogravity is characterized by a universal *symmetry* (and not a covariance), the Poincaré-Santilli isosymmetry $\hat{P}(3.1)$ for the gravity of matter with isounit $\hat{I}^{ext}_{grav}(x)$, the isodual isosymmetry $\hat{P}^d(3.1)$ for the gravity of antimatter, and the isoselfdual symmetry $\hat{P}(3.1) \times \hat{P}^d(3.1)$ for the gravity of matter-antimatter systems;

ii) All conventional field equations, such as the Einstein-Hilbert and other field equations, can be formulated via the Minkowski-Santilli isogeometry since the latter preserves all the tools of the conventional Riemannian geometry, such as the Christoffel's symbols, covariant derivative, etc. [15];

iii) Isogravitation is isocanonical at the classical level and isounitarity at the operator level, thus resolving the catastrophic inconsistencies of Theorems 1.5.1 and 1.5.2;

iv) An axiomatically consistent operator version of gravity always existed and merely crept in unnoticed through the 20-th century because gravity is imbedded where nobody looked for, in the *unit* of relativistic quantum mechanics, and it is given by isorelativistic hadronic mechanics outlined in the next section.

v) The basic feature permitting the above advances is the abandonment of curvature for the characterization of gravity (namely, curvature characterized by metric $g^{ext}(x)$ referred to the unit I) and its replacement with *isoflatness,* namely, the verification of the axioms of flatness in isospacetime, while preserving conventional curvature in its projection on conventional spacetime (or, equivalently, curvature characterized by the $g(x) = \hat{T}^{ext}_{grav}(x) \times \eta$ referred to the isounit $\hat{I}_{grav}(x)$ in which case curvature becomes null due to the inter-relation $\hat{I}^{ext}_{grav}(x) = 1/\hat{T}^{ext}_{grav}(x))$ [26].

A resolution of numerous controversies on classical formulations of gravity then follows from the above main features, such as:

a) The resolution of the century old controversy on the lack of existence of consistent total conservation laws for gravitation on a Riemannian space, which controversy is resolved under the universal $\hat{P}(3.1)$ symmetry by mere visual verification that the generators of the conventional and isotopic Poincaré symmetry are the same (since they represent conserved quantities in the absence and in the presence of gravity);

b) The controversy on the fact that gravity on a Riemannian space admits a well defined "Euclidean", but not "Minkowskian" limit, which controversy is trivially resolved by isogravity via the limit

$$\hat{I}^{ext}_{grav}(x) \rightarrow I; \tag{3.5.50}$$

c) The resolution of the controversy that Einstein's gravitation predicts a value of the bending of light that is twice the experimental value, one for curvature and one for newtonian attraction, which controversy is evidently resolved by the elimination of curvature as the origin of the bending, as necessary in any case for the free fall of a body along a straight radial line in which no curvature of any type is conceivably possible or credible; and other controversies.

A resolution of the controversies on quantum gravity can be seen from the property that relativistic hadronic mechanics of the preceding section *is* a quantum formulation of gravity whenever $\hat{T} = \hat{T}_{grav}$.

Such a form of operator gravity is as axiomatically consistent as conventional relativistic quantum mechanics because the two formulations coincide, by construction, at the abstract, realization-free level.

As an illustration, whenever

$$\hat{T}^{ext}_{grav} = Diag.(g^{ext}_{11}, g^{ext}_{22}, g^{ext}_{33}, g^{ext}_{44}), \quad g_{\mu\mu} > 0, \tag{3.5.51}$$

the Dirac-Santilli isoequation (3.5.41) provides a direct representation of the conventional electromagnetic interactions experienced by an electron, represented by the vector potential A_μ, plus gravitational interactions represented by the isogamma matrices.

Once curvature is abandoned in favor of the broader isoflatness, the axiomatic incompatibilities existing between gravity and electroweak interactions are resolved because:

i) isogravity possesses, at the abstract level, the *same* Poincaré invariance of electroweak interactions;

ii) isogravity can be formulated on the *same* flat isospace of electroweak theories; and

iii) isogravity admits positive and negative energies in the *same* way as it occurs for electroweak theories.

An axiomatically consistent *iso-grand-unification* then follows, as studied in Chapter 14.

Note that the above grand-unification requires the prior *geometric unification of the special and general relativities,* that is achieved precisely by isorelativity and its underlying iso-Minkowskian geometry.

In fact, special and general relativities are merely differentiated in isospecial relativity by the explicit realization of the unit. In particular, *black holes are now characterized by the zeros of the isounit* [7]

$$\hat{I}^{ext}_{grav}(x) = 0. \tag{3.5.52}$$

The above formulation recovers all conventional results on gravitational singularities, such as the singularities of the Schwarzschild's metric, since they are all described by the gravitational content $\hat{T}_{grav}(x)$ of $g(x) = \hat{T}_{grav}(x) \times \eta$, since η is flat.

This illustrates again that *all conventional results of gravitation, including experimental verifications, can be reformulated in invariant form via isorelativity.*

Moreover, the problematic aspects of general relativity mentioned earlier refer to the *exterior gravitational problem.* Perhaps greater problematic aspects exist in gravitation on a Riemannian space for *interior gravitational problems,* e.g., because of the lack of characterization of basic features, such as the density of the interior problem, the locally varying speed of light, etc.

These additional problematic aspects are also resolved by isorelativity due to the unrestricted character of the functional dependence of the isometric that, therefore, permits a direct geometrization of the density, local variation of the speed of light, etc.

The above lines constitute only the initial aspects of isogravitation since its most important branch is *interior isogravitation* as characterized by isounit and isotopic elements of the illustrative type

$$\hat{I}^{int}_{grav} = 1/\hat{T}^{int}_{grav} > 0, \tag{3.5.53a}$$

$$\hat{T}^{int}_{grav} = Diag.(g^{int}_{11}/n_1^2, g^{int}_{22}/n_2^2, g^{int}_{33}/n_3^2, g^{int}_{44}/n_4^2), \tag{3.5.53b}$$

permitting a *geometric representation directly via the isometric of the actual shape of the body considered, in the above case an ellipsoid with semiaxes* n_1^2, n_2^2, n_3^2*, as well as the (average) interior density* n_4^2 *with consequential representation of the (average value of the) interior speed of light* $C = c/n_4$.

A most important point is that the invariance of interior isogravitation under the Poincaré-Santilli isosymmetry persists in its totality since the latter symmetry is completely independent from the explicit value of the isounit or isotopic element, and solely depends on their positive-definite character.

Needless to say, isounit (3.4.53) is merely illustrative because a more accurate interior isounit has a much more complex functional dependence with a locally varying density, light speed and other characteristics as they occur in reality.

Explicit forms of these more adequate models depends on the astrophysical body considered, e.g., whether gaseous, solid or a mixture of both, and their study is left to the interested reader.

It should also be noted that *gravitational singularities should be solely referred to interior models* evidently because exterior descriptions of type (3.5.52) are a mere approximation or a geometric abstraction.

In fact, *gravitational singularities existing for exterior models are not necessarily confirmed by the corresponding interior formulations.* Consequently, the current views on black holes could well result to be pseudo-scientific beliefs because the only scientific statement that can be proffered at this time without raising issue of scientific ethics is that *the gravitational features of large and hyperdense aggregations of matter, whether characterizing a "black" or "brown" hole, are basically unresolved at this time.*

Needless to say, exterior isogravitation is a particular case of the interior formulation. Consequently, from now on, unless otherwise specified isogravitation will be referred to the interior form.

The cosmological implications are also intriguing and will be studied in Chapter 6. It should be indicated that numerous formulations of gravitation in flat *Minkowski* space exist in the literature, such as Ref. [79] and papers quoted therein. However, these formulations have no connection with isogravity since the background space of the former is conventional, while that of the latter is a geometric unification of the Minkowskian and Riemannian spaces.

It is hoped that readers with young minds of any age admit the incontrovertible character of the limitations of special and general relativities and participate in the laborious efforts toward new vistas because any lack of participation in new frontiers of science, whether for personal academic interest or other reason, is a gift of scientific priorities to others.

Appendix 3.A
Universal Enveloping Isoassociative Algebras

The main structural component of Lie's theory is its *universal enveloping associative algebra* $\xi(L)$ of a Lie algebra L. In fact, Lie algebras can be obtained as the attached antisymmetric part $[\xi(L)]^- \approx L$; the infinite dimensional basis of $\xi(l)$ permit the exponentiation to a finite transformation group G; and the representation theory is crucially dependent on the right and/or left modular associative action originally defined on G.

In Section 3.2.9B we have reviewed the rudiments of the *universal enveloping isoassociative algebras* $\hat{\xi}(L)$ of a Lie-Santilli isoalgebra $\hat{L}$. It is easy to see that all features occurring for $\xi(L)$ carry over to the covering isoform $\hat{\xi}(L)$.[30]

In this appendix we would like to outline a more technical definition of universal enveloping isoassociative algebras since they are at the foundations of the unification of simple Lie algebras of dimension N into a single Lie-Santilli isoalgebra of the same dimension (Section 3.2.13).

With reference to Figure **??**, the envelop $\xi(L)$ can be defined as the (ξ, τ) where ξ is an associative algebra and τ is a homomorphism of L into the antisymmetric algebra ξ^- attached to ξ such that: if ξ' is another associative algebra and τ' is another homomorphism of L into ξ'^-, a unique isomorphism γ exists between ξ and ξ' in such a way that the diagram in the l.h.s of Figure **??** is commutative. The above definition evidently expresses the uniqueness of the Lie algebra L up to local isomorphism, and illustrates the origin of the name "universal" enveloping algebra of L.

With reference to the r.h.s. diagram of Figure **??**, the universal enveloping isoassociative algebra $\hat{\xi}(L)$ of a Lie algebra L was introduced in Ref. [4] as the set $\{(\xi, \tau), i, \hat{\xi}, \hat{\tau}\}$ where: (ξ, τ) is a conventional envelope of L; i is an isotopic mapping $L \to i(L) = \hat{L} \sim L$; $\hat{\xi}$ is an associative algebra generally nonisomorphic to ξ; $\hat{\tau}$ is a homomorphism of $\hat{L}$ into $\hat{\xi}^-$; such that: if $\hat{\xi}'$ is another associative algebra and $\hat{\tau}'$ is another homomorphism of $\hat{L}$ into $\hat{\xi}'^-$, there exists a unique

[30] We use the denomination $\hat{\xi}(L)$ rather than $\hat{\xi}(\hat{L})$ to stress the fact that the generators of ξ are those of L and not of $\hat{L}$, a requirement that is essential for consistent physical applications because the generators of L represent ordinary physical quantities (such as total energy, total linear momentum, etc.) that, as such, cannot be changed by isotopies.

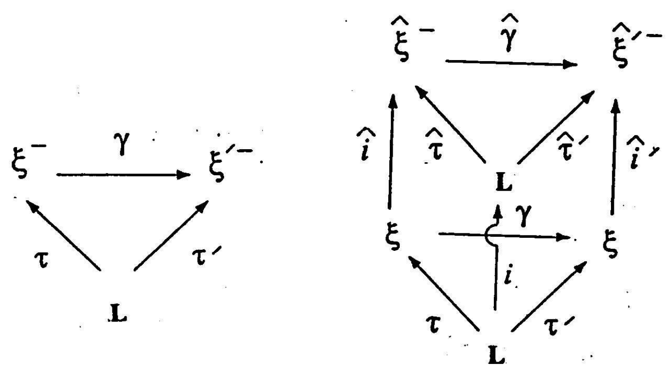

Figure 3.A.1. A schematic view of the universal enveloping associative algebra of a Lie algebra L and its lifting for the Lie-Santilli isoalgebra $\hat{L}$ according to the original proposal [4] of 1978.

isomorphism $\hat{\gamma}$ of $\hat{\xi}$ into $\hat{\xi}'$ with $\hat{\tau}' = \gamma(\hat{\tau})$ and two unique isotopies $i(\xi) = \hat{\xi}$ and $i(\hat{\xi}) = \hat{\xi}'$.

A primary objective of the above definition of isoenvelope is the *lack of uniqueness of the Lie algebra characterized by the isoenvelope* or, equivalently, the *characterization of a family of generally nonisomorphic Lie algebras via the use of only one basis.* The above definition of isoenvelope also explains in more details the variety of realization of the simple 3-dimensional Lie-Santilli isoalgebra $\hat{L}_3$ provided in Eq. (3.2.236), and may be of assistance in extending the same classification to other isoalgebras.

The above notion of isoenvelope represents the essential mathematical structure of hadronic mechanics, namely, the preservation of the conventional basis, i.e., the set of observables of quantum mechanics, and the generalization of the operations on them via an infinite number of isotopies so as to admit a new class of interactions structurally beyond the possibilities of quantum mechanics.

Appendix 3.B
Recent Advances in the TSSFN Isotopology

In Section 3.2.7 we introduced the elements of the *Tsagas-Sourlas-Santilli-Falcón-Núñez isotopology* (or TSSFN Isotopology for short). In this appendix we outline recent advances on the isotopology by the Spanish mathematicians R. M. Falcón Ganfornina and J. Núñez Valdés [24,25].

PROPOSITION 3.2.B1: Consider a mathematical structure

$$(E, +, \times, \circ, \bullet, \ldots),$$

if we construct an isotopic lifting such that:

a) Both primaries $*$, $\widehat{I}$ *and secondaries* $\star$, $\widehat{S}$ *isotopic elements are used.*

b) $(E, \star, *, \ldots)$ *is a structure of the same type as the initial, which is endowed with isounits* S, I, $\ldots$, *with respect to* $\star, *, \ldots$, *respectively.*

c) I *is an unit with respect to* $*$ *in the corresponding general set* V, *being* $T = \widehat{I}^{-I} \in V$ *the associated isotopic element.*

Then, by defining in the isotopic level the operations:

$$\widehat{a}\widehat{+}\widehat{b} = \widehat{a \star b}; \quad \widehat{a}\widehat{\times}\widehat{b} = \widehat{a * b}; \quad \ldots \tag{3.B.1}$$

And being defined in the projection level:

$$\overline{\widehat{a}} = a * \widehat{I}; \quad \alpha\overline{\widehat{+}}\beta = ((\alpha * T) \star (\beta * T)) * \widehat{I}; \quad \alpha\overline{\widehat{\times}}\beta = \alpha * T * \beta; \ \ldots \tag{3.B.2}$$

It is obtained that the isostructure $(\overline{\widehat{E}}, \overline{\widehat{+}}, \overline{\widehat{\times}}, \ldots)$ *is of the same type as the initial one.*

The study in Refs. [24,25] is made by taking into consideration both isotopic and projection levels. Equivalent results related to injective isotopies are also obtained. In the first place, Proposition 3.2.A1 is verified for topological spaces and for their elements and basic properties: isotopologies, isoclosed sets, isoopen sets, T_2, etc:

A *topological isospace* is every isospace endowed with a topological space structure. If, besides, such an isospace is an isotopic projection of a topological space, it is called *isotopological isospace.*

Similarly, they are defined concepts of *(iso)boundary isopoint, closure of a set, closed set, isointerior isopoint, interior of a set, open set, (iso)Hausdorff isospace and second countable isospace, among others.*

PROPOSITION 3.2.B2: The space from which any topological isospace in the isotopic level is obtained can be endowed with the final topology relative to the mapping **I**.

The isotopic projection of a topological space is an isotopological isospace in the projection level. If such a projection is injective, then every topological isospace in such a level is, in fact, isotopological.

Similar results are obtained for the concepts of (iso)boundary isopoint, isointerior isopoint and (iso)Hausdorff isospace.

Next, Refs. [24,25] generalize Kadeisvili's isocontinuity [19]. Particularly, the basic isofield can be endowed with an isoorder, according to the following procedure.

Let $\widehat{K}$ be an isofield associated with a field K, endowed with an order $\leq$, by using an isotopology which preserves the inverse element with respect to the addition. We define the *isoorder* $\widehat{\leq}$ as $\widehat{a}\widehat{\leq}\widehat{b}$ if and only if $a \leq b$. If the isotopy is injective, the isoorder $\overline{\widehat{\leq}}$ en $\overline{\widehat{K}}$ is defined in the same way.

PROPOSITION 3.2.B3: The isoorders $\widehat{\leq}$ *and* $\overline{\widehat{\leq}}$ *are orders over* $\widehat{K}$ *and* $\overline{\widehat{K}}$*, of the same type as* $\leq$.

Let $\widehat{U}$ be a $\widehat{R}$ isovectorspace with isonorm $\widehat{||}.\widehat{||} \equiv \widehat{||.||}$ and isoorder $\widehat{\leq}$, obtained from an isotopy compatible with respect to each one of the initial operations. It will be said that an isoreal isofunction $\widehat{f}$ of $\widehat{U}$ is *isocontinuous in* $\widehat{X} \in \widehat{U}$, if for all $\widehat{\epsilon}\widehat{>}\widehat{S}$, there exists $\widehat{\delta}\widehat{>}\widehat{S}$ such that for all $\widehat{Y} \in \widehat{U}$ with $\widehat{||}\widehat{X}-\widehat{Y}\widehat{||}\widehat{<}\widehat{\delta}$, it is verified that $\widehat{|}\widehat{f}(\widehat{X})-\widehat{f}(\widehat{Y})\widehat{|}\widehat{<}\widehat{\epsilon}$. We will say that $\widehat{f}$ is *isocontinuous in* $\widehat{U}$ if it is isocontinuous in $\widehat{X}$, for all $\widehat{X} \in \widehat{U}$. Finally, when dealing with injective isotopies, the isocontinuity in the projection level is defined in a similar way.

PROPOSITION 3.2.B4: The isocontinuity in $\widehat{U}$ *is equivalent to the continuity in* U*. In the case of injective isotopies, both ones are equivalent to the one in* $\overline{\widehat{U}}$*.*

The isocontinuity on isotopological isospaces is also analyzed:

An *isocontinuous isomapping* in the isotopic level between two topological isospaces $\widehat{M}$ and $\widehat{N}$ is every isomapping $\widehat{f} : \widehat{M} \to \widehat{N}$ preserving closures. The definition in the projection level is given in a similar way.

PROPOSITION 3.2.B5: They are verified that:

a) $\widehat{f}$ *is isocontinuous if and only if the mapping* f *from which comes from is continuous. That result is similar in the projection level by using injective isotopies.*

b) *Every isoconstant isomapping is isocontinuous.*

c) *Isocontinuity is preserved by both topological composition and product.*

Finally, the analysis of (iso)(pseudo)metric isospaces is also concreted:

PROPOSITION 3.2.B6: Let $\widehat{M}$ *be a* $\widehat{K}$ *isovectorspace, isotopic lifting of a vectorspace* M*, endowed with a (pseudo)metric* d *defined on an ordered field* K*, by using an isotopy which preserves the inverse element and compatible with respect to the addition in* K*. Then, the isofunction* $\widehat{d}$ *is an iso(pseudo)metric.*

Let $(\widehat{M}, d')$ be an (iso)(pseudo)metric $\widehat{K}$ isovectorspace, endowed with an iso-order $\widehat{\leq}$. $B_{d'}(\widehat{X_0}, \widehat{\epsilon}) = \{\widehat{X} \in \widehat{M} : d'(\widehat{X}, \widehat{X_0}) \widehat{<} \widehat{\epsilon}\}$ is called *metric ball* with center $\widehat{X_0} \in \widehat{M}$ and radius $\widehat{\epsilon} \widehat{>} \widehat{S}$. If M is endowed with a (pseudo)metric d, with $\widehat{d} = d'$, then every metric ball $B_{d'} = B_{\widehat{d}} = \widehat{B_d}$ in $\widehat{M}$, which is isotopic lifting of a metric ball B_d in M, is called *metric isoball* in $\widehat{M}$.

PROPOSITION 3.2.B7: Under conditions of Proposition XXX, if $B_d(X_0, \epsilon)$ *is a metric ball in* M*, then* $\widehat{B_d(X_0, \epsilon)} = B_{\widehat{d}}(\widehat{X_0}, \widehat{\epsilon})$ *is a metric ball in* $\widehat{M}$*.*

A *metric neighborhood of an isopoint* $\widehat{X} \in \widehat{M}$ is a subset $\widehat{A} \subseteq \widehat{M}$ containing a metric ball centered in $\widehat{X}$. The set of metric neighborhoods of $\widehat{X}$ is denoted by $\widehat{\aleph}^{d'}_{\widehat{X}}$. Finally, if d' is the iso-Euclidean isodistance over $\widehat{R}^n$, the associated metric neighborhoods are called *iso-Euclidean neighborhoods*.

PROPOSITION 3.2.B8: Let d' *and* d'' *two (iso)(pseudo)metrics over an isovectorspace* $\widehat{M}$*. It is verified that* $\widehat{\aleph}^{d'}_{\widehat{X}} = \widehat{\aleph}^{d''}_{\widehat{X}}$ *if and only if every metric ball* $B_{d'}(\widehat{X}, \widehat{\epsilon})$ *contains a ball* $B_{d''}(\widehat{X}, \widehat{\rho})$ *and every ball* $B_{d''}(\widehat{X}, \widehat{\delta})$ *contains a ball* $B_{d'}(\widehat{X}, \widehat{\mu})$*.*

PROPOSITION 3.2.B9: Every isospace endowed with an (iso)(pseudo)metric is an isotopological isospace.

PROPOSITION 3.2.B10: Let $\widehat{f} : (\widehat{M}, d') \rightarrow (\widehat{N}, d'')$ *be an isomapping between* $\widehat{K}$*-isospaces endowed with (iso)(pseudo)metric and let us consider* $\widehat{X} \in \widehat{M}$*. Then,* $\widehat{f}$ *is isocontinuous in* $\widehat{X}$ *if and only if for all* $\widehat{\epsilon}\widehat{>}\widehat{S}$ *there exists* $\widehat{\delta} \in \widehat{K}$ *such that* $\widehat{\delta}\widehat{>}\widehat{S}$*, and if* $\widehat{Y} \in B_{d'}(\widehat{X}, \widehat{\delta})$*, then it is verified that* $\widehat{f}(\widehat{Y}) \in B_{d''}(\widehat{f}(\widehat{X}), \widehat{\epsilon})$*.*

PROPOSITION 3.2.B11: Let $\widehat{f} : \widehat{M} \rightarrow \widehat{N}$ *be an isomapping between two isotopological isospaces* $\widehat{M}$ *and* $\widehat{N}$*. If conditions of the definition of isocontinuity are satisfied, then* $\widehat{f}$ *is isocontinuous if and only if* $\widehat{f}^{-1}(\widehat{U})$ *is an isoopen of* $\widehat{M}$*, for all isoopen* $\widehat{U}$ *of* $\widehat{N}$*.*

Appendix 3.C
Recent Advances on the Lie-Santilli Isotheory

In Section 3.2.9 we have outlined the rudiments of the *Lie-Santilli isotheory.* It may be useful for the mathematically oriented reader to outline recent developments achieved by the Spanish mathematicians R. M. Falcón Ganfornina and J. Núñez Valdés [24,25,43] in the field beyond those presented in monographs [2,6,36,37].

Falcón and Núñez introduced in 2001 [37] a new construction model of isotopies which was similar to the one proposed by Santilli in 1978 although in its multivalued version presented by the same author later on [6] (see Chapter 4) because based on the use of several isolaws and isounits as operations existing in the initial mathematical structure. Such a model, which from now on will be called MCIM (*isoproduct construction model based on the multiplication*), was later generalized in Refs. [24,25,43]. In a schematic way, Santilli's isotopies can be described with the following diagram:

$$\begin{array}{ccc} \text{Conventional Level} & \longrightarrow & \begin{array}{c}\text{General Level}\\ (V,*,\star,\ldots)\\ \cup \end{array} \\ (E,+,\times,\ldots) & & (E,\star,*,\ldots) \\ \downarrow & \sharp & \downarrow \mathbf{I} \\ \text{Projection Level} & \overset{\pi}{\longleftarrow} & \text{Isotopic level} \\ (\overline{\widehat{E}},\overline{\widehat{+}},\overline{\widehat{\times}},\ldots) & & (\widehat{E},\widehat{+},\widehat{\times},\ldots) \end{array}$$

where, by construction:

a) The mapping $\mathbf{I} : (E,\star,*,\ldots) \to (\widehat{E},\widehat{+},\widehat{\times},\ldots) : X \to \widehat{X}$ is an isomorphism.

b) The *isotopic projection* is onto:
$\pi : (\widehat{E},\widehat{+},\widehat{\times},\ldots) \to (\overline{\widehat{E}},\overline{\widehat{+}},\overline{\widehat{\times}},\ldots) : \widehat{a} \to \pi(\widehat{a}) = \overline{\widehat{a}} = a * \widehat{I}.$

c) $\widehat{a}\widehat{+}\widehat{b} = \widehat{a \star b}; \quad \widehat{a}\widehat{\times}\widehat{b} = \widehat{a * b}; \ldots.$

d) $\overline{\widehat{a}} = a * \widehat{I}; \quad \alpha\overline{\widehat{+}}\beta = ((\alpha * T) \star (\beta * T)) * \widehat{I}; \quad \alpha\overline{\widehat{\times}}\beta = \alpha * T * \beta; \ldots.$

PROPOSITION 3.2.C1: The following properties are verified:

a) The isotopic projection associated with each injective isotopic lifting is an isomorphism.

b) If the isotopic lifting used is compatible with respect to all of initial operations, then the isostructure $\widehat{E}$ is isomorphic to the initial E.

c) The relation of being isotopically equivalents is of equivalence.

d) Every isotopy $\pi \circ \mathbf{I} : (E, +, \times, \circ, \bullet, \ldots) \to (\overline{\widehat{E}}, \overline{\widehat{+}}, \overline{\widehat{\times}}, \overline{\widehat{\circ}}, \overline{\widehat{\bullet}}, \ldots)$ can be considered as an isotopic lifting which follows the MCIM, that is, every mathematical isostructure is an isostructure with respect to the multiplication.

Then, it has a perfect sense to considerer each one of the isostructures which result of applying the MCIM to conventional structures. Particularly, we can consider the construction of *Santilli's isoalgebras* (as the isotopic lifting of each algebra, which is endowed with a structure of algebra).

PROPOSITION 3.2.C2: Let U be a K-algebra and let $\widehat{U}$ be a $\widehat{K}$-isovector-space. If a $K(a, \star, *)$-algebra $(U, \diamond, \Box, \cdot)$ is used in the general level, then the isotopic lifting $\widehat{U}$ corresponding to the isotopy of primary elements $\widehat{I}$ and $\Box$ and secondary ones $\widehat{S}$ and $\diamond$, when MCIM is used, has a structure of isoalgebra on $\widehat{K}$, and it preserves the initial type of the algebra.

A particular type of isoalgebra is the *Lie-Santilli isoalgebra* [4]. Particularly, if $\overline{\widehat{U}}$ is the isotopic projection of a Lie-Santilli isoalgebra,

$$\widehat{I} = \widehat{I}(x, dx, d^2x, t, T, \mu, \tau, \ldots)$$

is an isounit and a basis $\overline{\widehat{U}}$, $\{\overline{\widehat{e_1}}, \ldots, \overline{\widehat{e_n}}\}$ is fixed, where $\overline{\widehat{e_i}} \,\widehat{\cdot}\, \overline{\widehat{e_j}} = \sum \overline{\widehat{c_{ij}^h}} \,\overline{\widehat{\bullet}}\, \overline{\widehat{e_h}}$, $\forall\, 1 \le i, j \le n$, then coefficients $\overline{\widehat{c_{ij}^h}} \in \overline{\widehat{K}}$ are the *Maurer-Cartan coefficients* of the isoalgebra, which constitute a generalization of the conventional case, since they are not constants in general, but functions dependent of the factors of $\widehat{I}$.

Another interesting isoalgebra is the *Santilli's Lie-admissible algebra* [4], that is, the isoalgebra $\widehat{U}$ such that with the commutator bracket $[.,.]_{\widehat{U}} : [\widehat{X}, \widehat{Y}]_{\widehat{U}} = (\widehat{X} \widehat{\cdot} \widehat{Y}) - (\widehat{Y} \widehat{\cdot} \widehat{X})$ is an isotopic Lie isoalgebra. The following result is satisfied:

PROPOSITION 3.2.C3: Under conditions of Proposition XXX, let us suppose that the law $\widehat{\circ}$ of the isoalgebra $\widehat{U}$ is defined according $\widehat{X} \widehat{\circ} \widehat{Y} = (X \circ Y) \Box \widehat{I}$, for all $X, Y \in U$. If U is a Lie (admissible) algebra, then $\widehat{U}$ is a Lie isoadmissible isoalgebra.

In this way, Santilli's Lie-admissible isoalgebras inherit the usual properties of conventional (admissible) Lie algebras. In the same way, usual structures related with such algebras have also their analogue ones when isotopies are used.

For instance, an *isoideal* of a Lie isoalgebra $\widehat{U}$ is every isotopic lifting of an ideal $\Im$ of U, which is by itself an ideal. In particular, the *center* of a Lie isoalgebra $\widehat{U}$, $\{\widehat{X} \in \widehat{U}$ such that $\widehat{X}\widehat{\cdot}\widehat{Y} = \widehat{S},\ \forall \widehat{Y} \in \widehat{U}\}$, is an isoideal of $\widehat{U}$. In fact, it is verified the following result:

PROPOSITION 3.2.C4: Let $\widehat{U}$ be a Lie isoalgebra associated with a Lie algebra U and let $\Im$ be an ideal of U. Then, the corresponding isotopic lifting $\widehat{\Im}$ is an isoideal of $\widehat{U}$.

An isoideal $\widehat{\Im}$ of a Lie isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$, is called *isocommutative* if $\widehat{X}\widehat{\cdot}\widehat{Y} = \widehat{S}$, for all $\widehat{X} \in \widehat{\Im}$ and for all $\widehat{Y} \in \widehat{U}$, being $\widehat{U}$ *isocommutative* if it is so as an isoideal.

PROPOSITION 3.2.C5: $\widehat{U}$ is isocommutative if and only if U is commutative.

Lie-Santilli isoalgebras can also be introduced as follows. Given an $\widehat{K}$-isoassociative isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$, the commutator in $\widehat{U}$ associated with $\widehat{\cdot}$: $[\widehat{X}, \widehat{Y}]_S = (\widehat{X}\widehat{\cdot}\widehat{Y}) - (\widehat{Y}\widehat{\cdot}\widehat{X})$, for all $\widehat{X}, \widehat{Y} \in \widehat{U}$ is denominated *Lie-Santilli bracket product* $[.,.]_S$ with respect to $\widehat{\cdot}$. The isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, [.,.]_S)$ is then denominated *Lie-Santilli algebra*.

DEFINITION 3.2.C6: Let $\widehat{U}$ be an $\widehat{K}$-isoassociative isoalgebra associated with a K-algebra U, under conditions of Proposition XXX. Then, the Lie-Santilli algebra associated with $\widehat{U}$ is a Lie isoalgebra if the algebra U is either associative or Lie admissible.

Apart from that, a Lie-Santilli isoalgebra $\widehat{U}$ is said to be *isosimple* if, being an isotopy of a simple Lie algebra, it is not isocommutative and the only isoideals which contains are trivial. In an analogous way, $\widehat{U}$ is called *isosemisimple* if, being an isotopy of a semisimple Lie algebra, it does not contain non trivial isocommutative isoideals. Note that, this definition involves that every isosemisimple Lie isoalgebra is also isosimple. Moreover, it is verified:

PROPOSITION 3.2.C7: Under conditions of Proposition XXX, the isotopic lifting of a (semi)simple Lie algebra is an iso(semi)simple Lie isoalgebra. Particularly, every isosemisimple Lie isoalgebra is a direct sum of isosimple Lie isoalgebras.

A lie-Santilli isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ is said to be *isosolvable* if, being an isotopy of a solvable Lie algebra, in the *isosolvability series*

$$\widehat{U}_1 = \widehat{U}, \quad \widehat{U}_2 = \widehat{U}\widehat{\cdot}\widehat{U}, \quad \widehat{U}_3 = \widehat{U}_2\widehat{\cdot}\widehat{U}_2, \ldots, \widehat{U}_i = \widehat{U}_{i-1}\widehat{\cdot}\widehat{U}_{i-1}, \ldots$$

there exists a natural integer n such that $\widehat{U}_n = \{\widehat{S}\}$. The minor of such integers is called *isosolvability index* of the isoalgebra.

PROPOSITION 3.2.C8: Under conditions of Proposition XXX, the isotopic lifting of a solvable Lie algebra is an isosolvable Lie isoalgebra.

An easy example of isosolvable Lie isoalgebras are the isocommutative isotopic Lie isoalgebras, since they verify, by definition, that $\widehat{U}\widehat{\cdot}\widehat{U} = \widehat{U}_2 = \{\widehat{S}\}$. It implies that every nonzero isocommutative Lie isoalgebra has an isosolvability index equals 2, being 1 the corresponding to the trivial isoalgebra $\{\widehat{S}\}$.

PROPOSITION 3.2.C9: Let $\widehat{U}$ be a Lie isoalgebra associated with a Lie algebra U. Under conditions of Proposition XXX, they are verified:

1) *$\widehat{U}_i$ is an isoideal of $\widehat{U}$ and of $\widehat{U}_{i-1}$, for all $i \in N$.*

2) *If $\widehat{U}$ is isosolvable and U is solvable, then every isosubalgebra of $\widehat{U}$ is isosolvable.*

3) *The intersection and the product of a finite number of isosolvable isoideals of $\widehat{U}$ are isosolvable isoideals. Moreover, under conditions of Proposition XXX, the sum of a finite number of isosolvable isoideals is also an isosolvable isoideal.*

By using this last result it can be deduced that the sum of all isosolvable isoideals of $\widehat{U}$ is another isosolvable isoideal, which is called *isoradical* of $\widehat{U}$. Note that it is different from the *radical* of $\widehat{U}$, which would be the sum of all solvable ideals of $\widehat{U}$. The isoradical is denoted by *isorad* $\widehat{U}$, not to be confused with *rad* $\widehat{U}$, and it will always contain $\{\widehat{S}\}$, because this last one is a trivial isosolvable isoideal of every isoalgebra. Note also that as every isosolvable isoideal of $\widehat{U}$ is a solvable ideal of $\widehat{U}$, then *isorad* $\widehat{U} \subset$ *rad* $\widehat{U}$. So, if $\widehat{U}$ is isosolvable, then $\widehat{U} =$ *isorad* $\widehat{U} =$ *rad* $\widehat{U}$, due to $\widehat{U}$ is solvable in particular.

PROPOSITION 3.2.C10: If $\widehat{U}$ is a semisimple Lie isoalgebra over a field of zero characteristic, then isorad $\widehat{U} = \{\widehat{S}\}$.

A Lie-Santilli isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ is called *isonilpotent* if, being an isotopy of a nilpotent Lie algebra, in the series

$$\widehat{U}^1 = \widehat{U}, \quad \widehat{U}^2 = \widehat{U} \widehat{\cdot} \widehat{U}, \quad \widehat{U}^3 = \widehat{U}^2 \widehat{\cdot} \widehat{U}, \ldots, \widehat{U}^i = \widehat{U}^{i-1} \widehat{\cdot} \widehat{U}, \ldots$$

(which is called *isonilpotency series*), there exists a natural integer n such that $\widehat{U}^n = \{\widehat{S}\}$. The minor of such integers is denominated *nilpotency index* of the isoalgebra.

As an immediate consequence of this definition it is deduced that every isonilpotent Lie isoalgebra is isosolvable and that every nonzero isocommutative Lie isoalgebra has an isonilpotency index equals 2, being 1 the corresponding of the isoalgebra $\{\widehat{S}\}$. Moreover, they are verified:

PROPOSITION 3.2.C11: Under conditions of Proposition XXX, the isotopic lifting of a nilpotent Lie algebra is an isonilpotent isotopic Lie isoalgebra.

PROPOSITION 3.2.C12: Let $\widehat{U}$ be a Lie isoalgebra associated with a Lie algebra U. They are verified:

1) Under conditions of Proposition XXX, the sum of a finite number of isonilpotent isoideals of $\widehat{U}$ is another isonilpotent isoideal.

2) If $\widehat{U}$ is also isonilpotent and U is nilpotent, then

   (a) Every isosubalgebra of $\widehat{U}$ is isonilpotent.

   (b) Under conditions of Proposition XXX, if $\widehat{U}$ is nonzero isonilpotent, then its center is non null.

In a similar way as the case isosolvable, the result (1) involves that the sum of all isonilpotent isoideals of $\widehat{U}$ is another isonilpotent isoideal, which is denoted by *isonihil-radical* of $\widehat{U}$, to be distinguished from the nihil-radical of $\widehat{U}$, which is the sum of the radicals ideals. It will be represented by *isonil-rad* $\widehat{U}$, which allows to distinguish it from the *nil-rad* $\widehat{U}$. It is immediate that *isonil-rad* $\widehat{U} \subset$ *nil-rad* $\widehat{U} \cap$ *isorad* $\widehat{U} \subset$ *nil-rad* $\widehat{U} \subset$ *rad* $\widehat{U}$.

Apart from that, it is possible to relate an isosolvable isotopic Lie isoalgebra with its derived Lie isoalgebra, by using the following:

PROPOSITION 3.2.C13: Under conditions of Proposition XXX, a Lie isotopic isoalgebra is isosolvable if and only if its derived Lie isoalgebra is isonilpotent.

Finally, an isonilpotent Lie isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ is called *isofiliform* if, being an isotopy of a filiform Lie algebra, it is verified that

$$\dim \widehat{U}^2 = n - 2, \ \ \ldots, \ \ \dim \widehat{U}^i = n - i, \ \ \ldots, \ \ \dim \widehat{U}^n = 0,$$

where $\dim \widehat{U} = n$.

Note that the theory related with a filiform Lie algebra U is based on the use of a basis of such an algebra. So, starting from a basis $\{e_1, \ldots, e_n\}$ of U, which is preferably an *adapted basis*, we can deal with lots of concepts of it, such as dimensions of U and of elements of the nilpotency series, invariants i and j of U and, in general, the resting properties, starting from its structure coefficients, which are, in fact, responsible for the complete study of filiform Lie algebras.

Appendix 3.D
Relativistic Formulation of the Galilei Symmetry

As indicated in Section 3.5.1, special relativity has remained unsettled after one century of studies, even in the arena of its original conception, namely, point–particles and electromagnetic waves propagating in vacuum. A reason of the ongoing debates is connected to the alternative of Lorentz invariance for the two-ways light experiments conducted to date, and the Galilean invariance expected for one-way light experiments.

The alternative of Lorentzian vs Galilean treatments is obscured by the fact that the former applies for relativistic speeds while the latter is not perceived as such. This limitation was resolved in the early 1970s by the *relativistic formulation of the Galilean symmetry and relativity* proposed by P. Roman, J. J. Aghassi and R. M. Santilli [76-78], and known as *Galileo-Roman-Santikki-Aghassi symmetry and relativity*.[31]

In short, the alternative as to whether the ultimate relativity is of Lorentzian or Galilean type is far from being resolved. It is an easy prediction that such an alternative will not be resolved in these volumes. Consequently, in this appendix we can merely review the main ideas of the Galileo-Roman-Santilli-Aghassi (GRSA) symmetry, and leave the resolution of the alternative to future generations.

By assuming an in depth knowledge of the Galileo symmetry and its scalar extension (that we cannot possibly review here), the GRSA symmetry is based on the following assumptions:

1) The carrier space is given by the Kronecker product of the conventional Minkowski space $M(x, \eta, R)$ times a one-dimensional space $U(u)$ where u represents the proper time normalized to the dimension of length for reason clarified below,

$$S_{tot} = M(3.1) \times U(u) \qquad (3.D1)$$

2) The GRSA symmetry is characterized by the following transformations

$$SO_o(3.1): \quad x^\mu \rightarrow x'^{\mu\,\prime} = \Lambda^\mu_\nu \times x^\nu, \quad \Lambda^B_\alpha eta \times \Lambda^\rho_\beta = \eta_a lpha^r ho, \qquad (3.D2a)$$

$$T^a_4: \,: x^\mu \rightarrow x'^\mu = x^\mu + a^\mu, \qquad (3.D2b)$$

[31] These studies were conducted while the author was Associate Professor of Physics at Boston University under partial financial support by the USAFOSR here gratefully acknowledged.

$$T_4^b: \quad x^\mu \to x'^\mu = x^\mu + b^\mu \times u \tag{3.D2c}$$

$$T_1^\sigma: \quad u \to u' = u + \sigma, \tag{3.D2d}$$

where: Eqs. (3.D2a) are the (connected) conventional Lorentz transformations; Eqs. (3.D2b) are the conventional translations (with a^μ constants); Eqs. (3.D2c) and (3.D2d) are the new transformations with b^μ and σ non-null parameters, b^μ being dimensionless and σ having the dimension of length. Eqs. (3.D2c) were originally called *relativistic Galilean boosts,* [76] and here called *GRSA boosts,* since they are indeed a relativistic extension of the conventional nonrelativistic boosts. Eq. (3.D2d) was originally called the *relativistic Galilean time translation* [76],and it is here called the *GRSA time translation.*

3) The GRSA symmetry is then fifteen-dimensional and its connected component is written

$$GR = \{SO_o(3.1) \times T_4^b\} \times \{T_4^a \times T_1^\sigma\}, \tag{3.D3}$$

where one should note: the presence of the Poincaré group as a subgroup; the presence of the conventional Galileo group as a subgroup; and the separation of conventional translations from the Lorentz symmetry and their association to the new variable u.

Group (3.D3) admits as an invariant subgroup the group $T_4^a \times T_4^b \times T_1^\sigma$. Hence, *the GRSA group (3.D3) is an extension of the restricted Lorentz group, but not of the Poincaré group,* even though the latter is also an extension of the Lorentz group. These are central features for the understanding of the differences between the Galileo symmetry, the Poincaré symmetry and the GRSA symmetry.

The conventional Galileo group requires a scalar extension for its dynamical application, and the same occurs for the GRSA group, thus leading to the covering

$$\widetilde{GR} = T_1^\theta \times \{SL(2.C) \times T_4^b\} \times \{T_4^a \times T_1^\sigma\}, \tag{3.D4}$$

where θ is the usual phase factor.

By denoting the generators of $SL(2.C)$ with $J_{\mu\nu}$, the generators of T_4^a with $P_m u$, the generators of T_4^b with Q_μ, and the generators of T_1^σ with S, we have the following Lie algebra

$$[J_{\mu\nu}, J_{\rho\sigma}] = i \times (\eta_{\nu\rho} \times J_{\mu\sigma} - \eta_{mu\rho} \times J_{\nu\sigma} - \eta_{\mu\sigma} \times J_{\rho\nu} + \eta_{\nu\sigma} \times J_{\rho\mu}), \tag{3.D5a}$$

$$[P_\mu, J_{\rho\sigma}] = i \times (\eta_{\mu\rho} \times P_\sigma - \eta_{\mu\sigma} \times P_\rho), \tag{3.D5b}$$

$$[Q_\rho, J_{\mu\nu}] = i \times (\eta_{\mu\rho} \times Q_\nu - \eta_{\nu\rho} \times Q_\mu), \tag{3.D5c}$$

$$[P_\mu, Q_\nu] = i \times \eta_{\mu\nu} \times \ell^{-1}, \tag{3.D5d}$$

$$[S, Q_\nu] = i \times P_\nu, \tag{3.D5e}$$

$$[P_\mu, P_\nu] = [Q_\mu . Q_\nu] = [J_{\mu\nu}, S] = [P\mu, S] = 0, \tag{3.D5f}$$

where ℓ is the parameter originating from the scalar extension.

The physical interpretation is based on the following main aspects. Dynamics is assumed to verify the GRSA symmetry, with the Poincaré symmetry characterizing kinematics. Under such an assumption, the GRSA symmetry allows the introduction of a fully consistent *relativistic spacetime position operator* that is absent in relativistic quantum mechanics, with explicit expression

$$X_\mu = -\ell \times Q_\mu. \tag{3.D6}$$

In fact, the above interpretation is fully supported by commutation rules (3.D5).

Eq. (3.D6) introduces quite automatically a *universal length,* with the significant feature that *systems with different fundamental lengths are independent of each other.* The main dynamical invariant is no longer the familiar expression $P_\mu \times P^\mu = m^2$, but it is given instead by the following relativistic extension of the Galilean invariant

$$P_\mu \times P^\mu + 2 \times \ell{-}1 \times S = inv. \tag{3.D7}$$

By assuming the value

$$P_\mu \times P^\mu + 2 \times \ell{-}1 \times S = 0, \tag{3.D8}$$

the Galileo-Roman symmetry allows the introduction of the *relativistic mass operator*

$$\mathcal{M}^\in = \in \times \ell^{-\infty} \times \mathcal{S}. \tag{3.D9}$$

Note that the above definition is confirmed by commutation rules [3.D5) as well as from the fact that the above mass operator is invariant and a Lorentz scalar, as it should be. In particular, the eigenvalue of the above mass operator is the conventional scalar m^2 (see Ref. [76] for details). For a number of additional intriguing features of the GRSA symmetry, such as the nonlocality of the position operator "spread over" an area of radius ℓ, we have to refer the interested reader to paper [76] for brevity.

In closing with personal comments and recollections of these studies conducted some 37 years ago, there is no doubt that the GRSA group has dramatically more dynamical capabilities than the conventional Poincaré group. Also, to my best recollection, we could find no experimental data contradicting the GRSA symmetry.

Yet, the novelty of the symmetry caused a real opposition furor among colleagues, namely, a reaction that has to be distinguished from proper scientific scrutiny. Part of the opposition was due to the political attachment to Einsteinian doctrines, but part was also due to the fact that the GRSA group required technical knowledge above the average of theoretical physicists of the time. [32]

[32] See the footnotes of Volume IV.

Such huge an opposition essentially forced the author to abandon the studies in the field, a decision that he regretted later, but could not change at that time due to the need in the 1970s for the author to secure an academic position so as to feed and shelter two children in tender age and his wife.

During the 37 years that have passed since that time, the author discovered numerous theories published in the best technical journals that, in reality, did verify the GRSA symmetry, but were published as verifying the conventional Poincaré symmetry. All attempts by the author for editorial corrections turned out as being useless. That was unfortunate for the fully deserved continuation of Paul Romans name in science.

In this way, the author was exposed for to the academic rage caused by novelty and, in so doing, he acquired the necessary strength to resist academic disruptions when he proposed the construction of hadronic mechanics in 1978 [4]. Also in this way, the human experience gained by the author during his studies of the GRSA symmetry and relativity proved as being crucial for the studies on hadronic mechanics againt hardly credible obstructions, oppositions and disruptions.

Yet, the author hopes that studies on the GRSA symmetry and relativity are indeed continued by new generations of physicists, not only because of the dramatic richness of content compared to the Poincaré sub-symmetry, but also because the GRSA symmetry and its easily derivable isotopic extension appear to possess the necessary ingredients for a solution of the numerous unresolved problems of special relativity, including compatibility with the ultimate frontier of knowledge: space.[33]

[33]In the author's view, these advances may happen only when society will one future day understand the importance of scientific democracy for qualified inquiries.

Appendix 3.E
Whitney's Studies of Lorentzian vs. Galilean Relativities

3.E.1 Foreword

In this appendix we report *ad litteram* the studies on the alternative between Lorentzian vs Galilean relativities conducted by Cynthia Kolb Whitney, Editor, Galilean Electrodynamics, 141 Rhinecliff Street, Arlington, MA 02476, email dwhitney@mit.edu

3.E.2 Introduction

The art of mathematical physics lies in modeling physical processes mathematically by introducing idealizations simple enough to make the mathematics not infeasible, while at the same time complete enough to avoid rendering the physics inapplicable. It is a tough job, and we will probably never complete it. The fact is: Nature is not constrained to adhere to *any* idealizations that we introduce. History has revealed this truth over and over. But here is a brief report on progress so far.

3.E.3 Newton

The first modern mathematical physicist was Sir Isaac Newton. Important features of Newton's theory include its Universal Time, which runs the same for all observers, regardless of any absolute motion or relative motion between them. That means Newton's theory embodies Galilean Relativity. That is why I begin with him.

In Newton's *Principia* [80], the universe of discourse consisted of material bodies, whether small like apples or large like planets, reduced to point particles, with reciprocal forces between such particles, and the orbits thereby created for the particles. This universe of discourse was in total contrast to that for scientists on the European continent, which consisted of a presumed fluid 'aether', with vortices within it that carried the particles in complicated vortical orbits.

The difference in world view embodies Newton's important contribution to natural philosophy: the idea that it is right and proper to stick to describing mathematically the observable facts, without injecting any unprovable mechanical explanations. Critics forever pressed Newton for such explanations for gravity. How could it act, at a distance, without any contact? In response he included with the second edition of his *Principia* the 'General Scholium', including the remarks: "But hitherto I have not been able to discover the cause of the properties of gravity from phenomena, and I frame no hypothesis; for whatever is not deduced from the phenomena is to be called a hypothesis; and hypotheses, whether metaphysical or physical, whether of occult qualities or mechanical, have no place in experimental philosophy... And in us it is enough that gravity does really exist, and act according to the laws which we have explained, and abundantly serves to account for all the motions of the celestial bodies, and our sea."

Or, perhaps more memorably, "*Hypothesis non fingo.*" This statement does not mean that such an explanation is *never* to be sought; it just means that the time for such explanation can only be later on, when more facts are known. And so until that time arrives, one should just do what one can better do. If that that does finally arrive, then hypotheses need no longer be avoided; they can be embraced and tested.

In Newton's day, his purely descriptive mathematical approach was exceedingly successful. It achieved an unexpected unification between terrestrial and celestial physics. It could solve in closed form any two-body problem, with any ratio of masses involved. Given modern computers, it can handle three, or however many more, bodies. One tiny detail that it cannot do is the perihelion advance for a planet in the solar system, which is actually observable for the planet nearest the Sun, Mercury. That is, Newton's theory gets most, but not all, of that perihelion advance. For this tiny problem, Newton's theory would one day yield to Einstein's General Relativity Theory (GRT). But more comment on that development comes later.

3.E.4 Maxwell

The next batch of phenomenology for mathematical physics to deal with was revealed with the discovery and study of electromagnetic phenomena. A lot of individuals were involved, but the one who really changed things was Maxwell. He achieved an amazing unification of electricity and magnetism into electromagnetic theory [81]. It is a little unclear if he knew what he had sacrificed to get there. There was no Galilean Relativity there. Did he realize that Universal Time was gone? We do not know.

Maxwell's universe of discourse included point particles, but it put more attention onto what was between the particles: electromagnetic fields. Some particles generated the fields, while other, much smaller particles, responded to the fields, as 'test particles', unable to react back on the sources. There is an asymmetry there: Maxwell's theory is not built for a two-body problem. Indeed, because of the radiation associated with acceleration, Maxwell's theory could not handle one particularly important two-body problem: the Hydrogen atom. In part because of that problem, a totally new branch of physics, Quantum Mechanics (QM), would arise. But more comment on that development comes later.

Near the end of his Treatise on Electricity and Magnetism, Maxwell referred to a letter from Gauss to Weber expressing the opinion that the real keystone of electrodynamics would be "the deduction of the force acting between electric particles in motion from the consideration of an action between them, not instantaneous, but propagated in time, in a similar manner to that of light." Gauss had not accomplished this, nor had Maxwell, nor had three others who had tried at the time Maxwell wrote; namely, Riemann, Clausius, and Betti. Maxwell attributed the lack of success of those three to prejudice against a hypothesis of a medium in which radiation of light and heat and electric action at a distance takes place. Maxwell was an aether man. Nevertheless, his later followers reformulated his theory without his aether, without his quaternion mathematics to represent that aether, and instead with the now-familiar field vectors. The feasibility of making the math description without requiring the aether hypothesis again illustrates Newton's point about *hypothesis non fingo*.

Later on, Liènard and Wiechert [82, 83] did something that seems to fulfill the description that Gauss envisioned: they formulated retarded potentials, from which retarded fields follow, and with the Lorentz force law, the retarded forces follow. Their approach embodied an idea later crystallized more clearly. The idea is this: Maxwell's theory involves parameters ε_0 and μ_0 for free-space electric permittivity and magnetic permeability. They have no dependence on source or observer motion. And they imply a wave speed $c = 1\big/\sqrt{\varepsilon_0\mu_0}$ that also cannot depend on source or observer motion. So the speed for potential and field retardation should also be c.

3.E.5 Einstein

Enter Einstein [84]. He elevated the idea that had emerged from Maxwell to the status of a Postulate — his famous 'Second Postulate', — which was the foundation for Special Relativity Theory (SRT). SRT does not have Galilean Relativity; it has Lorentzian relativity. Unlike Newton's theory, SRT does not have Universal Time; it has Relative Time. The idea of Relative Time is mind-boggling, and in fact leads to an extensive literature about 'paradoxes', especially about traveling twins, or trains, or clocks, or meter sticks, or buildings, *etc.*

Inasmuch as SRT is founded on Maxwell's theory, and Maxwell's theory cannot handle the Hydrogen atom, SRT is unlikely ever to be fully compatible with QM. Einstein was involved in the development of QM, through his Nobel-Prize winning work on the photoelectric effect, but he was not fond of QM, and in later years did not work so much on it. Instead, he mainly went back to SRT, embraced the Minkowski tensor formulation for it, and exploited the metric tensor therein to develop General Relativity Theory (GRT) [85].

GRT is believed to offer the explanatory hypothesis that Newton eschewed in saying "*Hypothesis non fingo*". GRT says that gravitational masses affect the metric tensor; *i.e.*, 'curve the spacetime', and responding masses travel paths that are straight in curved spacetime, or curved in flat spacetime.

Inasmuch as GRT is founded on SRT, and SRT is founded on Maxwell's theory, and Maxwell's theory cannot handle the Hydrogen atom, GRT is not likely ever to be fully compatible with QM. But scientists today do keep trying for that Holy Grail.

GRT has the same design weakness that Maxwell's theory: it is a field theory, and as such, is not designed for something so complicated as a two-body problem. Late in life, Einstein wrote about his misgivings in a letter to his friend Michel Angelo Besso: "I consider it quite possible that physics cannot be based on the field concept, *i.e.*, on continuous structures. In that case *nothing* remains of my entire castle in the air, gravitation theory included, [and the] rest of physics."

Maintaining such doubt is, I believe, the mark of a truly great scientist. Einstein's present-day followers generally do not harbor such doubts.

3.E.6 Reformulations

There have always been researchers questioning Einstein's Second Postulate, and evaluating alternatives to it. Ritz was an early [86], but not successful, example. Later, in the 1950's, began the work of P. Moon, D. Spencer, E. Moon, and many of Spencer's students [87–89]. Their work has been successful in producing a lot of very interesting results, if not in garnering all the recognition it deserves.

The key Moon-Spencer-Moon idea was a propagation process with continuing control by the source, even after the initiating 'emission' event, so that the light moves away from the source at speed c relative to that source, however arbitrarily the source itself may be moving. (This is *not* the Ritz postulate, which had the light moving at velocity $c + V$, where V was the velocity of the source at the moment of emission, and c is the velocity vector of the light if it had come from a stationary source at that moment.)

In any event, continuing control by the source implies that 'light', whatever it is, has a longitudinal extent (Of course! Light possesses wavelength, does it not?), and the longitudinal extent is expanding in time. That expansion naturally raises the question: exactly what *part* of the expanding light packet is it that moves at speed c relative to the source? The tacit hypothesis of Moon-Spencer-Moon is that the c-speed part is the leading tip of the light packet. It then follows that when a receiver is encountered, the entire longitudinal extent of the light packet must collapse instantly to the receiver. That means the trailing tail of the light packet must snap into the receiver at infinite speed. The infinite speed might be unacceptable for Einstein true believers, but maybe not for QM true believers.

3.E.7 Two Step Light

My own work [90–92] follows the Moon-Spencer-Moon lead, with one conceptual addition. My variation to the Moon-Spencer-Moon postulate is that the speed c relative to the source characterizes, not the leading tip of the light packet,

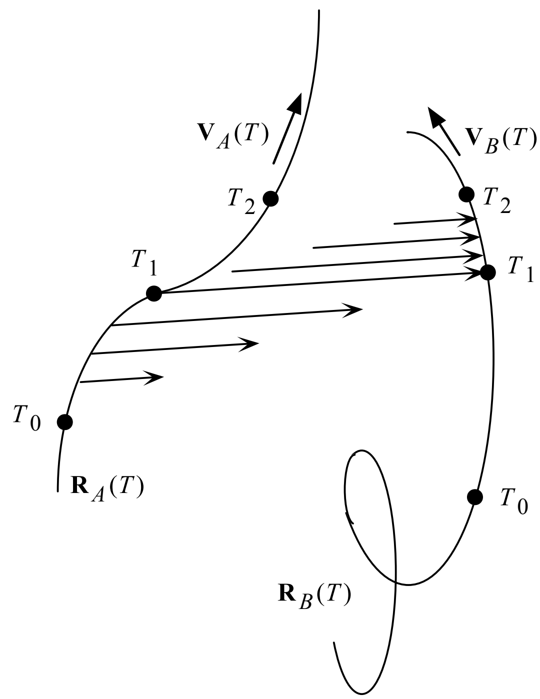

Figure 3.E.1. Illustration of Two-Step Light propagation, reprinted from [90] with permission.

but rather the mid point of the light packet. That means the leading tip must move relative to the source, not at c, but rather at $2c$. (A $2c$ anywhere is probably shocking to Einstein true-believers, but maybe not so shocking as an infinite speed would be.)

My variation on the Moon-Spencer-Moon theme introduces symmetry between light emission and absorption. The leading tip reaches the receiver in half the time for propagation at c, so there is time left for a completely symmetric absorption process, wherein the mid point of the light packet travels at speed c relative to the receiver, however arbitrarily that may move. That idea then means the tail end reels in at speed $2c$ relative to the receiver.

The fully revised light postulate is what I have called “Two-Step Light”. It is illustrated in Fig. 1. The T's are Universal Times: T_0 at the beginning of the scenario, T_1 at the mid point, and T_2 at the end. Particle A is the source, and particle B is the receiver (one of possibly many candidate receivers, selected by the accidental collision with the expanding light arrow at T_1). The mid points of the light arrows may be said to conform to the Moon-Spencer-Moon favored postulate in the expansion phase of the scenario, and then with the Einstein postulate in the collapse phase of the scenario.

How can light do all that? Stay in contact with a moving source? Switch control to a moving receiver? Stay in contact with a moving receiver? At this point, I resort to saying *hypothesis non fingo*. My first job is just to work out the implications of the Two-Step Light Postulate. It is a mundane task, involving no more than high-school algebra. It has been detailed in [90] and [91]. Here I shall just summarize results.

Consider the problem of processing data consisting of successive light signals from a moving source in order to estimate the speed V of that source. If the light propagates according to the Two-Step process, but the data gets processed under the assumption of the one-step Einstein postulate, then there will be a systematic error to the estimate. In fact, the estimate turns out to be

$$v = V/\left(1 + V^2/4c^2\right). \tag{3.E.1}$$

The estimate v is always less than V, and in fact is limited to c, which value occurs at $V = 2c$. Thus v has the property of any speed in Einstein's SRT. The obvious implication is that v is an Einsteinian speed, whereas V is a Galilean speed.

One is obviously invited to look also at a related construct

$$V^{\uparrow} = V/\left(1 - V^2/4c^2\right). \tag{3.E.2}$$

The superscript $\uparrow$ is present to call attention to the fact that $V^{\uparrow}$ has a singularity, which is located at $V = 2c$, or $v = c$. That is, $V^{\uparrow}$ has the property of the so-called 'covariant' or 'proper' velocity. Interestingly, past the singularity, it changes sign. This behavior mimics the behavior that SRT practitioners attribute to 'tachyons', or 'super-luminal particles': they are said to 'travel backwards in time'. The sign change is a mathematical description, while the 'travel backwards in time' is a literary description.

The relationships expressed by (3.E.1) and (3.E.2) can be inverted, to express V in terms of v or $V^{\uparrow}$. The definition $v = V/\left(1 + V^2/4c^2\right)$ rearranges to a quadratic equation

$$\left(v/4c^2\right) V^2 - V + v = 0,$$

which has solutions

$$V = \frac{1}{v/2c^2}\left(+1 \pm \sqrt{1 - v^2/c^2}\right).$$

Multiplying numerator and denominator by $\left(+1 \mp \sqrt{1 - v^2/c^2}\right)$ converts these to the form

$$V = v \Bigg/ \frac{1}{2}\left(1 \mp \sqrt{1 - v^2/c^2}\right), \tag{3.E.3}$$

which makes clear that for small v, V has one value much, much larger than v, and another value essentially equal to v.

The definition $V^{\uparrow} = V/\left(1 - V^2/4c^2\right)$ rearranges to a quadratic equation

$$\left(-V^{\uparrow}/4c^2\right) V^2 - V + V^{\uparrow} = 0,$$

which has solutions

$$V = \frac{1}{-V^{\uparrow}/2c^2}\left(+1 \pm \sqrt{1 - {V^{\uparrow}}^2/c^2}\right).$$

Multiplying numerator and denominator by $\left(+1 \mp \sqrt{1 + {V^{\uparrow}}^2/c^2}\right)$ converts these to the form

$$V = V^{\uparrow} \Big/ \frac{1}{2}\left(1 \mp \sqrt{1 + {V^{\uparrow}}^2/c^2}\right), \tag{3.E.4}$$

which makes clear that for small $V^{\uparrow}$, V has one value much larger in magnitude than $V^{\uparrow}$ (which is negative there), and another value essentially equal to $V^{\uparrow}$.

To see that v and $V^{\uparrow}$ are not only qualitatively *like* Einsteinian speed and covariant speed, but in fact quantitatively *equal* to them, one can do a bit more algebra. Substitute (3.E.3) into (3.E.2) and simplify to find

$$V^{\uparrow} = \mp v \Big/ \sqrt{1 - v^2/c^2}, \tag{3.E.5}$$

which is the definition of covariant speed familiar from SRT, made slightly more precise by inclusion of the minus sign for situations beyond the singularity.

Similarly, substitute (3.E.4) into (3.E.1) and simplify to find

$$v = \mp V^{\uparrow} \Big/ \sqrt{1 + V^{\uparrow}/c^2}, \tag{3.E.6}$$

which is again a relationship familiar from SRT, made slightly more precise by inclusion of the minus sign for situations beyond the singularity.

The information content of Eqs. (3.E.1)–(3.E.6) is displayed graphically in Fig. 2. Both plot axes denote multiples of nominal light speed c. Galilean particle speed V is the independent variable. To save space, it is the absolute value of $V^{\uparrow}$ that is plotted.

Speed stands here as a proxy for many other interesting things in SRT, like momentum, relativistic mass, *etc.* SRT only offers only two speed relationships; *i.e.*, (3.E.5) and (3.E.6), whereas Two Step Light offers six relationships; *i.e.* (3.E.1) through (3.E.6). This constitutes three times the information content. That means Two Step Light offers a lot more opportunities for better explaining all the interesting things in SRT.

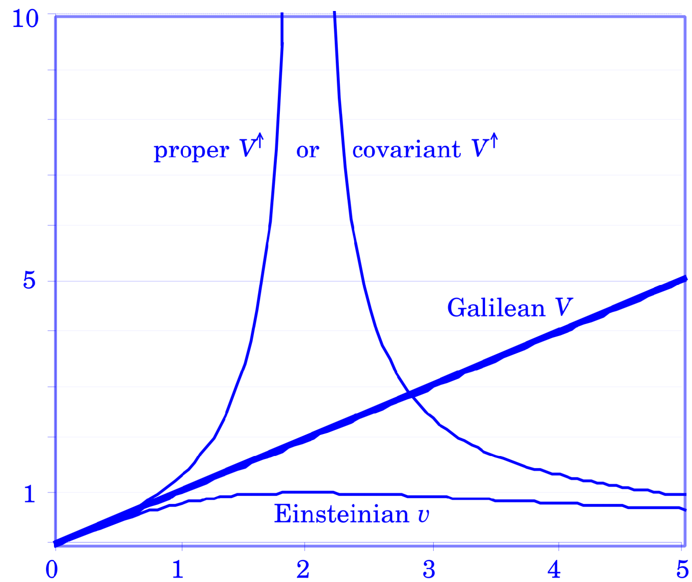

Figure 3.E.2. Numerical relationships among three speed concepts. Reprinted from [90] with permission.

3.E.8 Paradoxes Resolved

There are a great many peculiar-sounding results from SRT — that is why the word 'paradox' occurs so often in the SRT literature. But there are no paradoxes in physical reality, and there are none in Two Step Light theory. To illustrate this point, consider one rather obscure but very important case. I mentioned before the Liènard-Wiechert retarded potentials and fields and implied forces. They have a paradoxical property.

Expressed in Gaussian units [93], the Liènard-Wiechert scalar and vector potentials are

$$\Phi\left(\mathbf{x},t\right)=e\left[1/\kappa R\right]_{\text{retarded}} \quad \text{and} \quad A\left(\mathbf{x},t\right)=e\left[\boldsymbol{\beta}/\kappa R\right]_{\text{retarded}},$$

where $\kappa=1-\mathbf{n}\cdot\boldsymbol{\beta}$, with $\boldsymbol{\beta}$ being source velocity normalized by c, and $n=\mathbf{R}/R$ (a unit vector), and $\mathbf{R}=\mathbf{r}_{\text{observer}}\left(t\right)-\mathbf{r}_{\text{source}}\left(t-R/c\right)$ (an implicit definition for the terminology 'retarded'). The Liènard-Wiechert fields expressed in Gaussian units are then

$$\mathbf{E}\left(\mathbf{x},t\right)=e\left[\frac{\left(\mathbf{n}-\boldsymbol{\beta}\right)\left(1-\beta^2\right)}{\kappa^3R^2}+\frac{\mathbf{n}}{c\kappa^3R}\times\left(\left(\mathbf{n}-\boldsymbol{\beta}\right)\times\frac{d\boldsymbol{\beta}}{dt}\right)\right]_{\text{retarded}},$$

and $\mathbf{B}(\mathbf{x},t) = \mathbf{n}_{\text{retarded}} \times \mathbf{E}(\mathbf{x},t)$. The $1/R$ fields are radiation fields, and they make a Poynting vector that lies along $\mathbf{n}_{\text{retarded}}$:

$$\mathbf{P} = \mathbf{E}_{\text{radiative}} \times \mathbf{B}_{\text{radiative}} = \mathbf{E}_{\text{radiative}} \times (\mathbf{n}_{\text{retarded}} \times \mathbf{E}_{\text{radiative}}) = E^2_{\text{radiative}} \mathbf{n}_{\text{retarded}}.$$

But the $1/R^2$ fields are Coulomb-Ampère fields, and the Coulomb field does not lie along $\mathbf{n}_{\text{retarded}}$ as one might naively expect; instead, it lies along $(\mathbf{n} - \boldsymbol{\beta})_{\text{retarded}}$.

Consider the following scenario, designed specifically for an instructive exercise in *reductio ad absurdum*. A source executes a motion comprising two components: **1)** inertial motion at constant $\boldsymbol{\beta}$, plus **2)** oscillatory motion at small amplitude and high frequency, so that there exists a small velocity $\Delta\boldsymbol{\beta}_{\text{retarded}}$ and a not-so-small acceleration $d\Delta\boldsymbol{\beta}/dt|_{\text{retarded}}$. Observe that the radiation and the Coulomb attraction/repulsion come from different directions. The radiation comes along $\mathbf{n}_{\text{retarded}}$ from the retarded source position, but the Coulomb attraction/repulsion lies along $(\mathbf{n} - \boldsymbol{\beta})_{\text{retarded}}$, which is basically $(\mathbf{n}_{\text{retarded}})_{\text{projected}}$, and lies nearly along $\mathbf{n}_{\text{present}}$. This behavior seems peculiar. Particularly from the perspective of modern Quantum Electrodynamics (QED), all electromagnetic effects are mediated by photons — real ones for radiation and virtual ones for Coulomb-Ampere forces. How can these so-similar photons come from different directions?

Two-Step Light theory resolves the directionality paradox inherent in the Linard-Wiechert fields. Because of the various $2c$'s in the mathematics, the radiation direction $\mathbf{n}_{\text{retarded}}$ changes to $\mathbf{n}_{\text{half retarded}}$, and the Coulomb attraction/repulsion direction $(\mathbf{n}_{\text{retarded}})_{\text{projected}}$ changes to $(\mathbf{n}_{\text{retarded}})_{\text{half projected}}$. These two directions are now physically the same; namely the source-to-receiver direction at the mid point of the scenario, *i.e.* $\mathbf{n}_{\text{mid point}}$. The potentials and fields become:

$$\Phi(\mathbf{x},t) = e\,[1/R]_{\text{mid point}} \quad \text{and} \quad \mathbf{A}(\mathbf{x},t) = e\,[\mathbf{V}/cR]_{\text{mid point}},$$

$$\mathbf{E}(\mathbf{x},t) = e\left[\frac{\mathbf{n}}{R^2} + \frac{\mathbf{n}}{cR} \times \left(\mathbf{n} \times \frac{d\mathbf{V}}{cdt}\right)\right]_{\text{mid point}} \text{ and } \mathbf{B}(\mathbf{x},t) = \mathbf{n}_{\text{mid point}} \times \mathbf{E}(\mathbf{x},t).$$

What is so important about the field formulations consistent with Two Step Light is the forces that they imply in a two-body system, such as the Hydrogen atom. The attractive forces are not central. They impose a torque on the system, and through that, a mechanism for energy input into the system. This can work against the energy loss due to radiation reaction. This can provide an approach for understanding atoms that is completely different from QM. One need not postulate the value of Planck's constant and the nature of its involvement in the mathematics of 'probability' waves. One can derive Planck's constant. And one can uncover a tremendous amount of previously unrecognized regularity in chemical data. Refs. [91, 92] go into all this in some detail.

3.E.9 Conclusions

About SRT: A symbol is missing from the language of SRT (namely, the Galilean speed V). As a result, Einsteinian speed v often gets conflated with Galilean speed V. Any conflation of physical concepts can cause confusion and misinterpretations of results. That is why the SRT literature has to discuss so many 'paradoxes'.

About Two Step Light: Two Step Light is a 'covering' theory; it contains all the variables and relationships familiar from SRT, but it also contains other variables and relationships as well. Users who are comfortable with the familiar need not give anything up, and users who are curious about the rest can readily make use of it.

About relativities: If one accepts Two Step Light as an explanation for SRT, then one can describe any situation of interest in terms of Galiean V and Galilean coordinate transformations. That is, one is free to use Galilean relativity rather than Lorentzian relativity if one wishes.

About QM: Like SRT, QM has required unnecessary abandonment of rationality. And there is a lot of phenomenology out there that simply is not treated by present-day QM. So it is worth re-doing QM in a different way.

About philosophy: Today's QM is rightly understood as a theory not so much of 'things', but rather of 'knowledge': what we can 'know', given our means of knowing anything about what 'is'. SRT should be understood that way too. It isn't necessarily about what 'is'; it is about what we *think*, given what data we can take, and what algorithms we allow ourselves to apply in processing that data.

Appendix 3.F
Rapoport Studies on Geometry, Torsion, Statistics, Diffusion and Isotopies

3.F.1 Introduction

In this appendix we report *ad litteram* the studies on geometry, torsion, statistics, diffusion and isotopies by Diego Lucio Rapoport of the Department of Sciences and Technology, Universidad Nacional de Quilmes, Buenos Aires, Argentina, email ¡diego.rapoport@gmail.com¿.

It is appropriate to start by quoting Prof. Santilli (see Section 6.1, Volume IV of this series): "a first meaning of the novel hadronic mechanics is that of providing the first known methods for quantitative studies of the interplay between matter and the underlying substratum. The understanding is that space is the final frontier of human knowledge, with potential outcomes beyond the most vivid science fiction of today". In this almost prophetic observation, Prof. Santilli has pointed out the essential role of the substratum, its geometrical structure and the link with consciousness. In the present appendix, which we owe to the kind invitation of Prof. Santilli, we shall present similar views, specifically in presenting both quantum and hadronic mechanics as space-time fluctuations, and we shall discuss the role of the substratum. As for the problem of human knowledge, we shall very briefly indicate on how the present approach may be related to the fundamental problem of consciousness, which is that of self-reference.

A central problem of contemporary physics is the distinct world views provided by QM and GR (short for quantum mechanics and general relativity, respectively), and more generally of gravitation. In a series of articles [94–97, 115] and references therein, we have presented an unification between space-time structures, Brownian motions, fluid dynamics and QM. The starting point is the unification of space-time geometry and classical statistical theory, which has been possible due to a complementarity of the objects characterizing the Brownian motion, i.e. the noise tensor which produces a metric, and the drift vector field which describes the average velocity of the Brownian, in jointly describing both the space-time geometry and the stochastic processes. These space-time structures can be defined starting from flat Euclidean or Minkowski space-time, and they have in addition to a metric a torsion tensor which is formed from the metric conjugate of the drift vector field. The key to this unification lies in that the Laplacian operator defined by this geometrical structure is the differential generator of the Brownian motions; stochastic analysis which deals with the transformation rules of classical observables on diffusion paths ensures that this unification is valid in both directions [116]. Thus, in this equivalence, one can choose the Brownian motions as the original structures determining a space-time structure, or conversely, the space-time structures produce a Brownian motion process. Space-time geometries with torsion have lead to an extension of the theory of gravitation which was first explored in joint work by Einstein with Cartan [98], so that the foundations for the gravitational field, for the special case in which the torsion reduces to its trace, can be found in these Brownian motions. Furthermore, in [95] we have shown that the relativistic quantum potential coincides, up to a conformal factor, with the metric scalar curvature. In this setting we are lead to conceive that there is no actual propagation of disturbances but instead an holistic modification of the whole space-time structure due to an initial perturbation which provides for the Brownian process modification of the original configuration. Furthermore, the present theory which has a kinetic Brownian motion generation of the geometries, is related to Le Sage's proposal of a Universe filled with all pervading tiny particles moving in all directions as a pushing (in contrast with Newton's pulling force) source for the gravitational field [129]. Le Sage's perspective was found to be compatible with cosmological observations by H. Arp [130]. This analysis stems from the assumption of a non-constant mass in GR which goes back to Hoyle and Narlikar, which in another perspective developed by Wu and Lin generates rotational forces [131]. These rotational forces can be ascribed to the drift trace-torsion vector field of the Brownian processes through the Hodge duality transformation [96], or still to the vorticity generated by this vector field. In our present theory, motions in space *and* time are fractal, they generate the gravitational field, and furthermore they generate rotational fields, in contrast with the pulling force of Newton's theory and the pushing force of Le Sage, or in the realm of the neutron, the Coulomb force. Furthermore, in our construction the drift has built-in terms given by the conjugate of electromagnetic-like potential 1-forms, whose associated intensity two-form generate vorticity, i.e. angular momentum; these terms include the Hertz potential which is the basis for the construction of superluminal solutions of Maxwell's equations; see [95] and references therein. So the present geometries are very different from the metric geometries of general relativity and are not in conflict with present cosmological observations.

The space-time geometrical structures of this theory can be introduced by the Einstein λ transformations on the tetrad fields [98, 95], from which the usual Weyl scale transformations on the metric can be derived, but contrarily to Weyl geometries, these structures have torsion and they are integrable in contrast with Weyl's theory; we have called these connections as RCW structures (short for Riemann-Cartan-Weyl) [94–97]. This construction is a special case of the construction of

Riemannian or Lorentzian metrics presented in Section 3.5.3, in which Santilli generalized isotopic unit takes a diagonal form with equal elements given by (the square of) a scale function, while the number field, the differential and integral calculus are the usual ones of practice in differential geometry; these restrictions will be lifted to work with a full isotopic theory for HM in extending the theory developed for QM; in distinction with HM, the usual scale transformations do not depend on anything but the space-time coordinates, thus excluding the more general non-linear non-hamiltonian case contemplated by HM. In distinction with GR which due to the lack of a source leads to inconsistencies discussed in Section 1.4, a theory based on torsion and in particular in the case of a so-called absolute parallelism in which the torsion is derived from the differential of the cotetrad field (the so-called Weitzenbock spaces), has a geometrically defined energy-momentum tensor which is built from the torsion tensor [113, 134]. Furthermore, the trace-torsion has built-in electromagnetic potential terms. We must recall that in Section 1.4 it was proved that gravitational mass has partially an electromagnetic origin. So our original setup in terms of torsion fields which can be non-null in flat Minkowski or Euclidean spaces (while in these spaces curvature is null), does not lead in principle to the inconsistencies observed before. There are other differences between the present approach and GR which we would like to discuss. In the latter theory, the space-time structure is absolute in the sense that it is defined without going through a self-referential characterization. With the introduction of torsion, and especially in the case of the trivial metric with null associated curvature tensor, we are introducing a self-referential characterization of the geometry since the definition of the manifold by the torsion, is through the concept of locus of a point (be that temporal or spatial). Indeed, space and time can only be distinguished if we can distinguish inhomogenities, and this is the intent of torsion, to measure the dislocation (in space and time) in the manifold [142]. Thus all these theories stem from a geometrical operation which has a logical background related to the concept of distinction (and more fundamentally, the concept of identity, which is prior to that of distinction) and its implementation through the operation of comparison by parallel transport with the affine connection with non-vanishing torsion.[34] In comparison, in GR there is also an operation of distinction carried out by the parallel transport of pair of vector fields with the Levi-Civita metric connection yielding a trivial difference, i.e. the torsion is null and infinitesimal parallelograms trivially close, so that it does not lead to the appearence of inhomogenities as resulting from this primitive

[34]This can be further related with multivalued logics and the appearance of time waves related to paradoxes, which in a cognitive systems approach yields the Schroedinger representation; furthermore this conception leads to the notion of reentrance of a space-time domain into itself, as a self-referential cybernetic system, and ultimately to multidimensional time; this may ultimately be linked to semiotics and its role in biology [134].

operation of distinction; these are realized through the curvature derived from the metric. But to close this discussion, we refer again to the inconsistencies that an approach based on the curvature viz a viz the present approach which places the appearence of spacetime in terms of deformations of the vacuum, and as such, has the same genesis as Isorelativity developed by Prof. Santilli and presented in Section 3.5.5.

We have shown that this approach leads to non-relativistic QM both in configuration space [96] and in the projective Hilbert state-space through the stochastic Schroedinger equation [98] (in the latter case, it was proved shown that this geometry is related to the reduction of the wave function can be described by decoherence through noise [96, 98]), and further to Maxwell's equation and its equivalence with the Dirac-Hestenes equation of relativistic QM [95, 115]. The fact that non-relativistic QM can be linked to torsion fields was unveiled recently [96]. In fact, torsion fields have been considered to be as providing deviations of GR outside the reach of present precision measurements [112]. It turns out that quantum wave-functions verifying linear or non-linear Schroedinger equations are another universal, or if wished, mundane examples of torsion fields. We shall show in the present article, that this approach extends to the strong interactions as described by HM and thus that the isotopic lift of the Schroedinger wave function is also a source for torsion, albeit one which incorporates the full non-linearity and non-hamiltonian character of the strong interactions. The quantum random ensembles which generate the quantum geometries, or which dually can be seen as generated by them, in the case of the Schroedinger equation can be associated with harmonic oscillators with disordered random phase and amplitude first proposed by Planck, which have the same energy spectrum as the one derived originally by Schroedinger [146]. The probabilities of these ensembles are classical since they are associated with classical Brownian motions in the configuration and projective Hilbert-state manifolds, in sharp contrast with the Copenhagen interpretation of QM which is constructed in terms of single system description, and they are related to the scalar amplitude of the spinor field in the case of the Dirac field, and in terms of the modulus of the complex wave function in the non-relativistic case [95, 96, 115]. We would like to recall at this stage that Khrennikov has proved that Kolmogorov's axiomatics of classical probability theory, in a contextual approach which means an a-priori consideration of a complex of physical conditions, permits the reconstruction of quantum theory [117]. Thus, Khrennikov's theory places the validity of quantum theory in ensembles, in distinction with the Copenhagen interpretation, and is known as the Vaxho interpretation of quantum mechanics. In the present approach we obtain both a geometrical characterization of the quantum domain through random ensembles performing Brownian motions which generate the space and time geometries, and additionally a characterization for single systems through the topological

Bohr-Sommerfeld invariants associated with the trace-torsion by introducing the concept of Pfaffian system developed by Kiehn in his geometro-topological theory of processes [132], specifically applied to the trace-torsion one-form [134]. Most remarkably, in our setting another relevant example of these space-time geometries is provided by viscous fluids obeying the invariant Navier-Stokes equations of fluid-dynamics, or alternatively the kinematical dynamo equation for the passive transport of magnetic fields on fluids [94, 97]. This is of importance with respect to cosmology, since cosmological observations have registered turbulent large-scale structures which are described in terms of the Navier-Stokes equations [135].

There have been numerous attempts to relate non-relativistic QM to diffusion equations; the most notable of them is Stochastic Mechanics due to Nelson [102]. Already Schroedinger proposed in 1930–32 that his equation should be related to the theory of Brownian motions (most probably as a late reaction to his previous acceptance of the single system probabilistic Copenhagen interpretation), and further proposed a scheme he was not able to achieve, the so-called interpolation problem which requires to describe the Brownian motion and the wave functions in terms of interpolating the initial and final densities in a given time-interval [102]. More recently Nagasawa presented a solution to this interpolation problem and further elucidated that the Schroedinger equation is in fact a Boltzmann equation [107], and thus the generation of the space and time structures produced by the Brownian motions has a statistical origin.[35] Neither Nagasawa nor Nelson presented these Brownian motions as space-time structures, but rather as matter fields *on* the vacuum.[36] Furthermore, Kiehn has proved that the Schroedinger equation in spatial 2D can be exactly transformed into the Navier-Stokes equation for a compressible fluid, if we further take the kinematical viscosity ν to be $\frac{\hbar}{m}$ with m the mass of the electron [105]. We have argued in [96] that the Navier-Stokes equations share with the Schroedinger equation, that both have a RCW geometry at their basis: While in the Navier-Stokes equations the trace-torsion

[35]We have discussed in [96] that the solution of the interpolation problem leads to consider time to be more than a classical parameter, but an active operational variable, as recent experiments have shown [136] which have elicited theoretical studies in [145]; other experiments that suggest an active role of time are further discussed in [96].

[36]Another developments following Nelson's approach, in terms of an initial fractal structure of space-time and the introduction of Nelson's forward and backward stochastic derivatives, was developed by Nottale in his Scale Theory of Relativity [114]. Remarkably, his approach has promoted the Schroedinger equation to be valid for large scale structures, and predicted the existence of exo-solar planets which were observationally verified to exist [106]. This may further support the idea that the RCW structures introduced in the vacuum by scale transformations, are valid independently of the scale in which the associated Brownian motions and equations of QM are posited. Nottale's covariant derivative operator turns to be a particular case of our RCW laplacian [96]. We would like to mention also the important developments of a theory of space-time with a Cantorian structure being elaborated in numerous articles by M. El Naschie [137] and a theory of fractals and stochastic processes of QM which has been elaborated by G. Ord [138].

is $\frac{-1}{2\nu}u$ with u the time-dependent velocity one-form of the viscous fluid, in the Schroedinger equation, the trace-torsion one-form incorporates the logarithmic differential of the wave function – just like in Nottale's theory [114] – and further incorporates electromagnetic potential terms in the trace-torsion one-form. This correspondence between trace-torsion one-forms is what lies at the base of Kiehn's correspondance, with an important addendum: While in the approach of the Schroedinger equation the probability density is related to the Schroedinger scale factor (in incorporating the complex phase) and the Born formula turns out to be a formula and not an hypothesis, under the transformation to the Navier-Stokes equations it turns out that the probability density of non-relativistic quantum mechanics, is the enstrophy density of the fluid, i.e. the square of the vorticity, which thus plays a *geometrical* role that substitutes the probability density. Thus, in this approach, while there exist virtual paths sustaining the random behaviour of particles (as is the case also of the Navier-Stokes equations) and interference such as in the two-slit experiments can be interpreted as a superposition of Brownian paths [107], the probability density has a purely geometrical fluid-dynamical meaning. This is of great relevance with regards to the fundamental role that the vorticity, i.e. the fluid's particles angular-momentum has as an organizing structure of the geometry of space and time. In spite that the torsion tensor in this theory is naturally restricted to its trace and thus generates a differential one-form, in the non-propagating torsion theories it is interpreted that the vanishing of the completely skew-symmetric torsion implies the absence of spin and angular momentum densities [112], it is precisely the role of the vorticity to introduce angular momentum into the present theory.

To explain the fundamental kinematical role of torsion in QM and classical mechanics of systems with Lie group symmetries, we note that if we consider as configuration space a Lie group, there is a canonical connection whose torsion tensor coefficients are non other than the coefficients of the Lie-algebra under the Lie bracket operation [128]. Thus a Lie group symmetry is characterized by the torsion tensor for the canonical connection. Thus the Lie-Santilli isotopic theory implies a deformation of the torsion tensor of the canonical connection by the generalized unit [19, 20, 22, 46, 73, 108–110].[37] With regards to another role of torsion in classical mechanics, it appears as describing friction, or more generally, non-anholonomic terms which produce additional terms in the equations of motion, which were obliterated by contemporary physics with the exception of Birkhoffian mechanics and discussed in Sections 1.2.4, 3.1, 3.3 and 4.1.2 by Prof. Santilli, which originated in the monographs [150]. In fact the attention

[37]The introduction of this generalized unit, in contrast with the basic unit of mathematics and physics, establishes a relation between these new units and physical processes which is unknown to mathematics, and is presently developed in terms of an arithmetic of forms which follows from the principle of distinction previously alluded, the multivalued logics associated to it and self-reference [134].

of this author to HM at an early stage, stemmed from his work (jointly with S. Sternberg) in classical mechanical systems with angular momentum, which could be formulated without lagrangians nor hamiltonians, and furthermore could not be reduced to the canonical form of conservative systems [155]. Further in common to HM and torsion geometries, is that the latter are associated to angular momentum densities [112], while in HM the isotopic unit incorporates spin-up spin-down couplings such as in the Rutherford-Santilli model of the neutron [108, 141, 70]. Possible relations between torsion as spin or angular momentum densities can be ventured in relation with anomalous spin interactions of the proton, and magnetic resonances [139]. Furthermore, it has been shown that completely skew-symmetric torsion can produce a spin flip of high energy fermionic matter at very high densities, and that in this situation helicity can be identified with spin [133]. An intrinsic macroscopic angular momentum would be the evidence of this phenomena.This may be of relevance when taking in consideration the time periodicity of the fine structure of histograms and its relation to macroscopic angular momentum which we have discussed in [96] and others we shall discuss in this article.

To understand the need of carrying the extensions produced by the isotopic lifts, it is based in the fact that the isotopic lift of Relativity due to Santilli (see [73]) is applicable for the electromagnetic and weak interactions but not applicable for the case of hadrons. These have a charge radius of 1 fm (10^{-13} cm) which is the radius of the strong interactions. Unlike the electromagnetic and weak interactions a necessary condition to activate the strong interaction is that hadrons enter into a condition of mutual interpenetration. In view of the developments below, we would like to stress that the modification of the symmetries of particles under conditions of possible fusion, is the first step for the usual developments of fusion theories which have been represented in terms of diffusion processes that overcome the Coulomb repulsive potential which impedes the fusion [122]; Brownian motions and other stochastic processes also appear in a phenomenological approach to the many body problem in particle and nuclear physics, but with no hint as to the possibility of an underlying space-time structure [151]. The basic idea goes back to the foundational works of Smoluchowski (independently of A. Einstein's work in the subject) in Brownian motion [123]. In the case of fusion theories, we have a gas of neutrons (which have an internal structure) and electrons, or an hadron gas; in these cases the fused particles are considered to be alike a compressible fluid with an unstable neck in its fused drops which have to be stabilized to achieve effective fusion; we can see here the figure of deformed symmetries. Thus, the situation for the application of Brownian motion to fusion is a natural extension to the subatomic scale of the original theory. We finally notice that the models for fusion in terms of diffusion do not require QM nor QCD [122]. In contrast, HM stems from symmetry group transformations that describe the contact fusion processes that deform the neutron structure, and lead to the isotopic Schroedinger equation which in this article, together with the isotopic Heisenberg representation, will be applied to establish a link between the RCW geometries, fusion processes and diffusions. The reason for the use of the iso-Heisenberg representation, is that in Santilli's theory, the isotopic lift of the symmetries in carried out in terms of the Heisenberg-Santilli isorepresentation, where its connection with classical mechanics under the quantization rules including the isotopic lift is transparent. Similarly to QM it will turn out to be that this quantization that leads to HM can be framed in another terms, i.e. Brownian motions appear to be quantum representations with no need of a quantization of classical mechanics, which can nevertheless be achieved by taking in account the fluctuations represented by the noise tensor of these random motions.

3.F.2 Riemann-Cartan-Weyl Geometries

In this section we follow [94, 95]. In this appendix M denotes a smooth connected compact orientable n-dimensional manifold (without boundary). While in our initial works, we took for M to be space-time, there is no intrinsic reason for this limitation, in fact if can be an arbitrary configuration manifold and still a phase-space associated to a dynamical system. The paradigmatical example of the latter, is the projective space associated to a finite-dimensional Hilbert-space of a quantum mechanical system [96, 98]. We shall further provide M with an affine connection, or still by a covariant derivative operator ∇ which we assume to be compatible with a given metric g on M, i.e. $\nabla g = 0$. Here, the metric can be the Minkowski degenerate metric, or an arbitrary positive-definite (i.e. Riemannian) metric. Given a coordinate chart (x^α) $(\alpha = 1, \ldots, n)$ of M, a system of functions on M (the Christoffel symbols of ∇) are defined by $\nabla_{\frac{\partial}{\partial x^\beta}} \frac{\partial}{\partial x^\gamma} = \Gamma(x)^\alpha_{\beta\gamma} \frac{\partial}{\partial x^\alpha}$. The Christoffel coefficients of ∇ can be decomposed as:

$$\Gamma^\alpha_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} + \frac{1}{2} K^\alpha_{\beta\gamma}. \tag{3.F.1}$$

The first term in (3.F.1) stands for the metric Christoffel coefficients of the Levi-Civita connection ∇^g associated to g, i.e. $\left\{ {\alpha \atop \beta\gamma} \right\} = \frac{1}{2}(\frac{\partial}{\partial x^\beta} g_{\nu\gamma} + \frac{\partial}{\partial x^\gamma} g_{\beta\nu} - \frac{\partial}{\partial x^\nu} g_{\beta\gamma}) g^{\alpha\nu}$, and

$$K^\alpha_{\beta\gamma} = T^\alpha_{\beta\gamma} + S^\alpha_{\beta\gamma} + S^\alpha_{\gamma\beta} \tag{3.F.2}$$

is the cotorsion tensor, with $S^\alpha_{\beta\gamma} = g^{\alpha\nu} g_{\beta\kappa} T^\kappa_{\nu\gamma}$, and $T^\alpha_{\beta\gamma} = (\Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta})$ is the skew-symmetric torsion tensor. We are interested in (one-half) the Laplacian operator associated to ∇, i.e. the operator acting on smooth functions on M defined as

$$H(\nabla) := 1/2 \nabla^2 = 1/2 g^{\alpha\beta} \nabla_\alpha \nabla_\beta. \tag{3.F.3}$$

A straightforward computation shows that $H(\nabla)$ only depends in the trace of the torsion tensor and g, since it is

$$H(\nabla) = 1/2\triangle_g + \hat{Q} \equiv H(g, Q), \tag{3.F.4}$$

with $Q := Q_\beta dx^\beta = T^\nu_{\nu\beta} dx^\beta$ the trace-torsion one-form and $\hat{Q}$ is the vector field associated to Q via g (the so-called g conjugate vector field to the one-form Q, i.e.

$$\hat{Q}(f) = g(Q, df), \tag{3.F.5}$$

for any smooth function f defined on M. Finally, $\triangle_g$ is the Laplace-Beltrami operator of g:

$$\triangle_g = g^{\alpha\beta}\nabla^g_{\frac{\partial}{\partial x^\alpha}}\nabla^g_{\frac{\partial}{\partial x^\beta}} = g^{\alpha\beta}\frac{\partial^2}{\partial x^\alpha \partial x^\beta} - g^{\alpha\beta}\begin{Bmatrix} \gamma \\ \alpha\beta \end{Bmatrix}\frac{\partial}{\partial x^\gamma}. \tag{3.F.6}$$

In this expression the partial derivatives are taken with respect to the Levi-Civita connection. Therefore, assuming that g is non-degenerate, we have defined a one-to-one mapping

$$\nabla \rightsquigarrow H(g, Q) = 1/2\triangle_g + \hat{Q}$$

between the space of g-compatible linear connections ∇ with Christoffel coefficients of the form

$$\Gamma^\alpha_{\beta\gamma} = \begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} + \frac{2}{(n-1)}\left\{\delta^\alpha_\beta\, Q_\gamma - g_{\beta\gamma}\, Q^\alpha\right\}, \quad n \neq 1 \tag{3.F.7}$$

and the space of elliptic second order differential operators on functions. The extensions of this laplacian to differential forms and in particular, to fluid-dynamics, has been presented in [94, 97].

3.F.3 Riemann-Cartan-Weyl Diffusions

In this section we shall recall the correspondence between RCW connections defined by (3.F.7) and diffusion processes of scalar fields having $H(g, Q)$ as its diffrential generator. Thus, naturally we have called these processes as *RCW diffusion processes.* For the extensions to describe the diffusion processes of differential forms, see [94, 97]. For the sake of generality, in the following we shall further assume that $Q = Q(\tau, x)$ is a time-dependent 1-form. In this setting τ is the universal time variable due to Stuckelberg [101]; for a very sharp account of the relation of this time to Einsten's time, t, we refer to Horwitz et al. [118]. The stochastic flow associated to the diffusion generated by $H(g, Q)$ has for sample paths the continuous curves $\tau \mapsto x(\tau) \in M$ satisfying the Itô invariant non-degenerate s.d.e. (stochastic differential equation)

$$dx(\tau) = \sigma(x(\tau))dW(\tau) + \hat{Q}(\tau, x(\tau))d\tau. \tag{3.F.8}$$

In this expression, $\sigma : M \times R^m \to TM$ is such that $\sigma(x) : R^m \to TM$ is linear for any $x \in M$, the noise tensor, so that we write $\sigma(x) = (\sigma_i^\alpha(x))$ $(1 \leq \alpha \leq n, 1 \leq i \leq m)$ which satisfies

$$\sigma_i^\alpha \sigma_i^\beta = g^{\alpha\beta}, \tag{3.F.9}$$

where $g = (g^{\alpha\beta})$ is the expression for the metric in covariant form, and $\{W(\tau), \tau \geq 0\}$ is a standard Wiener process on R^m, with zero mean with respect to the standard centered Gaussian function, and covariance given by $\mathrm{diag}(\tau, \ldots, \tau)$; finally, $dW(\tau) = W(\tau + d\tau) - W(\tau)$ is an increment. Now, it is important to remark that m can be arbitrary, i.e. we can take noise tensors defined on different spaces, and obtain the essentially the same diffusion process [116]. In regards to the equivalence between the stochastic and the geometric picture, this enhances the fact that there is a freedom in the stochastic picture, which if chosen as the originator of the equivalence, points out to a more fundamental basis of the stochastic description. This is satisfactory, since it is impossible to identify all the sources for noise, and in particular those coming from the vacuum, which we take as the source for the randomness. Note that in taking the drift and the diffusion tensor as the original objects to build the geometry, the latter is derived from objects which are associated to *collective* phenomena. Note that if we start with Eq. (3.F.8), we can reconstruct the associated RCW connection by using Eq. (3.F.9) and the fact that the trace-torsion is the g-conjugate of the drift, i.e., in simple words, by lowering indexes of $\hat{Q}$ to obtain Q. We shall not go into the details of these constructions, which relies heavily on stochastic analysis on smooth manifolds [116].

Observations 1. Note that in the above construction of the s.d.e. all terms corresponding to the Levi-Civita connection $\left\{ {\alpha \atop \beta\gamma} \right\}$ have disappeared completely. In fact one can start with a Laplacian written without these terms, say

$$H := 1/2 g^{\alpha\beta} \frac{\partial^2}{\partial x^\alpha \partial x^\beta} + \hat{Q}^\alpha \partial_\alpha, \tag{3.F.10}$$

and rewrite it as

$$\frac{1}{2} \triangle_g + \tilde{b}^\alpha \partial_\alpha \tag{3.F.11}$$

with

$$\tilde{b}^\alpha = \hat{Q}^\alpha + \frac{1}{2} g^{\beta\gamma} \left\{ {\alpha \atop \beta\gamma} \right\}; \tag{3.F.12}$$

we then redefine the connection $\nabla = (\Gamma^\gamma_{\alpha\beta})$ to be compatible with g and such that $\tilde{b}^\alpha = \frac{1}{2}[g^{\beta\gamma} \left\{ {\alpha \atop \beta\gamma} \right\} - \Gamma^\alpha_{\beta\gamma}]$ so that finally our original RCW laplacian $H(\nabla)$ takes the form $H(g, \tilde{b})$ of Eq. (3.F.4) and the s.d.e. is given by (3.F.8); c.f. pages 285–289 in Ideda & Watanabe [116]. From this follows that we can write the laplacians either with the Levi-Civita covariant derivative or the usual derivative for characterizing the diffusion processes corresponding to the Schroedinger equation; this is also valid for the iso-Schroedinger equations, starting by producing the isotopic lift of the differential operator, or further, the isotopic lift of the covariant derivative operator, the isocovariant differential introduced in Section 3.2.9.C above.

3.F.4 RCW Geometries, Brownian Motions and the Schroedinger Equation

We have shown that we can represent the space-time quantum geometries for the relativistic diffusion associated with the invariant distribution, so that $Q = \frac{1}{2}d\ln\rho$, with $\rho = \psi^2$ and $H(g, Q)$ has a self-adjoint extension for which we can construct the quantum geometry on state-space and still the stochastic extension of the Schroedinger equation defined by this operator on taking the analytical continuation on the time variable for the evolution parameter [96]. In this section which retakes the solution of the Schroedinger problem of interpolation by Nagasawa [107], we shall present the equivalence between RCW geometries, their Brownian motions and the Schroedinger equation which is a different approach to taking the analytical continuation in time, which by the way, has a very important significance in terms of considering time to be an active variable; see [96]. We shall now present the construction of non-relativistic QM with the restriction that the Hodge decomposition of the trace-torsion restricts to its exact component, excluding thus the electromagnetic potential terms of the full trace-torsion which we considered in [95, 96]. So that we take $Q = Q(t, x) = d\ln f_t(x)$ where $f(t, x) = f_t(x)$ is a function defined on the configuration manifold given by $[a, b] \times M$, where M is a 3-dimensional manifold provided with a metric, g. The construction applies as well to the general case as well, as we shall show further below. The scheme to determine f will be to manifest the time-reversal invariance of the Schroedinger representation in terms of a forward in time diffusion process and its time-reversed representation for the original equations for creation and annihilation diffusion processes produced when there is no background torsion field, whose explicit form and relation to f we shall determine in the sequel. From now onwards, the exterior differential, the divergence operator and the laplacian will act on the M manifold variables only, so that we shall write their action on fields, say $df_t(x)$, to signal that the exterior differential acts only on the x variables of M. We should remark that in this context, the time-variable t of non-relativistic theory and the evolution parameter τ, are identical [118]. Let

$$L = \frac{\partial}{\partial t} + \frac{1}{2}\triangle_g = \frac{\partial}{\partial t} + H(g, 0). \tag{3.F.13}$$

Let $p(s, x; t, y)$ be the weak fundamental solution of

$$L\phi + c\phi = 0. \tag{3.F.14}$$

The interpretation of this equation as one of creation (whenever $c > 0$) and annihilation ($c < 0$) of particles is warranted by the Feynman-Kac representation for the solution of this equation [107]. Then $\phi = \phi(t, x)$ satisfies the equation

$$\phi(s, x) = \int_M p(s, x; t, y)\phi(t, y)dy, \tag{3.F.15}$$

where for the sake of simplicity, we shall write in the sequel $dy = \mathrm{vol_g(y)} = \sqrt{\det(\mathrm{g})}dy^1 \wedge \ldots \wedge dy^3$. Note that we can start for data with a given function $\phi(a, x)$, and with the knowledge of $p(s, x; a, y)$ we define $\phi(t, x) = \int_M p(t, x; a, y)dy$. Next we define

$$q(s, x; t, y) = \frac{1}{\phi(s, x)} p(s, x; t, y)\phi(t, y), \tag{3.F.16}$$

which is a transition probability density, i.e.

$$\int_M q(s, x; t, y)dy = 1, \tag{3.F.17}$$

while

$$\int_M p(s, x; t, y)dy \neq 1. \tag{3.F.18}$$

Having chosen the function $\phi(t, x)$ in terms of which we have defined the probability density $q(s, x; t, y)$ we shall further assume that we can choose a second bounded non-negative measurable function $\breve{\phi}(a, x)$ on M such that

$$\int_M \phi(a, x)\breve{\phi}(a, x)dx = 1, \tag{3.F.19}$$

We further extend it to $[a, b] \times M$ by defining

$$\breve{\phi}(t, y) = \int \breve{\phi}(a, x)p(a, x; t, y)dx, \forall (t, y) \in [a, b] \times M, \tag{3.F.20}$$

where $p(s, x; t, y)$ is the fundamental solution of Eq. (3.F.14).

Let $\{X_t \in M, \mathcal{Q}\}$ be the time-inhomogeneous diffusion process in M with the transition probability density $q(s, x; t, y)$ and a prescribed initial distribution density

$$\mu(a, x) = \breve{\phi}(t = a, x)\phi(t = a, x) \equiv \breve{\phi}_a(x)\phi_a(x). \tag{3.F.21}$$

The finite-dimensional distribution of the process $\{X_t \in M, t \in [a, b]\}$ with probability measure on the space of paths which we denote as Q; for $a = t_0 < t_1 <$

$\ldots < t_n = b$, it is given by

$$E_Q[f(X_a, X_{t_1}, \ldots, X_{t_n-1}, X_b)] = \int_M dx_0 \mu(a, x_0) q(a, x_0; t_1, x_1) dx_1 \ldots q(t_1, x_1; t_2, x_2) dx_2 \ldots q(t_{n-1}, x_{n-1}, b, x_n) dx_n f(x_0, x_1, \ldots, x_{n-1}, x_n) := [\mu_a q >> \quad (3.F.22)$$

which is the Kolmogorov forward in time (and thus time-irreversible) representation for the diffusion process with initial distribution $\mu_a(x_0) = \mu(a, x_0)$, which using Eq. (3.F.16) can still be rewritten as

$$\int_M dx_0 \mu_a(x_0) \frac{1}{\phi_a(x_0)} p(a, x_0; t_1, x_1) \phi_{t_1}(x_1) dx_1 \frac{1}{\phi_{t_1}(x_1)} dx_1 p(t_1, x_1; t_2, x_2) \phi_{t_2}(x_2) dx_2 \ldots \frac{1}{\phi(t_{n-1}, x_{n-1})} p(t_{n-1}, x_{n-1}; b, x_n) \phi_b(x_n) dx_n f(x_0, \ldots, x_n) \quad (3.F.23)$$

which in account of $\mu_a(x_0) = \breve{\phi}_a(x_0)\phi_a(x_0)$ and Eq. (3.F.16) can be written in the time-reversible form

$$\int_M \breve{\phi}_a(x_0) dx_0 p(a, x_0; t_1, x_1) dx_1 p(t_1, x_1; t_2, x_2) dx_2 \ldots p(t_{n-1}, x_{n-1}; b, x_n) \phi_b(x_n) dx_n f(x_0, \ldots, x_n) \quad (3.F.24)$$

which we write as

$$= [\breve{\phi}_a p >><< p\phi_b]. \quad (3.F.25)$$

This is the *formally* time-symmetric Schroedinger representation with the transition (but not probability) density p. Here, the formal time symmetry is seen in the fact that this equation can be read in any direction, preserving the physical sense of transition. This representation, in distinction with the Kolmogorov representation, does *not* have the Markov property.

We define the adjoint transition probability density $\breve{q}(s, x; t, y)$ with the $\breve{\phi}$-transformation

$$\breve{q}(s, x; t, y) = \breve{\phi}(s, x) p(s, x; t, y) \frac{1}{\breve{\phi}(t, y)} \quad (3.F.26)$$

which satisfies the Chapmann-Kolmogorov equation and the time-reversed normalization

$$\int_M dx \breve{q}(s, x; t, y) = 1. \quad (3.F.27)$$

We get

$$E_{\breve{Q}}[f(X_a, X_{t_1}, \ldots, X_b)] = \int_M f(x_0, \ldots, x_n) \breve{q}(a, x_0; t_1, x_1) dx_1 \breve{q}(t_1, x_1; t_2, x_2) dx_2 \ldots \breve{q}(t_{n-1}, x_{n-1}; b, x_n) \breve{\phi}(b, x_n) \phi(b, x_n) dx_n, \quad (3.F.28)$$

which has a form non-invariant in time, i.e. reading from right to left, as

$$<< \breve{q}\hat{\phi}_b\phi_b] =<< \breve{q}\hat{\mu}_b], \tag{3.F.29}$$

which is the time-reversed representation for the final distribution $\mu_b(x) = \breve{\phi}_b(x)\phi_b(x)$. Now, starting from this last expression and rewriting it in a similar form that is in the forward process but now with $\breve{\phi}$ instead of ϕ, we get

$$\int_M dx_0\breve{\phi}_a(x_0)p(a,x_0;t_1,x_1)\frac{1}{\breve{\phi}_{t_1}(x_1))}dx_1\breve{\phi}(t_1,x_1)p(t_1,x_1;t_2,x_2)\frac{1}{\breve{\phi}_{t_2}(x_2)}dx_2$$
$$\dots dx_{n-1}\breve{\phi}(t_{n-1},x_{n-1})p(t_{n-1},x_{n-1};b,x_n)$$
$$\frac{1}{\breve{\phi}(b,x_n)}\breve{\phi}_b(x_n)\phi(b,x_n)dx_nf(x_0,\dots,x_n) \tag{3.F.30}$$

which coincides with the time-reversible Schroedinger representation

$$[\breve{\phi}_a p >><< p\phi_b].$$

We therefore have three equivalent representations for the diffusion process: the forward in time Kolmogorov representation, the backward Kolmogorov representation, which are both naturally irreversible in time, and the time-reversible Schroedinger representation, so that we can write succintly,

$$[\mu_a q >>= [\breve{\phi}_a p >><< p\phi_b] =<< \breve{q}\mu_b], \text{with } \mu_a = \phi_a\breve{\phi}_a,\ \mu_b = \phi_b\breve{\phi}_b. \tag{3.F.31}$$

In addition of this formal identity,we have to establish the relations between the equations that have led to them. We first note, that in the Schroedinger representation, which is formally time-reversible, we have an interpolation of states between the initial data $\breve{\phi}_a(x)$ and the final data, $\phi_b(x)$. The information for this interpolation is given by a filtration of interpolation $\mathcal{F}_a^r \cup \mathcal{F}_b^s$, which is given in terms of the filtration for the forward Kolmogorov representation $\mathcal{F} = \mathcal{F}_a^t, t \in [a,b]$ which is used for prediction starting with the initial density $\phi_a\breve{\phi}_a = \mu_a$ and the filtration $\mathcal{F}_t^b$ for retrodiction for the time-reversed process with initial distribution μ_b.

We observe that q and $\breve{q}$ are in time-dependent duality with respect to the measure

$$\mu_t(x)dx = \breve{\phi}_t(x)\phi_t(x)dx. \tag{3.F.32}$$

We shall now extend the state-space of the diffusion process to $[a,b] \times M$, to be able to transform the time-inhomogeneous processes into time-homogeneous processes, while the stochastic dynamics still takes place exclusively in M. This will allow us to define the duality of the processes to be with respect to $\mu_t(x)dtdx$ and to determine the form of the exact term of the trace-torsion, and ultimately, to establish the relation between the diffusion processes and Schroedinger equations, both for potential linear and non-linear in the wave-functions. If we define time-homogeneous semigroups of the processes on $\{(t, X_t) \in [a, b] \times M\}$ by

$$P_r f(s,x) = \begin{cases} Q_{s,s+r} f(s,x), & s \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{3.F.33}$$

and

$$\breve{P}_r g(t,y) = \begin{cases} g Q_{t-r,t}(t,y), & r \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{3.F.34}$$

then

$$< g, P_r f >_{\mu_t dtdx} = < \breve{P}_r g, f >_{\mu_t dtdx}, \tag{3.F.35}$$

which is the duality of $\{(t, X_t)\}$ with respect to the $\mu_t dtdx$ density. We remark here that we have an augmented density by integrating with respect to time t. Consequently, if in our spacetime case we define for $a_t(x), \hat{a}_t(x)$ time-dependent one-forms on M (to be determined later)

$$B\alpha : = \frac{\partial \alpha}{\partial t} + H(g, a_t)\alpha_t, \tag{3.F.36}$$

$$B^0 \mu : = -\frac{\partial \mu}{\partial t} + H(g, a_t)^\dagger \mu_t, \tag{3.F.37}$$

and its adjoint operators

$$\breve{B}\beta = -\frac{\partial \beta}{\partial t} - H(g, \breve{a}_t)^\dagger \beta_t, \tag{3.F.38}$$

$$(\breve{B})^0 \mu_t = \frac{\partial \mu_t}{\partial t} - H(g, \breve{a}_t)^\dagger \mu_t, \tag{3.F.39}$$

where by $H(g, \breve{a}_t)^\dagger$ we mean the vol_g-adjoint of this operator, i.e. $H(g, \breve{a}_t)^\dagger \mu_t = \frac{1}{2}\triangle_g \mu_t - \text{div}_{\text{g}}(\mu \breve{a}_t)$. From [96, 107] follows that the duality of space-time processes

$$< B\alpha, \beta >_{\mu_t(x)dtdx} = < \alpha, \breve{B}\beta >_{\mu_t(x)dtdx}, \tag{3.F.40}$$

is equivalent to

$$a_t(x) + \breve{a}_t(x) = d \ln \mu_t(x) \equiv d \ln (\phi_t(x)\breve{\phi}_t(x)), \tag{3.F.41}$$

$$B^0 \mu_t(x) = 0. \tag{3.F.42}$$

The latter equation being the Fokker-Planck equation for the diffusion with trace-torsion given by $a + A$, then the Fokker-Planck equation for the adjoint (time-reversed) process is valid, i.e.

$$(\breve{B})^0 \mu_t(x) = 0. \tag{3.F.43}$$

Substracting Eqs. (3.F.39) and (3.F.40) we get the final form of the duality condition

$$\frac{\partial \mu}{\partial t} + \mathrm{div_g}[(\frac{a_t - \breve{a}_t}{2})\mu_t)] \quad = \quad 0, \text{ for } \mu_t(x) = \breve{\phi}_t(x)\phi_t(x). \tag{3.F.44}$$

Therefore, we can establish that the duality conditions of the diffusion equation in the Kolmogorov representation and its time reversed diffusion lead to the following conditions on the additional elements of the drift vector fields:

$$a_t(x) + \breve{a}_t(x) = d \ \ln \ \mu_t(x) \equiv d \ \ln \ (\phi_t(x)\breve{\phi}_t(x)), \tag{3.F.45}$$

$$\frac{\partial \mu}{\partial t} + \mathrm{div_g}[(\frac{a_t(x) - \breve{a}_t(x)}{2})\mu_t(x)] = 0. \tag{3.F.46}$$

If we assume that $a_t - \hat{a}_t$ is an exact one-form, i.e., there exists a time-dependent differentiable function $S(t,x) = S_t(x)$ defined on $[a,b] \times M$ such that for $t \in [a,b]$,

$$a_t(x) - \breve{a}_t(x) = d \ \ln \ \frac{\phi_t(x)}{\breve{\phi}_t(x)} = 2dS_t(x) \tag{3.F.47}$$

which together with

$$a_t(x) + \breve{a}_t(x) = d \ \ln \ \mu_t(x), \tag{3.F.48}$$

implies that on $D(t,x)$ we have

$$a_t(x) \quad = \quad d \ \ln \ \phi_t(x), \tag{3.F.49}$$

$$\breve{a}_t(x) \quad = \quad d \ \ln \ \breve{\phi}_t(x). \tag{3.F.50}$$

Introduce now $R_t(x) = R(t,x) = \frac{1}{2}\ln\phi_t(x)\breve{\phi}_t(x)$ and $S_t(x) = S(t,x) = \frac{1}{2}\ln\frac{\phi_t(x)}{\breve{\phi}_t(x)}$, so that

$$a_t(x) \quad = \quad d(R_t(x) + S_t(x)), \tag{3.F.51}$$

$$\breve{a}_t(x) \quad = \quad d(R_t(x) - S_t(x)), \tag{3.F.52}$$

and Eq. (3.F.46) takes the form

$$\frac{\partial R}{\partial t} + \frac{1}{2}\triangle_g S_t + g(dS_t, dR_t) = 0. \tag{3.F.53}$$

Remarks. We have mentioned the fact that there is a hidden active role of time in QM [145], which in the above construction is built-in the very definition of the probability density in terms of a *final* and initial distributions. This back action of time appears to be not exclusive of QM. In the theory of growth of sea shells due to Santilli and Illert, it was shown that it cannot be explained by Minkowskian nor Euclidean geometry, but their isotopic lifts and their duals, and this requires the introduction of time duality and four-fold time [148]; this model has been further applied to diverse problems of morphology in biology by Reverberi [149]. We further note that the time-dependent function S on the 3-space manifold, is defined by Eq. (3.F.47) up to addition of an arbitrary function of t, and when further below we shall take this function as defining the complex phase of the quantum Schroedinger wave, this will introduce the quantum-phase indetermination of the quantum evolution, as we discussed already in the setting of geometry of the quantum state-space [96, 98].

Therefore, together with the three different time-homogeneous representations $\{(t, X_t), t \in [a, b], X_t \in M\}$ of a time-inhomogeneous diffusion process $\{X_t, Q)$ on M we have three equivalent dynamical descriptions. One description, with creation and killing described by the scalar field $c(t, x)$ and the diffusion equation describing it is given by a creation-destruction potential in the trace-torsion background given by an electromagnetic potential

$$\frac{\partial p}{\partial t} + H(g, 0)(x)p + c(t, x)p = 0; \tag{3.F.54}$$

the second description has an additional trace-torsion $a(t, x)$, a 1-form on $R \times M$

$$\frac{\partial q}{\partial t} + H(g, a_t)q = 0. \tag{3.F.55}$$

while the third description is the adjoint time-reversed of the first representation given by $\breve{\phi}$ satisfying the diffusion equation on the background with no torsion, i.e.

$$-\frac{\partial \breve{\phi}}{\partial t} + H(g, 0)\breve{\phi} + c\breve{\phi} = 0. \tag{3.F.56}$$

The second representation for the full trace-torsion diffusion forward in time Kolmogorov representation, we need to adopt the description in terms of the fundamental solution q of

$$\frac{\partial q}{\partial t} + H(g, a_t)q = 0, \tag{3.F.57}$$

for which one must start with the initial distribution $\mu_a(x) = \breve{\phi}_a(x)\phi_a(x)$. This is a time t-irreversible representation in the real world, where q describes the real transition and μ_a gives the initial distribution. If in addition one traces the diffusion backwards with reversed time t, with $t \in [a, b]$ running backwards, one needs for this the final distribution $\mu_b(x) = \breve{\phi}_b(x)\phi_b(x)$ and the time t reversed probability density $\hat{q}(s, x; t, y)$ which is the fundamental solution of the equation

$$-\frac{\partial \breve{q}}{\partial t} + H(g, \breve{a}_t)\hat{q} = 0, \tag{3.F.58}$$

with additional trace-torsion one-form on $R \times M$ given by $\hat{a}$, where

$$\breve{a}_t + a_t = d\ln\mu_t(x), \text{ with } \mu_t = \phi_t\breve{\phi}_t, \tag{3.F.59}$$

where the diffusion process in the time-irreversible forward Kolmogorov representation is given by the Ito s.d.e

$$dX_t^i = \sigma_j^i(X_t)dW_t^j + a^i(t, X_t)dt, \tag{3.F.60}$$

and the backward representation for the diffusion process is given by

$$dX_t^i = \sigma_j^i(X_t)dW_t^j + \breve{a}^i(t, X_t)dt, \tag{3.F.61}$$

where $a, \breve{a}$ are given by the Eqs. (3.F.51), (3.F.52), and $(\sigma\sigma^\dagger)^{\alpha\beta} = g^{\alpha\beta}$.

We follow Schroedinger in pointing that ϕ and $\breve{\phi}$ separately satisfy the creation and killing equations, while in quantum mechanics ψ and $\bar{\psi}$ are the complex-valued counterparts of ϕ and $\hat{\phi}$, respectively, they are not arbitrary but

$$\phi\breve{\phi} = \psi\bar{\psi}. \tag{3.F.62}$$

Thus, in the following, this Born formula, once the equations for ψ are determined, will be a consequence of the constructions, and not an hypothesis on the random basis of non-relativistic mechanics.

Therefore, the equations of motion given by the Ito s.d.e.

$$dX_t^i = \mathrm{grad_g}\phi^i(t, X_t)dt + \sigma_j^i(X_t)dW_t^j, \tag{3.F.63}$$

which are equivalent to

$$\frac{\partial u}{\partial t} + H(g, a_t)u = 0 \tag{3.F.64}$$

with $a_t(x) = d\ln\phi_t(x) = d(R_t(x) + S_t(x))$, determines the motion of the ensemble of non-relativistic particles. Note that this equivalence requires only the Laplacian for the RCW connection with the forward trace-torsion full one-form

$$Q(t, x) = d\ln\phi_t(x) = d(R_t(x) + S_t(x)). \tag{3.F.65}$$

In distinction with Stochastic Mechanics due to Nelson [102], and contemporary ellaborations of this applied to astrophysics as the theory of Scale Relativity due to Nottale [114, 106], we only need the form of the trace-torsion for the forward Kolmogorov representation, and this turns to be equivalent to the Schroedinger representation which interpolates in time-symmetric form between this forward process and its time dual with trace-torsion one-form given by $\breve{a}_t(x) = d\ln\breve{\phi}_t(x) = d(R_t(x) - S_t(x))$.

Finally, let us how this is related to the Schroedinger equation. Consider now the Schroedinger equations for the complex-valued wave function ψ and its complex conjugate $\bar{\psi}$, i.e. introducing $i = \sqrt{-1}$, we write them in the form

$$i\frac{\partial\psi}{\partial t} + H(g,0)\psi - V\psi \;=\; 0, \tag{3.F.66}$$

$$-i\frac{\partial\bar{\psi}}{\partial t} + H(g,0)\bar{\psi} - V\bar{\psi} \;=\; 0, \tag{3.F.67}$$

which are identical to the usual forms. So, we have the imaginary factor appearing in the time t, which we confront with the diffusion equations generated by the RCW connection with null trace-torsion, i.e. the system

$$\frac{\partial\phi}{\partial t} + H(g,0)\phi + c\phi \;=\; 0, \tag{3.F.68}$$

$$\frac{-\partial\breve{\phi}}{\partial t} + H(g,0)\breve{\phi} + c\breve{\phi} \;=\; 0, \tag{3.F.69}$$

and the diffusion equations determined by both the RCW connections with trace-torsion a and $\breve{a}$, i.e.

$$\frac{\partial q}{\partial t} + H(g,a_t)q \;=\; 0, \tag{3.F.70}$$

$$\frac{-\partial\breve{q}}{\partial t} + H(g,\breve{a}_t)\hat{q} \;=\; 0, \tag{3.F.71}$$

which are equivalent to the single equation

$$\frac{\partial q}{\partial t} + H(g,d\ln\phi_t)q \;=\; 0. \tag{3.F.72}$$

If we introduce a complex structure on the two-dimensional real-space with coordinates (R,S), i.e. we consider

$$\psi = e^{R+iS}, \;\; \psi = e^{R-iS}, \tag{3.F.73}$$

viz a viz $\phi = e^{R+S}$, $\breve{\phi} = e^{R-S}$, with $\psi\bar{\psi} = \phi\breve{\phi}$, then for a wave-function differentiable in t and twice-differentiable in the space variables, then, ψ satisfies the Schroedinger equation if and only if (R,S) satisfy the difference between the Fokker-Planck equations, i.e.

$$\frac{\partial R}{\partial t} + g(dS_t, dR_t) + \frac{1}{2}\triangle_g S_t = 0, \tag{3.F.74}$$

and

$$V = -\frac{\partial S}{\partial t} + H(g,dR_t)R_t - \frac{1}{2}g(dS_t,dS_t), \tag{3.F.75}$$

which follows from substituting ψ in the Schroedinger equation and further dividing by ψ and taking the real part and imaginary parts, to obtain the former and latter equations, respectively.

Conversely, if we take the coordinate space given by $(\phi, \breve{\phi})$, both non-negative functions, and consider the domain $D = D(s,x) = \{(s,x) : 0 < \breve{\phi}(s,x)\phi(s,x)\} \subset [a,b] \times M$ and define $R = \frac{1}{2}\ln\phi\breve{\phi}, S = \frac{1}{2}\ln\frac{\phi}{\breve{\phi}}$, with R, S having the same differentiabilty properties that previously ψ, then $\phi = e^{R+S}$ satisfies in D the equation

$$\frac{\partial\phi}{\partial t} + H(g,0)\phi + c\phi = 0, \tag{3.F.76}$$

if and only if

$$\begin{aligned} -c &= [-\frac{\partial S}{\partial t} + H(g,dR_t)R_t - \frac{1}{2}g(dS_t,dS_t)] \\ &+ [\frac{\partial R}{\partial t} + H(g,dR_t)S_t] + [2\frac{\partial S}{\partial t} + g(dS_t,dS_t)], \end{aligned} \tag{3.F.77}$$

while $\breve{\phi} = e^{R-S}$ satisfies in D the equation

$$-\frac{\partial\phi}{\partial t} + H(g,0)\breve{\phi} + c\breve{\phi} = 0, \tag{3.F.78}$$

if and only if

$$\begin{aligned} -c &= [-\frac{\partial S}{\partial t} + H(g,dR_t)R_t - \frac{1}{2}g(dS_t,dS_t)] \\ &- [\frac{\partial R}{\partial t} + H(g,dR_t)S_t] + [2\frac{\partial S}{\partial t} + g(dS_t,dS_t)]. \end{aligned} \tag{3.F.79}$$

Notice that $\phi, \breve{\phi}$ can be both negative or positive. So if we define $\psi = e^{R+iS}$, it then defines in weak form the Schroedinger equation in D with

$$V = -c - 2\frac{\partial S}{\partial t} - g(dS_t, dS_t). \tag{3.F.80}$$

Remarks. We note that from Eq. (3.F.80) follows that we can choose S in a way such that either c is independent of S and thus V is a potential which is non-linear in the sense that it depends on the phase of the wave function ψ and thus the Schroedinger equation with this choice becomes non-linear dependent of ψ, or conversely, we can make the alternative choice of c depending non-linearly on S, and thus the creation-annihilation of particles in the diffusion equation is non-linear, and consequently the Schroedinger equation has a potential V which does not depend on ψ. It is important for further developments in this article that the non-linear Schroedinger equation can be turned into the iso-linear iso-Schroedinger equation by taking the non-linear terms of the potential into the isotopic generalized unit. Indeed, the recovery of linearity in isohilbert space is achieved by the embedding of the nonlinear terms in the isounit as shown in [46]; see Eqs. (3.4.42) and (3.4.43).

3.F.4.1 Santilli-Lie Isotopies of the Differential Calculus and Metric Structures, and the Iso-Schroedinger Equation

To present the iso-Schroedinger equation, we need the Santilli-Lie-isotopic differential calculus [109, 46] and the isotopic lift of manifolds, the so-called isomanifolds, due to Tsagas and Sourlas [22]; we shall follow here the notations of Section 3.2 above. We start by considering the manifold M to be a vector space with local coordinates, which for simplicity we shall from now fix them to be a *contravariant* system, $x = (x^i), i = 1, \ldots, n$, unit given by $I = \mathrm{diag}(1, \ldots, 1)$ and metric g which we assumed diagonalized. We shall lift this structure to a vector space $\hat{M}$ provided with isocoordinates $\hat{x}$, isometric $\hat{G}$ and defined on Santilli isonumber field $\hat{F}$, where F can be the real or complex numbers; we denote this isospace by $\hat{M}(\hat{x}, \hat{G}, \hat{F})$. The isocoordinates are introduced by the transformation $x \mapsto U \times x \times U^\dagger = x \times \hat{I} := \hat{x}$. To introduce the *contravariant* isometric $\hat{G}$ we start by considering the transformation[38]

$$g \mapsto U \times g \times U^\dagger = \hat{I} \times g := \hat{g}. \tag{3.F.81}$$

Yet from the Definition 3.2.3 follows that the isometric is more properly defined by $\hat{G} = \hat{g} \times \hat{I}$. Thus we have a transformed $M(x, g, F)$ into the isospace $\hat{M}(\hat{x}, \hat{G}, \hat{F})$. Thus the projection on $M(x, g, F)$ of the isometric in $\hat{M}(\hat{x}, \hat{G}, \hat{F})$ is defined by a contravariant tensor, $\hat{g} = (\hat{g}^{ij})$ with components

$$\hat{g}^{ij} = (\hat{I} \times g)^{ij}. \tag{3.F.82}$$

If we take $\hat{I} = \psi^2(x) \times I$ we then retrieve the Weyl scale transformations, with ψ a scale field depending only on the coordinates of M. If we start with g being the Euclidean or Minkowski metrics, we obtain the iso-Euclidean and iso-Minkowski metrics; in the case we start with a general metric as in GR, we obtain Isorelativity. We shall now proceed to identify the isotopic lift of the noise tensor σ which verifies Eq. (3.F.9), i.e. $\sigma \times \sigma^\dagger = g$. The non-unitary transform of (a diagonalized) σ is given by

$$\sigma \mapsto U \times \sigma \times U^\dagger = \sigma \times \hat{I} := \hat{\sigma}. \tag{3.F.83}$$

Then,

$$\hat{\sigma} \hat{\times} \hat{\sigma} = \sigma \times \hat{I} \times \hat{T} \times (\sigma \times \hat{I})^\dagger = (\sigma \times \sigma^\dagger) \times \hat{I} = g \times \hat{I} = \hat{g}. \tag{3.F.84}$$

[38]We shall assume, as usual, a diagonal metric.

Thus the isotopic lift of the noise tensor defined on $\hat{M}(\hat{x},\hat{G},\hat{R})$ is given by $\hat{\sigma}=\sigma\times\hat{I}$ which on projection to $M(\hat{x},\hat{G},R)$ we retrieve σ. We know follow the notations and definitions of Section 3.2.5 for the isotopic differential, and for isofunctions. We introduce the isotopic gradient operator of the isometric $\hat{G}$ (the $\hat{G}$-gradient, for short), $\widehat{\mathrm{grad}_{\hat{G}}}$ applied to the isotopic lift $\hat{f}(\hat{x})$ of a function $f(x)$ is defined by

$$\widehat{\mathrm{grad}_{\hat{G}}}\hat{f}(\hat{x})(\hat{v})=\hat{G}(\hat{d}\hat{f}(\hat{x})\hat{,}\hat{v}), \tag{3.F.85}$$

for any vector field $\hat{v}\in T_{\hat{x}}(\hat{M})$, $\hat{x}\in\hat{M}$; we have denoted the inner product as $\hat{,}$ to stress that the inner product is taken with respect to the product in $\hat{F}$. Hence, the operator $\widehat{\mathrm{grad}_{\hat{G}}}\hat{f}(\hat{x})$ can be thought as the isovector field on the tangent manifold to $\hat{M}(\hat{x},\hat{G},\hat{F})$ defined by

$$\hat{G}^{\alpha\beta}\hat{\times}\frac{\hat{\partial}\hat{f}(\hat{x})}{\hat{\partial}\hat{x}^{\alpha}}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}^{\beta}}=\hat{g}^{\alpha\beta}\hat{\times}\frac{\hat{\partial}\hat{f}(\hat{x})}{\hat{\partial}\hat{x}^{\alpha}}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}^{i}}\times\hat{I}. \tag{3.F.86}$$

Therefore, the projection on $\hat{M}(\hat{x},\hat{g},F)$ of the $\hat{G}$-gradient vector field of $\hat{f}(\hat{x})$ is the vector field with components

$$\hat{g}^{\alpha\beta}\hat{\times}\frac{\hat{\partial}\hat{f}(\hat{x})}{\hat{\partial}\hat{x}^{\alpha}}=\hat{g}^{\alpha\beta}\hat{\times}\frac{\hat{\partial}\hat{f}(\hat{x})}{\hat{\partial}\hat{x}^{\alpha}}. \tag{3.F.87}$$

This will be of importance for the determination of the drift vector field of the diffusion linked with the Santilli- iso-Schroedinger equation. We finally define the isolaplacian as

$$\hat{\triangle}_{\hat{g}}=\hat{g}^{\alpha\beta}\hat{\times}\hat{D}_{\frac{\hat{\partial}}{\hat{\partial}\hat{x}^{\alpha}}}\hat{\times}\hat{D}_{\frac{\hat{\partial}}{\hat{\partial}\hat{x}^{\beta}}}. \tag{3.F.88}$$

Here $\hat{D}_{\frac{\hat{\partial}}{\hat{\partial}\hat{x}^{\alpha}}}$ is defined accordingly with Definition 3.2.13 above, by (c.f. Eq. (3.F.6) above)

$$\hat{D}_{\frac{\hat{\partial}}{\hat{\partial}\hat{x}^{\alpha}}}\hat{X}^{\beta}=\frac{\hat{\partial}\hat{X}^{\beta}}{\hat{\partial}\hat{x}^{\beta}}+\widehat{\begin{Bmatrix}\beta\\ \gamma\alpha\end{Bmatrix}}\hat{\times}\hat{X}^{\gamma}, \tag{3.F.89}$$

and hence it is the isocovariant differential with respect to the Levi-Civita isoconnection with isoChristoffel coefficients

$$\widehat{\begin{Bmatrix}\alpha\\ \beta\gamma\end{Bmatrix}}=\frac{\hat{1}}{\hat{2}}\left(\frac{\hat{\partial}}{\hat{\partial}\hat{x}^{\beta}}\hat{g}_{\nu\gamma}+\frac{\hat{\partial}}{\hat{\partial}\hat{x}^{\gamma}}\hat{g}_{\beta\nu}-\frac{\hat{\partial}}{\hat{\partial}\hat{x}^{\nu}}\hat{g}_{\beta\gamma}\right)\hat{\times}\hat{g}^{\alpha\nu}. \tag{3.F.90}$$

We remark that from Observations 1 follows that alternatively we can define the more simpler laplacian by taking instead

$$\hat{\triangle}_{\hat{g}}=\hat{g}^{\alpha\beta}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}^{\alpha}}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}^{\beta}}. \tag{3.F.91}$$

In both cases we take $\hat{\sigma}$ for the corresponding isonoise term in the isodiffusion representation. The latter definition of the isolaplacian differs from the original one introduced in [22].

3.F.4.2 Diffusions and the Heisenberg Representation

Up to now we have set our theory in terms of the Schroedinger representation, since the original setting for this theory has to do with scale transformations as introduced by Einstein in his last work [100] while it was recognized previously by London that the wave function was related to the Weyl scale transformation [138], and these scale fields turned to be in the non-relativistic case, nothing else than the wave function of Schroedinger equation, both in the linear and the non-linear cases. Historically the operator theory of QM was introduced before the Schroedinger equation, who later proved the equivalence of the two. The ensuing dispute and rejection by Heisenberg of Schroedinger's equation is a dramatic chapter of the history of QM [125]. It turns out to be the case that we can connect the Brownian motion approach to QM and the operator formalism due to Heisenberg and Jordan, and its isotopic lift presented in Section 3.4.

Let us define the position operator as usual and the momentum operator by

$$q^k = x^k, \ \ p_{\mathcal{D}k} = \sigma \times \frac{\partial}{\partial x^k}, \tag{3.F.92}$$

which we call the diffusion quantization rule (the subscript $\mathcal{D}$ denotes diffusion) since we have a representation different to the usual quantization rule

$$p_k = -i \times \frac{\partial}{\partial x^k}, \tag{3.F.93}$$

with $\sigma = (\sigma_a^\alpha)$ the diffusion tensor verifying $(\sigma \times \sigma^\dagger)^{\alpha\beta} = g^{\alpha\beta}$ and substitute into the Hamiltonian function

$$H(p,q) = \frac{1}{2}\sum_{k=1}^{d}(p_k)^2 + \mathbf{v}(q), \tag{3.F.94}$$

this yields the formal generator of a diffusion semigroup in $C^2(R^d)$ or $L^2(R^d)$ which in our previous notation is written as $H(g,0)+v$. Thus, an operator algebra on $C^2(R^n)$ or $L^2(R^n)$ together with the postulate of the commutation relation (instead of the usual commutator relation of quantum mechanics $[p,q] = -i \times I$)

$$[p_{\mathcal{D}}, q] = p_{\mathcal{D}} \times q - q \times p_{\mathcal{D}} = \sigma \times I \tag{3.F.95}$$

this yields the diffusion equation

$$\frac{\partial \phi}{\partial t} \times \phi + \frac{1}{2}\sum_{k=1}^{d}(\sigma\frac{\partial}{\partial x^\alpha})^2 \times \phi + \mathbf{v} \times \phi = 0, \tag{3.F.96}$$

which coincides with the diffusion Eq. (3.F.54) provided that $c = \mathbf{v}$. Thus, in this approach, the operator formalism and the quantization postulates, allow to deduce the diffusion equation. If we start from either the diffusion process or the RCW geometry, without any quantization conditions we already have the equations of motion of the quantum system which are non other than the original diffusion equations, or equivalently, the Schroedinger equations. We stress the fact that these arguments are valid for both cases relative to the choice of the potential function V, i.e. if it depends nonlinearly on the wave function ψ, or acts linearly by multiplication on it. Further below, we shall use this modification of the Heisenberg representation of QM by the previous Heisenberg type representation for diffusion processes, to give an account of the diffusion processes that are associated with HM. This treatment differs from our original (inconsistent with respect to HM, as it turned to be proved in the later findings by Prof. Santilli) treatment of the relation between RCW geometries and diffusions presented in [119] in incorporating the isotopic lift of all structures.

Let us frame now isoquantization in terms of diffusion processes. Define isomomentum, $\hat{p}_{\mathcal{D}}$, by

$$\hat{p}_{\mathcal{D}k} = \hat{\sigma}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}^k}, \text{ with } \hat{\sigma} = \sigma \times \hat{I}, \tag{3.F.97}$$

so that the kinetic term of the iso-Hamiltonian is

$$\begin{aligned} \hat{p}_{\mathcal{D}}\hat{\times}\hat{p}^{\dagger}_{\mathcal{D}} &= \hat{\sigma}\hat{\times}\hat{\sigma}^{\dagger}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}} \\ &= \hat{g}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}} = \hat{\triangle}_{\hat{g}}. \end{aligned} \tag{3.F.98}$$

We finally check the consistency of the construction by proving that it can be achieved via the non-unitary transformation

$$\begin{aligned} p_{\mathcal{D}_j} \mapsto U \times p_{\mathcal{D}_j} \times U^{\dagger} = U \times \sigma \times \frac{\partial}{\partial x^j} \times U^{\dagger} \\ = \sigma \times \hat{I} \times \hat{T} \times \hat{I} \times \frac{\partial}{\partial x^j} = \hat{\sigma}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}^j} = \hat{p}_{\mathcal{D}j}. \end{aligned} \tag{3.F.99}$$

Note that we have achieved this isoquantization in terms of the following transformations. Firstly, we carried out the transformation

$$p = -i \times \frac{\partial}{\partial x} \to p_{\mathcal{D}} := \sigma \times \frac{\partial}{\partial x}, \tag{3.F.100}$$

to further produce its isotopic lift

$$\hat{p}_{\mathcal{D}} = \hat{\sigma}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}}. \tag{3.F.101}$$

Whenever the original diffusion tensor σ is the identity I, from Eq. (3.F.9) follows that the original metric g is Euclidean, we reach compatibility of the diffusion quantization with the Santilli-iso-Heisenberg representation given by taking the non-unitary transformation on the canonical commutation relations, which are given by

$$[\hat{q}^i \hat{,} \hat{p}_j] = \hat{i} \hat{\times} \hat{\delta}^i_j = i \times \delta^i_j \times \hat{I}, \tag{3.F.102}$$

together with

$$[\hat{r}^i \hat{,} \hat{r}^j] = [\hat{p}_i \hat{,} \hat{p}_j] = 0, \tag{3.F.103}$$

with the Santilli-iso-quantization rule [109, 46]

$$\hat{p}_j = -\hat{\imath} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}^j}. \tag{3.F.104}$$

Thus, from the quantization by the diffusion representation we retrieve the Santilli-iso-Heisenberg representation, with the difference that the diffusion noise tensor in the above construction need not be restricted to the identity.

Finally, we consider the isoHamiltonian operator

$$\hat{H} = \frac{\hat{1}}{\hat{2} \hat{\times} \hat{m}} \hat{\times} \hat{p}^{\hat{2}} + \hat{V}_0(\hat{t}, \hat{x}) + \hat{V}_k(\hat{t}, \hat{v}) \hat{\times} \hat{v}^k, \tag{3.F.105}$$

where $\hat{p}$ may be taken to be given either by the Santilli isoquantization rule

$$\hat{p}_k \hat{\times} |\hat{\psi}> = -\hat{i} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}^k} \hat{\times} |\hat{\psi}>, \tag{3.F.106}$$

or by the diffusion representation $\hat{p}_{\mathcal{D}}$. $\hat{V}_0(\hat{t}, \hat{x})$ and $\hat{V}_k(\hat{t}, \hat{v})$ are potential isofunctions, the latter dependent on the isovelocities. Then the iso-Schroedinger equation (or Schroedinger-Santilli isoequation) [109, 46] is

$$\begin{aligned} \hat{i} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{t}} |\hat{\psi}> &= \hat{H} \hat{\times} |\psi> \\ &= \hat{H}(\hat{t}, \hat{x}, \hat{p}) \times \hat{T}(\hat{t}, \hat{x}, \hat{\psi}, \hat{\partial} \hat{\psi}, \ldots) \times |\hat{\psi}>, \end{aligned} \tag{3.F.107}$$

where the wave isofunction $\hat{\psi}$ is an element in $(\hat{\mathcal{H}}, <|\hat{\times}|>, \hat{C}(\hat{c}, \hat{+}, \hat{\times}))$ satisfies

$$\hat{I} \hat{\times} |\hat{\psi}> = |\hat{\psi}> . \tag{3.F.108}$$

3.F.4.3 Hadronic Mechanics and Diffusion Processes

Finally, the components of drift isovector field, projected on $\hat{M}(\hat{x}, \hat{g}, R)$ in the isotopic lift of Eq. (3.F.63) is given by Eq. (3.F.87) with $\hat{f} = \hat{\ln}\hat{\phi}$, where $\hat{\phi}(\hat{x}) = \hat{e}^{\hat{\mathcal{R}}(\hat{x}) + \hat{\mathcal{S}}(\hat{x})}$ is the diffusion wave associated to the solution $\hat{\psi}(\hat{x}) = \hat{e}^{\hat{\mathcal{R}}(\hat{x}) + i\hat{\mathcal{S}}(\hat{x})}$ of the iso-Schroedinger equation, and its adjoint wave is $\breve{\phi}(x) = \hat{e}^{\hat{\mathcal{R}}(x) - \hat{\mathcal{S}}(x)}$. Hence, the drift isovector field has components

$$\hat{g}^{\alpha\beta}(\hat{x}) \hat{\times} \frac{\hat{\partial} \hat{\ln} \hat{\phi}(\hat{x})}{\hat{\partial} \hat{x}^{\alpha}} = \hat{g}^{\alpha\beta}(\hat{x}) \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}^{\alpha}} (\hat{\mathcal{R}}_{\hat{t}} \hat{+} \hat{\mathcal{S}}_{\hat{t}})(\hat{x}). \tag{3.F.109}$$

Finally, we shall write the isotopic lift of the stochastic differential equation for the iso-Schroedinger Eq. (3.F.107). Applying the non-unitary transformation to Eq. (3.F.63), we obtain the iso-equation on $\hat{M}(\hat{x}, \hat{G}, \hat{R})$ for $\hat{X}_{\hat{t}}$ given by

$$d\hat{X}^{i}_{\hat{t}} = ((\hat{g}^{\alpha\beta} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}^{\alpha}} (\hat{\mathcal{R}}_{\hat{t}} \hat{+} \mathcal{S}_{\hat{t}}))(\hat{X}_{\hat{t}}) \hat{\times} \hat{d}\hat{t} + \hat{\sigma}^{i}_{j}(\hat{X}_{\hat{t}}) \hat{\times} d\hat{W}^{j}_{\hat{t}}, \tag{3.F.110}$$

with $d\hat{W}_{\hat{t}} = \hat{W}(\hat{t} \hat{+} \hat{d}\hat{t}) \hat{-} \hat{W}(\hat{t})$ the increment of a iso-Wiener process $\hat{W}_{\hat{t}} = (\hat{W}^{1}_{\hat{t}}, \ldots, \hat{W}^{m}_{\hat{t}})$ with isoaverage equal to $\hat{0}$ and isocovariance given by $\hat{\delta}^{i}_{j} \hat{\times} \hat{t}$; i.e.,

$$\hat{1} \hat{/} (\hat{4} \hat{\times} \hat{\pi} \hat{\times} \hat{t})^{\hat{m} \hat{/} \hat{2}} \hat{\int} \hat{w}_{i} \hat{\times} \hat{e}^{-\hat{w}^{\hat{2}} \hat{/} \hat{4} \hat{\times} \hat{t}^{\hat{2}}} \hat{\times} \hat{d}\hat{w} = \hat{0}, \quad \forall i = 1, \ldots, m \tag{3.F.111}$$

and

$$\hat{1} \hat{/} (\hat{4} \hat{\times} \hat{\pi} \hat{\times} \hat{t})^{\hat{m} \hat{/} \hat{2}} \hat{\int} \hat{w}_{i} \hat{\times} \hat{w}_{j} \hat{\times} \hat{e}^{-\hat{w}^{\hat{2}} \hat{/} \hat{4} \hat{\times} \hat{t}^{\hat{2}}} \hat{\times} \hat{d}\hat{w} = \hat{\delta}^{i}_{j} \hat{\times} \hat{t}, \quad \forall i, j = 1, \ldots, m \tag{3.F.112}$$

and $\hat{\int}$ denotes the isotopic integral defined by $\hat{\int} \hat{d}\hat{x} = (\int \hat{T} \times \hat{I} \times dx) \times \hat{I} = (\int dx) \times \hat{I} = \hat{x}$. Thus, formally at least, we have

$$\hat{X}_{\hat{t}} = \hat{X}_{\hat{0}} \hat{+} \hat{\int}_{\hat{0}}^{\hat{t}} (\hat{g}^{\alpha\beta} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}^{\alpha}} (\hat{\mathcal{R}}_{\hat{s}} \hat{+} \mathcal{S}_{\hat{s}}))(\hat{X}_{\hat{s}}) \hat{\times} \hat{d}\hat{s} + \hat{\int}_{\hat{0}}^{\hat{t}} \hat{\sigma}^{i}_{j}(\hat{X}_{\hat{s}}) \hat{\times} d\hat{W}^{j}_{\hat{s}}. \tag{3.F.113}$$

The integral in the first term of Eq. (3.F.113) is an isotopic lift of the usual Riemann-Lebesgue integral [109d,19,20], while the second one is the isotopic lift of a stochastic Itô integral; we shall not present here in detail the definition of this last term, which follows from the notions of convergence in the isofunctional analysis elaborated by Kadeisvili [110] (see Section 3.2.6), and the usual definition of Itô stochastic integrals [102, 107, 116], nor the presentation of analytical conditions for their convergence which follows in principle from the isotopic lift of the usual conditions.

3.F.4.4 The Extension to the Many-Body Case

Up to know we have presented the case of the Schroedinger equation for an ensemble of one-particle systems on space-time. Of course, our previous constructions are also valid for the case of an ensemble of interacting multiparticle systems, so that the dimension of the configuration space is $3d + 1$, for indistinguishable d particles; the general case follows with minor alterations. If we start by constructing the theory as we did for an ensemble of one-particle systems (Schroedinger's "cloud of electrons"), we can still extend trivially to the general case, by considering a diffusion in the product configuration manifold with coordinates $X_t = (X_t^1, \ldots, X^d) \in M^d$, where M^d is the d Cartesian product of three dimensional space with coordinates $X_t^i = (x_t^{1,i}, x_t^{2,i}, x_t^{3,i}) \in M$, for all $i = 1, \ldots, d$. The distribution of this is $\mu_t = E_Q \circ X_t^{-1}$, which is a probability density in M^d. To obtain the distribution of the system on the three-dimensional space M, we need the distribution of the system X_t:

$$U_t^x := \frac{1}{d} \sum_{i=1}^{d} \delta_{x_i}, \tag{3.F.114}$$

which is the same as

$$U_t^x(B) = \frac{1}{d} \sum_{i=1}^{d} 1_B(X_t^i), \tag{3.F.115}$$

where $1_B(X_t^i)$ is the characteristic system for a measurable set B, equal to 1 if $X_t^i \in B$, for any $i = 1 \ldots, d$ and 0 otherwise. Then, the probability density for the interacting ensembles is given by

$$\mu_t^x(B) = E_Q[U_t^x(B)], \tag{3.F.116}$$

where E_Q is the mean taken with respect to the forward Kolmogorov representation presented above, is the probability distribution in the three-dimensional space; see [107]. Therefore, the geometrical-stochastic representation in actual space is constructable for a system of interacting ensembles of particles. Thus the criticism to the Schroedinger equation by the Copenhagen school, as to the unphysical character of the wave function since it was originally defined on a multiple-dimensional configuration space of interacting system of ensembles, is invalid [125].

3.F.5 Possible Empirical Evidence and Conclusions

We have shown that the Schroedinger and isoSchroedinger equation have an equivalent representation in terms of diffusion processes. This can be further extended to hadronic chemistry, as shown in Volume V of this series. This is an universal phenomenae since the applicability of the Schroedinger equation does not restrict to the microcospic realm, as already shown in the astrophysical theory due to Nottale [114]; this universality is associated with the fact that the Planck constant (or equivalently, the diffusion constant) is multivalued, or still, it is context dependent, inasmuch as the velocity of light has the same feature [46]. In the case of HM this can be seen transparently in the fact that the isotopic unit plays the role, upon quantization, of the Planck constant as can seen in Eqs. (3.F.107), (3.F.108)[39], or furthermore, by its product with the noise tensor of the underlying Brownian motions. In the galactic scales, this may explain the red-shift without introducing a big-bang hypothesis [46, 73]. An identical conclusion was reached by Arp in considering as a theoretical framework the Le Sage's model of a Universe filled with a gas of particles [130], in our theory, the zero-point fluctuations described by the Brownian motions defined by the wave functions, as well as by viscous fluids, spinor fields, or electromagnetic fields [95] (and which one can speculate as related to the so-called dark energy problem). A similar view has been proposed by Santilli in which the elementary constituents are the so-called aetherinos [149], while in Sidharth's work, they appear to be elementary quantized vortices related to quantum-mechanical Kerr-Newman black holes [119]. Thus, whether we examine the domains of linear or non-linear quantum mechanics, or still of hadronic mechanics, vortices and superconductivity (which is the case of the Rutherford-Santilli model of the neutron which is derived from the previous constructions) appear as universal coherent structures; superconductivity is usually related to a non-linear Schroedinger equation with a Landau-Ginzburg potential, which is just an example of the Brownian motions related to torsion fields with further noise related to the metric. Furthermore, atoms and molecules have spin-spin interactions which will produce a contribution to the torsion field; we have seen already that the torsion geometry exists in the realm of hadronic chemistry, since we can extend the construction to the many-body case. In distinction with the usual repulsive Coulomb potential in nuclear physics, the isotopic deformations of the nuclear symmetries yield attractive potentials such as the Hulten potential, which in the range of 10^{-13} cm yields the usual potential [19, 20, 22, 46, 70, 73, 108–110, 141] without the need of introducing any sort of parameters or extra potentials. In contrast with the ad-hoc postulates of randomness in the fusion models which are considered in the usual approaches [122, 123], in the present work randomness is intrinsic to space-time or alternatively a by product of it, and in the case of HM, these geometries incorporate at a foundational level, a generalized unit which incorporates all the features of the fusion process itself: the non-canonical, non-local and nonlinear overlapping of the wave functions of the ensembles which correspond to the separate ensembles under deformable collisions in which the particles lose their

[39]See Postulate 3.4.1.

pointlike structure, or in a hypercondensed plasma state, where the dynamics of the process may have a random behavior; outside of the domain of 10^{-13} cm, the hadronic fluctuations associated to the isolinear isoSchroedinger equation decay to the quantum fluctuations of the linear Schroedinger equation.

There are already empirical findings that may lead to validate the present view (see Volume IV of this series for experimental evidence in particle physics, nuclear physics, astrophysics and cosmology, and Volume V for experimental evidence in chemistry). In the last fifty years, a team of scientists at the Biophysics Institute of the Academy of Sciences of Russia, directed by S. Shnoll (and presently developed in a world net which includes Roger Nelson, Engineering Anomalies Research, Princeton University, B. Belousov, International Institute of Biophysics, Neuss (Germany), Dr. Wilker, Max-Planck Institute for Aeronomy, Lindau, and others), have carried out tens of thousands of experiments of very different nature and energy scales (α decay, biochemical reactions, gravitational waves antenna, etc.) in different points of the globe, and carried out a software analysis of the observed histograms and their fluctuations, to find out an amazing fit which is repeated with regularity of 24 hours, 27 days and the duration of a sidereal year. In these experiments the fine spectrum of their measurements reveal a non-random pattern. At points of Earth with the same local hour, these patterns are reproduced with the said periodicity. The only thing in common to these experiments is that they are occur in space-time, which has lead to conclude that they stem from space-time fluctuations, which may further be associated with cosmological fields. Furthermore, the histograms reveal a fractal structure; this structure is interpreted as appearing from an interference phenomena related to the cosmological field; we recall that diffusion processes present interference phenomena alike to, say, the two-slit experiment.[40] Measurements taken with collimators show fluctuations emerging from the rotation of the Earth around its axis or its circumsolar orbit, showing a sharp anisotropy of space. Furthermore, it is claimed that the spatial heterogeinity occurs in a scale of 10^{-13} cm, coincidently with the scale of the strong interactions [152]. Contrary to common belief, the Michelson-Morley did not provide a final dismissal of the aether, while Einstein in the course of his life supported the idea of its existence [154]. Thousands of interferometry experiments were carried out by D. Miller, Allais and others, and contemporarily very diverse setups have proved that there is a space anisotropy [153]. As a closing remark we would like to recall that Planck himself proposed the existence of ensembles of random phase oscillators having

[40]This fractal structure has been found to follow the pattern of the logarithmic Muller fractal, which is associated with the existence of a global scale for all structures in the Universe; see H. Muller, Free Energy - Global Scaling, *Raum& Zeit Special 1*, Ehlers-Verlag GmbH, ISBN 3-934-196-17-9; 2004. This leads to reinforce the thesis of time as an active field. Furthermore, the space and time Brownian motions can exist, in principle, in the different space and time scales warranted by these global scales.

the zero-point structure as the basis for quantum physics [146]. Thus, the apeiron would be related to the Brownian motions which we have presented in this work, and define the space and time geometries, or alternatively, are defined by them. So we are back to the idea due to Clifford, that there is no-thing but space and time configurations, instead of a separation between substratum and fields and particles appearing on it. Furthermore, what we perceive to be void, is the hyperdense source of actuality. The same conception has been proposed by Prof. Santilli in the main body of this volume.

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Chapter 4

LIE-ADMISSIBLE BRANCH OF HADRONIC MECHANICS AND ITS ISODUAL

NOTE; THIS CHAPTER MUST BE COMPLETED AND EDITED

4.1 INTRODUCTION

4.1.1 The Scientific Imbalance Caused by Irreversibility

As recalled in Chapter 1, physical, chemical or biological systems are called *irreversible* when their images under time reversal $t \rightarrow -t$ are prohibited by causality and/or other laws, as it is generally the case for nuclear transmutations, chemical reactions and organism growth.

Systems are called *reversible* when their time reversal images are as causal as the original ones, as it is the case for planetary and atomic structures when considered isolated from the rest of the universe, the structure of crystals, and other structures (see reprint volume [1] on irreversibility and vast literature quoted therein).

Another large scientific imbalance of the 20-th century studied in these monographs is the treatment of irreversible systems via the mathematical and physical formulations developed for reversible systems, since these formulations are themselves reversible, thus resulting in serious limitations in virtually all branches of science.

The problem is compounded by the fact that all used formulations are of Hamiltonian type, under the awareness that all known Hamiltonians are reversible over time (since all known potentials, such as the Coulomb potential $V(r)$, etc., are reversible).

This scientific imbalance was generally dismissed in the 20-th century with unsubstantiated statements, such as "irreversibility is a macroscopic occurrence that disappears when all bodies are reduced to their elementary constituents".

Figure 4.1. All energy releasing processes are irreversible over time. By contrast, all formulations of the 20th century are fully reversible over time, a limitation that is apparently responsible for the lack of industrial development of any really new form of energy for over half a century, as well as the lack of resolution of the environmental problems caused by fossil fuels combustion depicted in this figure. A primary objective of hadronic mechanics is, firstly, identify formulations that are structurally irreversible (a task addressed in this chapter), as a necessary premise for their quantitative treatment of irreversible process and the search of basically new energies (a task address in Volume II).

These academic beliefs have been disproved by Theorem 1.3.3 according to which *a classical irreversible system cannot be consistently decomposed into a finite number of elementary constituents all in reversible conditions and, vice-versa, a finite collection of elementary constituents all in reversible conditions cannot yield an irreversible macroscopic ensemble.*

The implications of the above theorem are quite profound because it establishes that, contrary to popular beliefs, *irreversibility originates at the most primitive levels of nature, that of elementary particles, and then propagates all the way to our macroscopic environment.*

In this chapter we study the contribution by the author that originated the field, as well as contributions by a number of independent authors. The presentation will mainly follow the recently published memoir [32]. Nevertheless, an in depth knowledge of the topic requires the study of (at least some of) the author's monographs [18–23,29] and those by independent authors [33–39].

The author would like to express his sincere appreciation to the *Italian Physical Society* for publishing memoir [32] in *Il Nuovo Cimento B* as a final presentation of studies in the field initiated by the author in the same Journal in paper [7] forty years earlier.

4.1.2 The Forgotten Legacy of Newton, Lagrange and Hamilton

The scientific imbalance on irreversibility was created in the early part of the 20-th century when, to achieve compatibility with quantum mechanics and special relativity, the entire universe was reduced to potential forces. Jointly, the analytic equations were "truncated" with the removal of the external terms.

In reality, Newton [2] *did not* propose his celebrated equations restricted to forces derivable from a potential $F = \partial V/\partial r$, but proposed them for the most general possible forces,

$$m_a \times \frac{dv_{ka}}{dt} = F_{ka}(t, r, v), \quad k = 1, 2, 3; \quad a = 1, 2, \ldots, N, \tag{4.1.1}$$

where the conventional associative product of numbers, matrices, operators, etc. is continued to be denoted hereon with the symbol $\times$ so as to distinguish it from numerous other products needed later on.

Similarly, to be compatible with Newton's equations, Lagrange [3] and Hamilton [4] decomposed Newton's force into a potential and a nonpotential component,they represented all potential forces with functions today known as the Lagrangian and the Hamiltonian, and proposed their celebrated equations with external terms,

$$\frac{d}{dt}\frac{\partial L(t,r,v)}{\partial v_a^k} - \frac{\partial L(t,r,v)}{\partial r_a^k} = F_{ak}(t, r, v), \tag{4.1.2a}$$

$$\frac{dr_a^k}{dt} = \frac{\partial H(t,r,p)}{\partial p_{ak}}, \quad \frac{dp_{ak}}{dt} = -\frac{\partial H(t,r,p)}{\partial r_a^k} + F_{ak}(t, r, p), \tag{4.1.2b}$$

$$L = \Sigma_a \frac{1}{2} \times m_a \times v_a^2 - V(t, r, v), \quad H = \Sigma_a \frac{\mathbf{p}_a^2}{2 \times m_a} + V(t, r, p), \tag{4.1.2c}$$

$$V = U(t, r)_{ak} \times v_a^k + U_o(t, r), \quad F(t, r, v) = F(t, r, p/m). \tag{4.1.2d}$$

More recently, Santilli [5] conducted comprehensive studies on the integrability conditions for the existence of a potential or a Lagrangian or a hamiltonian, called *conditions of variational selfadjointness.* These study permit the rigorous decomposition of Newtonian forces into a component that is variationally selfadjoint (SA) and a component that is not (NSA),

$$m_a \times \frac{dv_{ka}}{dt} = F_{ka}^{SA}(t, r, v) + F_{ka}^{NSA}(t, r, v). \tag{4.1.3}$$

Consequently, the true Lagrange and Hamilton equations can be more technically written

$$\left[\frac{d}{dt}\frac{\partial L(t,r,v)}{\partial v_a^k} - \frac{\partial L(t,r,v)}{\partial r_a^k}\right]^{SA} = F_{ak}^{NSA}(t, r, v), \tag{4.1.4a}$$

$$\left[\frac{dr_a^k}{dt} - \frac{\partial H(t,r,p)}{\partial p_{ak}}\right]^{SA} = 0, \quad \left[\frac{dp_{ak}}{dt} + \frac{\partial H(t,r,p)}{\partial r_a^k}\right]^{SA} = F_{ak}^{NSA}(t,r,p). \quad (4.1.4b)$$

The *forgotten legacy of Newton, Lagrange and Hamilton is that irreversibility originates precisely in the truncated NSA terms,* because all known potential-SA forces are reversible. The scientific imbalance of Section 1.3 is then due to the fact that no serious scientific study on irreversibility can be done with the truncated analytic equations and their operator counterpart, since these equations can only represent reversible systems.

Being born and educated in Italy, during his graduate studies at the University of Torino, the author had the opportunity of studying in the late 1960s the original works by Lagrange that were written precisely in Torino and most of them in Italian.

In this way, the author had the opportunity of verifying *Lagrange's analytic vision of representing irreversibility precisely via the external terms,* due to the impossibility of representing all possible physical events via the sole use of the Lagrangian, since the latter was solely conceived for the representation of reversible and potential events. As the reader can verify, Hamilton had, independently, the same vision.

Consequently, the truncation of the basic analytic equations caused the impossibility of a credible treatment of irreversibility at the purely classical level. The lack of a credible treatment of irreversibility then propagated at the subsequent operator level.

It then follows that *quantum mechanics cannot possibly be used for serious studies on irreversibility* because the discipline was constructed for the description of reversible quantized atomic orbits and not for irreversible systems.

In plain terms, while the validity of quantum mechanics for the arena of its original conception and verification is beyond scientific doubt, the assumption of quantum mechanics as the final operator theory for all conditions existing in the universe is outside the boundaries of serious science.

This establishes the need for the construction of a broadening (or generalization here called *lifting*) of quantum mechanics specifically conceived for quantitative studies of irreversibility. Since reversible systems are a *particular case* of irreversible ones, the broader mechanics must be a *covering* of quantum mechanics, that is, admitting the latter under a unique and unambiguous limit.

It is easy to see that the needed broader mechanics must also be a covering of the isotopic branch of hadronic mechanics studied in the preceding chapter, thus being a new branch of hadronic mechanics. In fact, isomechanics is itself structurally reversible due to the Hermiticity of both the Hamiltonian, $\hat{H} = \hat{H}^\dagger$, and of the isotopic element, $\hat{T} = \hat{T}^\dagger$, while a serious study of irreversible processes requires a *structurally irreversible mechanics,* that is, a mechanics that is irreversible for all possible reversible Hamiltonians.[1]

4.1.3 Early Representations of Irreversible Systems

As reviewed in Section 1.5.2, the brackets of the time evolution of an observable $A(r,p)$ in phase space according to the analytic equations with external terms,

$$\frac{dA}{dt} = (A, H, F) = \frac{\partial A}{\partial r_a^k} \times \frac{\partial H}{\partial p_{ka}} - \frac{\partial H}{\partial r_a^k} \times \frac{\partial A}{\partial p_{ka}} + \frac{\partial A}{\partial r_a^k} \times F_{ka}, \tag{4.1.5}$$

violate the right associative and scalar laws.

Therefore, the presence of external terms in the analytic equations causes not only the loss of *all* Lie algebras in the study of irreversibility, but actually causes the loss of all possible algebras as commonly understood in mathematics.

To resolve this problem, the author initiated a long scientific journey beginning with his graduate studies at the University of Torino, Italy, following the reading of Lagrange's papers.

The original argument [7–9], still valid today, is to select analytic equations characterizing brackets in the time evolution verifying the following conditions:

(1) The brackets of the time evolution must verify the right and left associative and scalar laws to characterize an algebra;

(2) Said brackets must not be invariant under time reversal as a necessary condition to represent irreversibility *ab initio;*

(3) Said algebra must be a covering of Lie algebras as a necessary condition to have a covering of the truncated analytic equations, namely, as a condition for the selected representation of irreversibility to admit reversibility as a particular case.

Condition (1) requires that said brackets must be bilinear, e.g., of the form (A, B) with properties

$$(n \times A, B) = n \times (A, B), \quad (A, m \times B) = m \times (A, B); \quad n, m \in C, \tag{4.1.6a}$$

$$(A \times B, C) = A \times (B, C), \quad (A, B \times C) = (A, B) \times C. \tag{4.1.6b}$$

Condition (2) requires that brackets (A, B) should not be totally antisymmetric as the conventional Poisson brackets,

$$(A, B) \neq -(B, A), \tag{4.1.7}$$

because time reversal is realized via the use of Hermitian conjugation.

[1] An exception to this general rule we shall study later on occurs when the isotopic elements is indeed Hermitian, but explicitly dependent on time and such that $\hat{T}(t, \ldots) \neq \hat{T}(-t, \ldots)$.

Condition (3) then implies that brackets (A, B) characterize *Lie-admissible algebras* in the sense of Albert [10], namely, the brackets are such that the attached antisymmetric algebra is Lie.[2]

$$[A, B]^* = (A, B) - (B, A) = Lie. \tag{4.1.8}$$

In particular, the latter condition implies that the new brackets are formed by the superposition of totally antisymmetric and totally symmetric brackets,

$$(A, B) = [A, B]^* + \{A, B\}^*. \tag{4.1.9}$$

It should be noted that the operator realization of brackets (A, B) is also *Jordan-admissible* in the sense of Albert [10], namely, the attached symmetric brackets $\{A, B\}^*$ characterize a *Jordan algebra.* Consequently, *hadronic mechanics provides a realization of Jordan's dream, that of seeing his algebra applied to physics.*

However, the reader should be aware that, for certain technical reasons beyond the scope of this monograph, the classical realizations of brackets (A, B) are Lie-admissible but not Jordan-admissible. Therefore, Jordan-admissibility appears to emerge exclusively for operator theories.[3]

After identifying the above lines, Santilli [9] proposed in 1967 the following *generalized analytic equations*

$$\frac{dr_a^k}{dt} = \alpha \times \frac{\partial H(t, r, p)}{\partial p_{ak}}, \quad \frac{dp_{ak}}{dt} = -\beta \times \frac{\partial H(t, r, p)}{\partial r_a^k}, \tag{4.1.10}$$

(where α and β are real non-null parameters) that are manifestly irreversible. The brackets of the time evolution are then given by

$$i \times \frac{dA}{dt} = (A, H) =$$

[2] More technically, a generally nonassociative algebra U with elements a, b, c, ... and abstract product ab is said to be Lie-admissible when the attached algebra U^- characterized by the product $[a, b] = ab - ba$ verifies the *Lie axioms*

$$[a, b] = -[b, a],$$
$$[[a, b], c] + [[b, c], a] + [[c, b], a] = 0.$$

[3] More technically, a generally nonassociative algebra U with elements a, b, c, ... and abstract product ab is said to be Jordan-admissible when the attached algebra U^+ characterized by the product $\{a, b\} = ab + ba$ verifies the *Jordan axioms*

$$\{a, b\} = \{b, a\},$$
$$\{\{a, b\}, a^2\} = \{a, \{b, a^2\}\}.$$

In classical realizations of the algebra U the first axiom of Jordan-admissibility is generally verified but the second is generally violated, while in operator realizations both axioms are generally verified.

$$= \alpha \times \frac{\partial A}{\partial r_a^k} \times \frac{\partial H}{\partial p_{ka}} - \beta \times \frac{\partial H}{\partial r_a^k} \times \frac{\partial A}{\partial p_{ka}}, \tag{4.1.11}$$

whose brackets are manifestly Lie-admissible, but *not* Jordan-admissible as the interested reader is encouraged to verify.

The above analytic equations characterize the time-rate of variation of the energy

$$\frac{dH}{dt} = (\alpha - \beta) \times \frac{\partial H}{\partial r_a^k} \times \frac{\partial H}{\partial p_{ka}}. \tag{4.1.12}$$

Also in 1967, Santilli [7,8] proposed an operator counterpart of the preceding classical setting consisting in the first known *Lie-admissible parametric generalization of Heisenberg's equation,* also called *deformed Heisenberg equations,*[4] in the following infinitesimal form

$$i \times \frac{dA}{dt} = (A, B) = p \times A \times H - q \times H \times A =$$

$$= m \times (A \times B - B \times A) + n \times (A \times B + B \times A), \tag{4.1.13a}$$

$$m = p + q, \quad n = q - p, \tag{4.1.13b}$$

where $p, q, p \pm q$ are non-null parameters, with finite counterpart

$$A(t) = e^{i \times H \times q} \times A(0) \times e^{-i \times p \times H}. \tag{4.1.14}$$

Brackets (A, B) are manifestly Lie-admissible with attached antisymmetric part

$$[A, B]^* = (A, B) - (B, A) = (p - q) \times [A, B]. \tag{4.1.15}$$

The same brackets are also Jordan-admissible in view of the property

$$\{A, B\}^* = (A, B) + (B, A) = (p + q) \times \{A, B\}, \tag{4.1.16}$$

The resulting time evolution is then manifestly irreversible (for $p \neq q$) with nonconservation of the energy

$$i \times \frac{dH}{dt} = (H, H) = (p - q) \times H \times H \neq 0, \tag{4.1.17}$$

as necessary for an open system.

Subsequently, Santilli realized that the above formulations are not invariant under their own time evolution (4.1.14) because Eqs. (4.1.11) are manifestly *nonunitary.*

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[4] As we shall soon see, the term "deformed" is used for formulations that are catastrophically inconsistent because dreaming to treat new theories with the mathematics of the old ones.

The application of nonunitary transforms to brackets (4.1.12) then led to the proposal in memoir [11,12] of 1978 of the following *Lie-admissible operator generalization of Heisenberg equations* in their infinitesimal form

$$i \times \frac{dA}{dt} = A \times P \times H - H \times Q \times A = (A, H)^*, \tag{4.1.18}$$

with finite counterpart

$$A(t) = e^{i \times H \times Q} \times A(0) \times e^{-i \times P \times H}, \tag{4.1.19}$$

under the subsidiary conditions needed for consistency, as we shall see,

$$P = Q^\dagger, \tag{4.1.20}$$

where P, Q and $P \pm Q$ are now nonsingular operators (or matrices), and Eq. (4.1.16b) is a basic consistency condition explained later in this section.

Eqs. (4.1.18)–(4.1.19) are the *fundamental equations of hadronic mechanics.* Their basic brackets are manifestly Lie-admissible and Jordan admissible with structure

$$(A, B)^* = A \times P \times B - B \times Q \times A =$$

$$= (A \times T \times B - B \times T \times A) + (A \times R \times B + B \times R \times A), \tag{4.1.21a}$$

$$T = P + Q, \quad R = Q - P. \tag{4.1.21b}$$

As indicated in Section 1.5.2, it is easy to see that the application of a nonunitary transform to the parametric brackets of Eqs. (4.1.11) leads precisely to the operator brackets of Eqs. (4.1.17),

$$U \times (p \times A \times B - q \times B \times A) \times U^\dagger = \hat{A} \times P \times \hat{B} - \hat{B} \times Q \times \hat{A}, \tag{4.1.22a}$$

$$U \times U^\dagger \neq I, P = p \times (U \times (U^\dagger)^{-1}, Q = q \times (U \times U^\dagger)^{-1}, \hat{A} = U \times A \times U^\dagger. \tag{4.1.22b}$$

In particular, the application of any (nonsingular) nonunitary transforms preserves the Lie-admissible and Jordan-admissible characters. Consequently, fundamental equations (4.1.18), (4.1.19) are “directly universal” in the sense of Lemma 1.5.2.

However, the above equations *are not invariant* under their own (nonunitary) time evolution,

$$U \times (\hat{A} \times P \times \hat{B} - \hat{B} \times Q \times \hat{A},) \times U^\dagger = \hat{A}' \times P' \times \hat{B}' - \hat{B}' \times Q' \times \hat{A}', \tag{4.1.23}$$

where the lack of invariance is expressed by the lack of preservation of the numerical values of the P, Q operators because, as we shall see shortly, these operators characterize new multiplications.

By comparison, quantum mechanical brackets are indeed invariant under the class of admitted transformations, the unitary transforms

$$W \times A \times B - B \times A) \times W^\dagger = A' \times B' - B' \times A', \tag{4.1.24a}$$

$$W \times W^\dagger = W^\dagger \times W = I, A' = W \times A \times W^\dagger, B' = W \times B \times W^\dagger, \tag{4.1.24b}$$

where the invariance we are here referring to is expressed by the preservation of the associative product, namely, $A \times B$ *is not* mapped into a different product, say $A' * B'$.

As known to experts of quantum mechanics (to qualify as such), simple invariance (4.1.24) is at the foundations of the majestic axiomatic consistency of quantum mechanics, including: the prediction of the same numerical values under the same conditions at different times; the preservation of Hermiticity and, thus, of observables over time; and other basic features.

Consequently, Lie-admissible and Jordan admissible equations (4.1.18)–(4.1.19) are afflicted by the catastrophic inconsistencies of Theorem 1.5.2, as it is the fate for all nonunitary theories some of which are listed in Section 1.5. In particular, said equations do not preserve numerical predictions under the same conditions but at different times, do not preserve Hermiticity, thus do not admit observables, and have other catastrophic inconsistencies studied in detail in Section 1.5.

Moreover, in the form presented above, the dynamical equations are not derivable from a variational principle. Consequently, they admit no known unique map from classical into operator formulations.

In view of these insufficiencies, said equations cannot be assumed in the above given form as the basic equations of any consistent physical theory.

4.2 ELEMENTS OF SANTILLI GENOMATHEMATICS AND ITS ISODUAL

4.2.1 Genounits, Genoproducts and their Isoduals

The "direct universality" of Eqs. (4.1.18), (4.1.19) voids any attempt at seeking further generalizations in the hope of achieving invariance, since any nontrivial generalization would suffer the loss of any algebra in the brackets of the time evolution, with consequential inability to achieve any physically meaningful theory, e.g., because of the inability to treat the spin of a proton under irreversible conditions.

This occurrence leaves no alternative other than that of seeking a yet *new mathematics* permitting Eqs. (4.1.18), (4.1.19) to achieve the needed invariance.

After numerous attempts and a futile search in the mathematical literature of the Cantabrigian area,[5] Santilli proposed in Refs. [11,12] of 1978 the construction of a new mathematics specifically conceived for the indicated task, that

[5]Conducted in the period 1977–1978.

eventually reached mathematical maturity for numbers only in paper [13] of 1993, mathematical maturity for the new differential calculus only in memoir [14] of 1996, and, finally, an invariant formulation of Lie-admissible equations only in paper [15] of 1997.

The new Lie-admissible mathematics is today known as *Santilli genomathematics*, where the prefix "geno" suggested in the original proposal [11,12] is used in the Greek meaning of "inducting" new axioms (as compared to the prefix "iso" of the preceding chapter denoting the preservation of the axioms).

The basic idea is to lift the isounits of the preceding chapter into a form that is still nowhere singular, but *non-Hermitian*, thus implying the existence of *two* different generalized units, today called *Santilli genounits* for the description of matter, that are generally written [13]

$$\hat{I}^{>} = 1/\hat{T}^{>}, \quad {}^{<}\hat{I} = 1/{}^{<}\hat{T}, \tag{4.2.1a}$$

$$\hat{I}^{>} \neq {}^{<}\hat{I}, \quad \hat{I}^{>} = ({}^{<}\hat{I})^{\dagger}, \tag{4.2.1b}$$

with two additional *isodual genounits* for the description of antimatter [14]

$$(\hat{I}^{>})^{d} = -(\hat{I}^{>})^{\dagger} = -{}^{<}\hat{I} = -1/{}^{<}\hat{T}, \quad ({}^{<}\hat{I})^{d} = -\hat{I}^{>} = -1/\hat{T}^{>}. \tag{4.2.2}$$

Jointly, all conventional and/or isotopic products $A\hat{\times}B$ among generic quantities (numbers, vector fields, operators, etc.) are lifted in such a form admitting the genounits as the correct left and right units at all levels, i.e.,

$$A > B = A \times \hat{T}^{>} \times B, \quad A > \hat{I}^{>} = \hat{I}^{>} > A = A, \tag{4.2.3a}$$

$$A < B = A \times^{<} \hat{T} \times B, \quad A <^{<} \hat{I} =^{<} \hat{I} < A = A, \tag{4.2.3b}$$

$$A >^{d} B = A \times \hat{T}^{>d} \times B, \quad A >^{d} \hat{I}^{>d} = \hat{I}^{>d} >^{d} A = A, \tag{4.2.3c}$$

$$A <^{d} B = A \times^{<} \hat{T}^{d} \times B, \quad A <^{d} {}^{<}\hat{I}^{d} = {}^{<}\hat{I}^{d} <^{d} A = A, \tag{4.2.3d}$$

for all elements A, B of the set considered.

As we shall see in Section 4.3, the above basic assumptions permit the representation of irreversibility with the most primitive possible quantities, the basic units and related products.

In particular, as we shall see in Section 4.3 and 4.4, genounits permit an invariant representation of the external forces in Lagrange's and Hamilton's equations (4.1.2). As such, genounits are generally dependent on time, coordinates, momenta, wavefunctions and any other needed variable, e.g., $\hat{I}^{>} = \hat{I}^{>}(t^{>}, r^{>}, p^{>}, \psi^{>}, \ldots)$.

In fact, the assumption of all *ordered product to the right* $>$ represents matter systems moving forward in time, the assumption of all *ordered products to the left* $<$ represents matter systems moving backward in time, with the irreversibility being represented *ab initio* by the inequality $A > B \neq A < B$. Similar representation of irreversible antimatter systems occurs via isodualities.

4.2.2 Genonumbers, Genofunctional Analysis and Their Isoduals

Genomathematics began to reach maturity with the discovery made, apparently for the first time in paper [13] of 1993, that *the axioms of a field still hold under the ordering of all products to the right or, independently, to the left.*

This unexpected property permitted the formulation of *new numbers,* that can be best introduced as a generalization of the isonumbers [18], although they can also be independently presented as follows:

DEFINITION 4.2.1 [13]: Let $F = F(a, +, \times)$ be a field of characteristic zero as per Definitions 2.1.1 and 3.2.1. Santilli's forward genofields are rings $\hat{F}^{>} = \hat{F}(\hat{a}^{>}, \hat{+}^{>}, \hat{\times}^{>})$ with elements

$$\hat{a}^{>} = a \times \hat{I}^{>}, \tag{4.2.4}$$

where $a \in F$, $\hat{I}^{>} = 1/\hat{T}^{>}$ is a non singular non-Hermitian quantity (number, matrix or operator) generally outside F and $\times$ is the ordinary product of F; the genosum $\hat{+}^{>}$ coincides with the ordinary sum $+$,

$$\hat{a}^{>}\hat{+}^{>}\hat{b}^{>} \equiv \hat{a}^{>} + \hat{b}^{>}, \ \ \forall \hat{a}^{>}, \ \hat{b}^{>} \in \hat{F}^{>}, \tag{4.2.5}$$

consequently, the additive forward genounit $\hat{0}^{>} \in \hat{F}$ coincides with the ordinary $0 \in F$; and the forward genoproduct $>$ is such that $\hat{I}^{>}$ is the right and left isounit of $\hat{F}^{>}$,

$$\hat{I}^{>}\hat{\times}\hat{a}^{>} = \hat{a}^{>} > \hat{I}^{>} \equiv \hat{a}^{>}, \ \ \forall \hat{a}^{>} \in \hat{F}^{>}. \tag{4.2.6}$$

Santilli's forward genofields verify the following properties:

1) For each element $\hat{a}^{>} \in \hat{F}^{>}$ there is an element $\hat{a}^{>-\hat{1}^{>}}$, called forward genoinverse, for which

$$\hat{a}^{>} > \hat{a}^{>-\hat{I}^{>}} = \hat{I}^{>}, \ \ \forall \hat{a}^{>} \in \hat{F}^{>}; \tag{4.2.7}$$

2) The genosum is commutative

$$\hat{a}^{>}\hat{+}^{>}\hat{b}^{>} = \hat{b}^{>}\hat{+}^{>}\hat{a}^{>}, \tag{4.2.8}$$

and associative

$$(\hat{a}^{>}\hat{+}^{>}\hat{b}^{>}) +^{>} \hat{c}^{>} = \hat{a}^{>}\hat{+}^{>}(\hat{b}^{>}\hat{+}^{>}\hat{c}^{>}), \ \ \forall \hat{a}, \hat{b}, \hat{c} \in \hat{F}; \tag{4.2.9}$$

3) The forward genoproduct is associative

$$\hat{a}^{>} > (\hat{b}^{>} > \hat{c}^{>}) = (\hat{a}^{>} > \hat{b}^{>}) > \hat{c}^{>}, \quad \forall \hat{a}^{>}, \hat{b}^{>}, \hat{c}^{>} \in \hat{F}^{>}, \tag{4.2.10}$$

but not necessarily commutative

$$\hat{a}^{>} > \hat{b}^{>} \neq \hat{b}^{>} > \hat{a}^{>}; \tag{4.2.11}$$

4) The set $\hat{F}^>$ is closed under the genosum,

$$\hat{a}^> \hat{+}^> \hat{b}^> = \hat{c}^> \in \hat{F}^>, \tag{4.2.12}$$

the forward genoproduct,

$$\hat{a}^> > \hat{b}^> = \hat{c}^> \in \hat{F}^>, \tag{4.2.13}$$

and right and left genodistributive compositions,

$$\hat{a}^> > (\hat{b}^> \hat{+}^> \hat{c}^>) = \hat{d}^> \in \hat{F}^>, \tag{4.2.14a}$$

$$(\hat{a}^> \hat{+}^> \hat{b}^>) > \hat{c}^> = \hat{d}^> \in \hat{F}^> \quad \forall \hat{a}^>, \hat{b}^>, \hat{c}^>, \hat{d}^> \in \hat{F}^>; \tag{4.2.14b}$$

5) The set $\hat{F}^>$ verifies the right and left genodistributive law

$$\hat{a}^> > (\hat{b}^> \hat{+}^> \hat{c}^>) = (\hat{a}^> \hat{+}^> \hat{b}^>) > \hat{c}^> = \hat{d}^>, \ \ \forall \hat{a}^>, \hat{b}^>, \hat{c}^>, \in \hat{F}^>. \tag{4.2.15}$$

In this way we have the forward genoreal numbers $\hat{R}^>$, the forward genocomplex numbers $\hat{C}^>$ and the forward genoquaternionic numbers $\hat{Q}C^>$ while the forward genooctonions $\hat{O}^>$ can indeed be formulated but they do not constitute genofields [14].

The backward genofields and the isodual forward and backward genofields are defined accordingly. Santilli's genofields are called of the first (second) kind when the genounit is (is not) an element of F.

The basic axiom-preserving character of genofields is illustrated by the following:

LEMMA 4.2.1 [13]: Genofields of first and second kind are fields (namely, they verify all axioms of a field).

Note that the conventional product "2 multiplied by 3" is not necessarily equal to 6 because, for isodual numbers with unit -1 it is given by -6 [13].

The same product "2 multiplied by 3" is not necessarily equal to $+6$ or -6 because, for the case of isonumbers, it can also be equal to an arbitrary number, or a matrix or an integrodifferential operator depending on the assumed isounit [13].

In this section we point out that "2 multiplied by 3" can be ordered to the right or to the left, and the result is not only arbitrary, but yielding different numerical results for different orderings, $2 > 3 \neq 2 < 3$, all this by continuing to verify the axioms of a field per each order [13].

Once the forward and backward genofields have been identified, the various branches of genomathematics can be constructed via simple compatibility arguments.

For specific applications to irreversible processes there is first the need to construct the *genofunctional analysis,* studied in Refs. [6,18] that we cannot review here for brevity. The reader is however warned that any elaboration of irreversible processes via Lie-admissible formulations based on conventional or isotopic functional analysis leads to catastrophic inconsistencies because it would be the same as elaborating quantum mechanical calculations with genomathematics.

As an illustration, Theorems 1.5.1 and 1.5.2 of catastrophic inconsistencies are activated unless one uses the ordinary differential calculus lifted, for ordinary motion in time of matter, into the following *forward genodifferentials and genoderivatives*

$$\hat{d}^{>} x = \hat{T}^{>}_{x} \times dx, \quad \frac{\hat{\partial}^{>}}{\hat{\partial}^{>} x} = \hat{I}^{>}_{x} \times \frac{\partial}{\partial x}, \text{ etc}, \tag{4.2.16}$$

with corresponding backward and isodual expressions here ignored.

Similarly, all conventional functions and isofunctions, such as isosinus, isocosinus, isolog, etc., have to be lifted in the genoform

$$\hat{f}^{>}(x^{>}) = f(\hat{x}^{>}) \times \hat{I}^{>}, \tag{4.2.17}$$

where one should note the necessity of the multiplication by the genounit as a condition for the result to be in $\hat{R}^{>}$, $\hat{C}^{>}$, or $\hat{O}^{>}$.

4.2.3 Genogeometries and Their Isoduals

Particularly intriguing are the *genogeometries* [16] (see also monographs [18] for detailed treatments). They are best characterized by a simple genotopy of the isogeometries, although they can be independently defined.

As an illustration, the *Minkowski-Santilli forward genospace* $\hat{M}^{>}(\hat{x}^{>}, \hat{\eta}^{>}, \hat{R}^{>})$ over the genoreal $\hat{R}^{>}$ is characterized by the following spacetime, *genocoordinates, genometric and genoinvariant*

$$\hat{x}^{>} = x\hat{I}^{>} = \{x^{\mu}\} \times \hat{I}^{>}, \quad \hat{\eta}^{>} = \hat{T}^{>} \times \eta, \quad \eta = Diag.(1, 1, 1, -1), \tag{4.2.18a}$$

$$\hat{x}^{>2^{>}} = \hat{x}^{>\mu} \hat{\times}^{>} \hat{\eta}^{>}_{\mu\nu} \hat{\times}^{>} \hat{x}^{>\nu} = (x^{\mu} \times \hat{\eta}^{>}_{\mu\nu} \times x^{\nu}) \times \hat{I}^{>}, \tag{4.2.18b}$$

where the first expression of the genoinvariant is on genospaces while the second is its projection in our spacetime.

Note that the Minkowski-Santilli genospace has, in general, an explicit dependence on spacetime coordinates. Consequently, it is equipped with the entire formalism of the conventional Riemannian spaces covariant derivative, Christoffel's symbols, Bianchi identity, etc. only lifted from the isotopic form of the preceding chapter into the genotopic form.

A most important feature is that *genospaces permit, apparently for the first time in scientific history, the representation of irreversibility directly via the basic genometric.* This is due to the fact that genometrics are nonsymmetric by conception, e.g.,

$$\hat{\eta}^{>}_{\mu\nu} \quad \neq \quad \hat{\eta}^{>}_{\nu\mu}. \tag{4.2.19}$$

Consequently, *genotopies permit the lifting of conventional symmetric metrics into nonsymmetric forms,*

$$\eta_{Symm}^{Minkow.} \quad \rightarrow \quad \hat{\eta}^{>}{}_{NonSymm}^{Minkow.-Sant.}. \tag{4.2.20}$$

Remarkably, *nonsymmetric metrics are indeed permitted by the axioms of conventional spaces* as illustrated by the invariance

$$(x^{\mu} \times \eta_{\mu\nu} \times x^{\nu}) \times I \equiv [x^{\mu} \times (\hat{T}^{>} \times \eta_{\mu\nu}) \times x^{\nu}] \times T^{>-1} \equiv$$
$$\equiv (x^{\mu} \times \hat{\eta}^{>}_{\mu\nu} \times x^{\nu}) \times \hat{I}^{>}, \tag{4.2.21}$$

where $\hat{T}^{>}$ is assumed in this simple illustration to be a complex number.

Interested readers can then work out backward genogeometries and the isodual forward and backward genogeometries with their underlying genofunctional analysis.

This basic geometric feature was not discovered until recently because hidden where nobody looked for, in the basic unit. However, this basic geometric advance in the representation of irreversibility required the prior discovery of basically new numbers, Santilli's genonumbers with nonsymmetric unit and ordered multiplication [14].

4.2.4 Santilli Lie-Admissible Theory and Its Isodual

Particularly important for irreversibility is the lifting of Lie's theory and Lie-Santilli's isotheories permitted by genomathematics, first identified by Ref. [11] of 1978 (and then studied in various works, e.g., [6,18–22]) via the following genotopies:

(1) The *forward and backward universal enveloping genoassociative algebra* $\hat{\xi}^{>}$, ${}^{<}\hat{\xi}$, with infinite-dimensional basis characterizing the *Poincaré-Birkhoff-Witt-Santilli genotheorem*

$$\hat{\xi}^{>} : \hat{I}^{>},\ \hat{X}_i,\ \hat{X}_i > \hat{X}_j,\ \hat{X}_i > \hat{X}_j > \hat{X}_k,\ \ldots,\ i \leq j \leq k, \tag{4.2.22a}$$

$$^{<}\hat{\xi} : \hat{I},\ {}^{<}\hat{X}_i,\ \hat{X}_i < \hat{X}_j,\ \hat{X}_i < \hat{X}_j < \hat{X}_k,\ \ldots,\ i \leq j \leq k; \tag{4.2.22b}$$

where the "hat" on the generators denotes their formulation on genospaces over genofields and their Hermiticity implies that $\hat{X}^{>} = ^{<} \hat{X} = \hat{X}$;

(2) The *Lie-Santilli genoalgebras* characterized by the universal, jointly Lie- and Jordan-admissible brackets,

$$^{<}\hat{L}^{>} : (\hat{X}_i \hat{,} \hat{X}_j) = \hat{X}_i < \hat{X}_j - \hat{X}_j > \hat{X}_i = C_{ij}^{k} \times \hat{X}_k, \tag{4.2.23}$$

here formulated in an invariant form (see below);

(3) The *Lie-Santilli genotransformation groups*

$$^{<}\hat{G}^{>} : \hat{A}(\hat{w}) = (\hat{e}^{\hat{i} \hat{\times} \hat{X} \hat{\times} \hat{w}})^{>} > \hat{A}(\hat{0}) <^{<} (\hat{e}^{-\hat{i} \hat{\times} \hat{w} \hat{\times} \hat{X}}) =$$

$$= (e^{i\times\hat{X}\times\hat{T}^{>}\times w}) \times A(0) \times (e^{-i\times w\times{}^{<}\hat{T}\times\hat{X}}), \tag{4.2.24}$$

where $\hat{w}^{>} \in \hat{R}^{>}$ are the *genoparameters;* the *genorepresentation theory,* etc.

4.2.5 Genosymmetries and Nonconservation Laws

The implications of the Santilli Lie-admissible theory are significant mathematically and physically. On mathematical grounds, the Lie-Santilli genoalgebras are "directly universal" and include as particular cases all known algebras, such as Lie, Jordan, Flexible algebras, power associative algebras, quantum, algebras, supersymmetric algebras, Kac-Moody algebras, etc. (Section 1.5).

Moreover, when computed on the *genobimodule*

$${}^{<}\hat{B}^{>} = {}^{<}\hat{\xi} \times \hat{\xi}^{>}, \tag{4.2.25}$$

Lie-admissible algebras verify all Lie axioms, while deviations from Lie algebras emerge only in their *projection* on the conventional bimodule

$${}^{<}B^{>} = {}^{<}\xi \times \xi^{>} \tag{4.2.26}$$

of Lie's theory (see Ref. [17] for the initiation of the genorepresentation theory of Lie-admissible algebras on bimodules).

This is due to the fact that the computation of the left action $A < B = A \times^{<} \hat{T} \times B$ on ${}^{<}\hat{\xi}$ (that is, with respect to the genounit ${}^{<}\hat{I} = 1/{}^{<}\hat{T}$) yields the save value as the computation of the conventional product $A \times B$ on ${}^{<}\xi$ (that is, with respect to the trivial unit I), and the same occurs for the value of $A > B$ on $\hat{\xi}^{>}$.

The above occurrences explain the reason the structure constant and the product in the r.h.s. of Eq. (4.2.23) are those of a conventional Lie algebra.

In this way, thanks to genomathematics, *Lie algebras acquire a towering significance in view of the possibility of reducing all possible irreversible systems to primitive Lie axioms.*

The physical implications of the Lie-Santilli genotheory are equally far reaching. In fact, Noether's theorem on the reduction of reversible conservation laws to primitive Lie symmetries can be lifted to the *reduction, this time, of irreversible nonconservation laws to primitive Lie-Santilli genosymmetries.*

As a matter of fact, this reduction was the very first motivation for the construction of the genotheory in memoir [12] (see also monographs [6,18,19,20]). The reader can then foresee similar liftings of all remaining physical aspects treated via Lie algebras.

The construction of the isodual Lie-Santilli genotheory is an instructive exercise for readers interested in learning the new methods.

4.3 LIE-ADMISSIBLE CLASSICAL MECHANICS FOR MATTER AND ITS ISODUAL FOR ANTIMATTER

4.3.1 Fundamental Ordering Assumption on Irreversibility

Another reason for the inability during the 20-th century for in depth studies of irreversibility is the general belief that motion in time has only two directions, forward and backward (Eddington historical time arrows). In reality, motion in time admits *four* different forms, all essential for serious studies in irreversibility, given by: 1) *motion forward to future time* characterized by the forward genotime $\hat{t}^{>}$; 2) *motion backward to past time* characterized by the backward genotime $^{<}\hat{t}$; 3) *motion backward from future time* characterized by the isodual forward genotime $\hat{t}^{>d}$; and 4) *motion forward from past time* characterized by the isodual backward genotime $^{<}\hat{t}^{d}$.

It is at this point where the *necessity* of both time reversal and isoduality appears in its full light. In fact, time reversal is only applicable to matter and, being represented with Hermitian conjugation, permits the transition from motion forward to motion backward in time, $\hat{t}^{>} \rightarrow ^{<}\hat{t} = (\hat{t}^{>})^{\dagger}$. If used alone, time reversal cannot identify all four directions of motions. The *only* additional conjugation known to this author that is applicable at all levels of study and is equivalent to charge conjugation, is isoduality [22].

The additional discovery of two complementary orderings of the product and related units, with corresponding isoduals versions, individually preserving the abstract axioms of a field has truly fundamental implications for irreversibility, since it permits the axiomatically consistent and invariant representation of irreversibility via the most ultimate and primitive axioms, those on the product and related unit. We, therefore, have the following:

FUNDAMENTAL ORDERING ASSUMPTION ON IRREVERSIBILITY [15]: *Dynamical equations for motion forward in time of matter (antimatter) systems are characterized by genoproducts to the right and related genounits (their isoduals), while dynamical equations for the motion backward in time of matter (antimatter) are characterized by genoproducts to the left and related genounits (their isoduals) under the condition that said genoproducts and genounits are interconnected by time reversal expressible for generic quantities A, B with the relation,*

$$(A > B)^{\dagger} = (A > \hat{T}^{>} \times B)^{\dagger} = B^{\dagger} \times (\hat{T}^{>})^{\dagger} \times A^{\dagger}, \tag{4.3.1}$$

namely,

$$\hat{T}^{>} = (^{<}\hat{T})^{\dagger} \tag{4.3.2}$$

thus recovering the fundamental complementary conditions (4.1.17) or (4.2.2).

Unless otherwise specified, from now on physical and chemical expression for irreversible processes will have no meaning without the selection of one of the indicated two possible orderings.

4.3.2 Newton-Santilli Genoequations and Their Isoduals

Recall that, for the case of isotopies, the basic Newtonian systems are given by those admitting nonconservative internal forces restricted by certain constraints to verify total conservation laws called *closed non-Hamiltonian systems* [6b,18].

For the case of the genotopies under consideration here, the basic Newtonian systems are the conventional nonconservative systems without subsidiary constraints, known as*open non-Hamiltonian systems*, with generic expression (1.3), in which case irreversibility is entirely characterized by nonselfadjoint forces, since all conservative forces are reversible.

As it is well known, the above equations are not derivable from any variational principle in the fixed frame of the observer [6], and this is the reason all conventional attempts for consistently quantizing nonconservative forces have failed for about one century. In turn, the lack of achievement of a consistent operator counterpart of nonconservative forces lead to the belief that they are "illusory" because they "disappear" at the particle level.

The studies presented in this paper have achieved the first and only physically consistent operator formulation of nonconservative forces known to the author. This goal was achieved by rewriting Newton's equations (1.3) into an identical form derivable from a variational principle. Still in turn, the latter objective was solely permitted by the novel genomathematics.

It is appropriate to recall that Newton was forced to discover new mathematics, the differential calculus, prior to being able to formulated his celebrated equations. Therefore, readers should not be surprised at the need for the new genodifferential calculus as a condition to represent all nonconservative Newton's systems from a variational principle.

Recall also from Section 3.1 that, contrary to popular beliefs, there exist *four* inequivalent directions of time. Consequently, time reversal alone cannot represent all these possible motions, and isoduality results to be the only known additional conjugation that, when combined with time reversal, can represent all possible time evolutions of both matter and antimatter.

The above setting implies the existence of four different new mechanics first formulated by Santilli in memoir [14] of 1996, and today known as *Newton-Santilli genomechanics,* namely:

A) *Forward genomechanics* for the representation of forward motion of matter systems;

B) *Backward genomechanics* for the representation of the time reversal image of matter systems;

C) *Isodual backward genomechanics* for the representation of motion backward in time of antimatter systems, and

D) *Isodual forward genomechanics* for the representation of time reversal antimatter systems.

These new mechanics are characterized by:

1) Four different times, *forward and backward genotimes for matter systems and the backward and forward isodual genotimes for antimatter systems*

$$\hat{t}^{>} = t \times \hat{I}_t^{>}, \quad -\hat{t}^{>}, \quad \hat{t}^{>d}, \quad -\hat{t}^{>d}, \tag{4.3.3}$$

with (nowhere singular and non-Hermitian) *forward and backward time genounits and their isoduals* (Note that, to verify the condition of non-Hermiticity, the time genounits can be *complex valued.*),

$$\hat{I}_t^{>} = 1/\hat{T}_t^{>}, \quad -\hat{I}_t^{>}, \quad \hat{I}^{>d}_{t}, \quad -\hat{I}^{>d}_{t}; \tag{4.3.4}$$

2) The *forward and backward genocoordinates and their isoduals*

$$\hat{x}^{>} = x \times \hat{I}_x^{>}, \quad -\hat{x}^{>}, \quad \hat{x}^{>d}, \quad -\hat{x}^{>d}, \tag{4.3.5}$$

with (nowhere singular non-Hermitian) *coordinate genounit*

$$\hat{I}_x^{>} = 1/\hat{T}_x^{>}, \quad -\hat{I}_x^{>}, \quad \hat{I}^{>d}_{x}, \quad -\hat{I}^{>d}_{x}, \tag{4.3.6}$$

with *forward and backward coordinate genospace and their isoduals* $\hat{S}_x^{>}$, etc., and related *forward coordinate genofield and their isoduals* $\hat{R}_x^{>}$, etc.;

3) The *forward and backward genospeeds and their isoduals*

$$\hat{v}^{>} = \hat{d}^{>}\hat{x}^{>}/\hat{d}^{>}\hat{t}^{>}, \quad -\hat{v}^{>}, \quad \hat{v}^{>d}, \quad -\hat{v}^{>d}, \tag{4.3.7}$$

with (nowhere singular and non-Hermitian) *speed genounit*

$$\hat{I}_v^{>} = 1/\hat{T}_v^{>}, \quad -\hat{I}_v^{>}, \quad \hat{I}^{>d}_{v}, \quad -\hat{I}^{>d}_{v}, \tag{4.3.8}$$

with related *forward speed backward genospaces and their isoduals* $\hat{S}_v^{>}$, etc., over *forward and backward speed genofields* $\hat{R}_v^{>}$, etc.

The above formalism then leads to the *forward genospace for matter systems*

$$\hat{S}_{tot}^{>} = \hat{S}_t^{>} \times \hat{S}_x^{>} \times \hat{S}_v^{>}, \tag{4.3.9}$$

defined over the *forward genofield*

$$\hat{R}_{tot}^{>} = \hat{R}_t^{>} \times \hat{R}_x^{>} \times \hat{R}_v^{>}, \tag{4.3.10}$$

with *total forward genounit*

$$\hat{I}_{tot}^{>} = \hat{I}_t^{>} \times \hat{I}_x^{>} \times \hat{I}_v^{>}, \tag{4.3.11}$$

and corresponding expressions for the remaining three spaces obtained via time reversal and isoduality.

The basic equations are given by:

I) The *forward Newton-Santilli genoequations for matter systems* [14], formulated via the genodifferential calculus,

$$\hat{m}_a^> > \frac{\hat{d}^> \hat{v}_{ka}^>}{\hat{d}^> \hat{t}^>} = -\frac{\hat{\partial}^> \hat{V}^>}{\hat{\partial}^> \hat{x}_a^{>k}}; \tag{4.3.12}$$

II) The *backward genoequations for matter systems* that are characterized by time reversal of the preceding ones;

III) the *backward isodual genoequations for antimatter systems* that are characterized by the isodual map of the backward genoequations,

$$^<\hat{m}_a^d < \frac{^<\hat{d}^{d<} \hat{v}_{ka}^d}{^<\hat{d}^{d<} \hat{t}^d} = -\frac{^<\hat{\partial}^{d<} \hat{V}^d}{^<\hat{\partial}^{d<} \hat{x}_a^{dk}}; \tag{4.3.13}$$

IV) the *forward isodual genoequations for antimatter systems* characterized by time reversal of the preceding isodual equations.

Newton-Santilli genoequations (4.3.12) are "directly universal" for the representation of all possible (well behaved) Eqs. (1.3) in the frame of the observer because they admit a multiple infinity of solution for any given nonselfadjoint force.

A simple representation occurs under the conditions assumed for simplicity,

$$N = \hat{I}_t^> = \hat{I}_v^> = 1, \tag{4.3.14}$$

for which Eqs. (3.12) can be explicitly written

$$\hat{m}^> > \frac{\hat{d}^> \hat{v}^>}{\hat{d}^> t} = m \times \frac{d\hat{v}^>}{dt} =$$

$$= m \times \frac{d}{dt} \frac{d(x \times \hat{I}_x^>)}{dt} = m \times \frac{dv}{dt} \times \hat{I}_x^> + m \times x \times \frac{d\hat{I}^>}{dt} = \hat{I}_x^> \times \frac{\partial V}{\partial x}, \tag{4.3.15}$$

from which we obtain the genorepresentation

$$F^{NSA} = -m \times x \times \frac{1}{\hat{I}_x^>} \times \frac{d\hat{I}_x^>}{dt}, \tag{4.3.16}$$

that always admit solutions here left to the interested reader since in the next section we shall show a much simpler, universal, *algebraic* solution.

As one can see, in Newton's equations the nonpotential forces are part of the applied force, while in the Newton-Santilli genoequations nonpotential forces are represented by the genounits, or, equivalently, by the genodifferential calculus, in a way essentially similar to the case of isotopies.

The main difference between iso- and geno-equations is that isounits are Hermitian, thus implying the equivalence of forward and backward motions, while genounits are non-Hermitian, thus implying irreversibility.

Note also that the topology underlying Newton's equations is the conventional, Euclidean, local-differential topology which, as such, can only represent point particles.

By contrast, the topology underlying the Newton-Santilli genoequations is given by a genotopy of the isotopology studied in the preceding chapter, thus permitting the representation of extended, nonspherical and deformable particles via forward genounits, e.g., of the type

$$\hat{I}^{>} = Diag.(n_1^2, n_2^2, n_3^2, n_4^2) \times \Gamma^{>}(t, r, v, \ldots), \tag{4.3.17}$$

where $n_k^2,\ k = 1, 2, 3$ represents the semiaxes of an ellipsoid, n_4^2 represents the density of the medium in which motion occurs (with more general nondiagonal realizations here omitted for simplicity), and $\Gamma^{>}$ constitutes a nonsymmetric matrix representing nonselfadjoint forces, namely, the contact interactions among extended constituents occurring for the motion forward in time.

4.3.3 Hamilton-Santilli Genomechanics and Its Isodual

In this section we show that, once rewritten in their identical genoform (4.3.12), Newton's equations for nonconservative systems are indeed derivable from a variational principle, with analytic equations possessing a Lie-admissible structure and Hamilton-Jacobi equations suitable for the first known consistent and unique operator map studied in the next section.

The most effective setting to introduce real-valued non-symmetric genounits is in the $6N$-dimensional *forward genospace (genocotangent bundle)* with local genocoordinates and their conjugates

$$\hat{a}^{>\mu} = a^{\rho} \times \hat{I}^{>\mu}_{1\ \rho},\ \ (\hat{a}^{>\mu}) = \begin{pmatrix} \hat{x}^{>k}_{\alpha} \\ \hat{p}^{>}_{k\alpha} \end{pmatrix} \tag{4.3.18}$$

and

$$\hat{R}^{>}_{\mu} = R_{\rho} \times \hat{I}^{>\rho}_{2\ \mu},\ (\hat{R}^{>}_{\mu}) = (\hat{p}_{k\alpha}, \hat{0}), \tag{4.3.19a}$$

$$\hat{I}^{>}_1 = 1/\hat{T}^{>}_1 = (\hat{I}^{>}_2)^T = (1/\hat{T}^{>}_2)^T, \tag{4.3.19b}$$

$$k = 1, 2, 3;\ \alpha = 1, 2, \ldots, N;\ \ \mu,\ \rho = 1, 2, \ldots 6N,$$

where the superscript T stands for transposed, and nowhere singular, real-valued and non-symmetric genometric and related invariant

$$\hat{\delta}^{>} = \hat{T}^{>}_{1\,6N\times 6N}\,\delta_{6N\times 6N} \times \delta_{6N\times 6N}, \tag{4.3.20a}$$

$$\hat{a}^{>\mu} > \hat{R}^{>}_{\mu} = \hat{a}^{>\rho} \times \hat{T}^{>\beta}_{1\ \rho} \times \hat{R}^{>}_{\beta} = a^{\rho} \times \hat{I}^{>\beta}_{2\ \rho} \times R_{\beta}. \qquad (4.3.20b)$$

In this case we have the following *genoaction principle* [14]

$$\hat{\delta}^{>}\hat{\mathcal{A}}^{>} = \hat{\delta}^{>} \hat{\int}^{>} [\hat{R}^{>}_{\mu} >_a \hat{d}^{>}\hat{a}^{>\mu} - \hat{H}^{>} >_t \hat{d}^{>}\hat{t}^{>}] =$$

$$= \delta \int [R_{\mu} \times \hat{T}^{>\mu}_{1\ \nu}(t, x, p, \ldots) \times d(a^{\beta} \times \hat{I}^{>\nu}_{1\ \beta}) - H \times dt] = 0, \qquad (4.3.21)$$

where the second expression is the projection on conventional spaces over conventional fields and we have assumed for simplicity that the time genounit is 1.

It is easy to prove that the above genoprinciple characterizes the following *forward Hamilton-Santilli genoequations,* (originally proposed in Ref. [11] of 1978 with conventional mathematics and in Ref. [14] of 1996 with genomathematics (see also Refs. [18,19,20])

$$\hat{\omega}^{>}_{\mu\nu} > \frac{\hat{d}^{>}\hat{a}^{\nu>}}{\hat{d}^{>}\hat{t}^{>}} - \frac{\hat{\partial}^{>}\hat{H}^{>}(\hat{a}^{>})}{\hat{\partial}^{>}\hat{a}^{\mu>}} =$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} dr/dt \\ dp/dt \end{pmatrix} - \begin{pmatrix} 1 & K \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} \partial H/\partial r \\ \partial H/\partial p \end{pmatrix} = 0, \qquad (4.3.22a)$$

$$\hat{\omega}^{>} = \Big(\frac{\hat{\partial}^{>}R^{>}_{\nu}}{\hat{\partial}^{>}\hat{a}^{\mu>}} - \frac{\hat{\partial}^{>}\hat{R}^{>}_{\mu}}{\hat{\partial}^{>}\hat{a}^{\nu>}}\Big) \times \hat{I}^{>} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \times \hat{I}^{>}, \qquad (4.3.22b)$$

$$K = F^{NSA}/(\partial H/\partial p), \qquad (4.3.22c)$$

where one should note the "direct universality" of the simple algebraic solution (3.22c).

The time evolution of a quantity $\hat{A}^{>}(\hat{a}^{>})$ on the forward geno-phase-space can be written in terms of the following brackets

$$\frac{\hat{d}^{>}\hat{A}^{>}}{\hat{d}^{>}t^{>}} = (\hat{A}^{>}, \hat{H}^{>}) = \frac{\hat{\partial}^{>}\hat{A}^{>}}{\hat{\partial}^{>}\hat{a}^{>\mu}} > \hat{\omega}^{\mu\nu>} > \frac{\hat{\partial}^{>}\hat{H}^{>}}{\hat{\partial}\hat{a}^{>\nu}} =$$

$$= \frac{\partial\hat{A}^{>}}{\partial\hat{a}^{>\mu}} \times S^{\,\mu\nu} \times \frac{\partial\hat{H}^{>}}{\partial\hat{a}^{>\nu}} =$$

$$= \Big(\frac{\partial\hat{A}^{>}}{\partial\hat{r}^{>k}_{\alpha}} \times \frac{\partial\hat{H}^{>}}{\partial\hat{p}^{>}_{ka}} - \frac{\partial\hat{A}^{>}}{\partial\hat{p}^{>}_{ka}} \times \frac{\partial\hat{H}^{>}}{\partial\hat{r}^{>k}_{a}}\Big) + \frac{\partial\hat{A}^{>}}{\partial\hat{p}^{>}_{ka}} \times F^{NSA}_{ka}, \qquad (4.3.23a)$$

$$S^{>\mu\nu} = \omega^{\mu\rho} \times \hat{I}^{2\,\mu}_{\rho}, \omega^{\mu\nu} = (||\omega_{\alpha\beta}||^{-1})^{\mu\nu}, \qquad (4.3.23b)$$

where $\omega^{\mu\nu}$ is the conventional Lie tensor and, consequently, $S^{\mu\nu}$ is Lie-admissible in the sense of Albert [7].

As one can see, the important consequence of genomathematics and its genodifferential calculus is that of turning the triple system (A, H, F^{NSA}) of Eqs. (1.5) in the bilinear form $(A\hat{,}B)$, thus characterizing a consistent algebra in the brackets of the time evolution.

This is the central purpose for which genomathematics was built (note that the multiplicative factors represented by K are fixed for each given system). The invariance of such a formulation will be proved shortly.

It is an instructive exercise for interested readers to prove that the brackets $(A\hat{,}B)$ are Lie-admissible, although not Jordan-admissible.

It is easy to verify that the above identical reformulation of Hamilton's historical time evolution correctly recovers the *time rate of variations of physical quantities* in general, and that of the energy in particular,

$$\frac{dA^>}{dt} = (A^>, H^>) = [\hat{A}^>, \hat{H}^>] + \frac{\partial \hat{A}^>}{\partial \hat{p}^>_{k\alpha}} \times F^{NSA}_{k\alpha}, \tag{4.3.24a}$$

$$\frac{dH}{dt} = [\hat{H}^>, \hat{H}^>] + \frac{\partial \hat{H}^>}{\partial \hat{p}^>_{k\alpha}} \times F^{NSA}_{ka} = v^k_\alpha \times F^{NSA}_{ka}. \tag{4.3.24b}$$

It is easy to show that genoaction principle (4.3.21) characterizes the following *Hamilton-Jacobi-Santilli genoequations* [14]

$$\frac{\hat{\partial}^> \mathcal{A}^>}{\hat{\partial}^> \hat{t}^>} + \hat{H}^> = 0, \tag{4.3.25a}$$

$$\Big(\frac{\hat{\partial}^> \mathcal{A}^>}{\hat{\partial}^> \hat{a}^{>\mu}}\Big) = \Big(\frac{\hat{\partial}^> \mathcal{A}^>}{\hat{\partial}^> x^{>k}_a}, \frac{\hat{\partial}^> \mathcal{A}^>}{\hat{\partial}^> p^>_{ka}}\Big) = (\hat{R}^>_\mu) = (\hat{p}^>_{ka}, \hat{0}), \tag{4.3.25b}$$

which confirm the property (crucial for genoquantization as shown below) that the genoaction is indeed independent of the linear momentum.

Note the *direct universality* of the Lie-admissible equations for the representation of all infinitely possible Newton equations (1.3) (universality) directly in the fixed frame of the experimenter (direct universality).

Note also that, *at the abstract, realization-free level, Hamilton-Santilli genoequations coincide* with Hamilton's equations without external terms, yet represent those with external terms.

The latter are reformulated via genomathematics as the only known way to achieve invariance and derivability from a variational principle while admitting a consistent algebra in the brackets of the time evolution [38].

Therefore, Hamilton-Santilli genoequations (3.6.66) are indeed irreversible for all possible reversible Hamiltonians, as desired. The origin of irreversibility rests in the contact nonpotential forces F^{NSA} according to Lagrange's and Hamilton's teaching that is merely reformulated in an invariant way.

The above Lie-admissible mechanics requires, for completeness, *three* additional formulations, the *backward genomechanics* for the description of *matter moving backward in time,* and the isoduals of both the forward and backward mechanics for the description of *antimatter.*

The construction of these additional mechanics is left to the interested reader for brevity.

4.4 LIE-ADMISSIBLE OPERATOR MECHANICS FOR MATTER AND ITS ISODUAL FOR ANTIMATTER

4.4.1 Basic Dynamical Equations

A simple genotopy of the naive or symplectic quantization applied to Eqs. (3.24) yields the *Lie-admissible branch of hadronic mechanics* [18] comprising four different formulations, the *forward and backward genomechanics for matter and their isoduals for antimatter.* The forward genomechanics for matter is characterized by the following main topics:

1) The nowhere singular (thus everywhere invertible) non-Hermitian *forward genounit* for the representation of all effects causing irreversibility, such as contact nonpotential interactions among extended particles, etc. (see the subsequent chapters for various realizations)

$$\hat{I}^{>} = 1/\hat{T}^{>} \neq (\hat{I}^{>})^{\dagger}, \tag{4.4.1}$$

with corresponding ordered product and genoreal $\hat{R}^{>}$ and genocomplex $\hat{C}^{>}$ genofields;

2) The *forward genotopic Hilbert space* $\hat{\mathcal{H}}^{>}$ with *forward genostates* $|\hat{\psi}^{>}>$ and *forward genoinner product*

$$<^{<}\hat{\psi}| > |\hat{\psi}^{>}> \times \hat{I}^{>} = <^{<}\hat{\psi}| \times \hat{T}^{>} \times |\hat{\psi}^{>}> \times \hat{I}^{>} \in \hat{C}^{>}, \tag{4.4.2}$$

and fundamental property

$$\hat{I}^{>} > |\hat{\psi}^{>}> = |\hat{\psi}^{>}>, \tag{4.4.3}$$

holding under the condition that $\hat{I}^{>}$ is indeed the correct unit for motion forward in time, and *forward genounitary transforms*

$$\hat{U}^{>} > (^{<}\hat{U})^{\dagger>} = (^{<}\hat{U})^{\dagger>} > \hat{U}^{>} = \hat{I}^{>}; \tag{4.4.4}$$

3) The fundamental Lie-admissible equations, first proposed in Ref. [12] of 1974 (p. 783, Eqs. (4.18.16)) as the foundations of hadronic mechanics, formulated on conventional spaces over conventional fields, and first formulated in Refs. [14,18]

of 1996 on genospaces and genodifferential calculus on genofields, today's known as *Heisenberg-Santilli genoequations*, that can be written in the finite form

$$\hat{A}(\hat{t}) = \hat{U}^{>} > \hat{A}(0) <^{<} \hat{U} = (\hat{e}_{>}^{\hat{i}\hat{\times}\hat{H}\hat{\times}\hat{t}}) > \hat{A}(\hat{0}) < ({}_{<}\hat{e}^{-\hat{i}\hat{\times}\hat{t}\hat{\times}\hat{H}}) =$$
$$= (e^{i\times\hat{H}\times\hat{T}^{>}\times t}) \times A(0) \times (e^{-i\times t\times{}^{<}\hat{T}\times\hat{H}}), \tag{4.4.5}$$

with corresponding infinitesimal version

$$\hat{i}\hat{\times}\frac{\hat{d}\hat{A}}{\hat{d}\hat{t}} = (\hat{A}\hat{;}\hat{H}) = \hat{A} < \hat{H} - \hat{H} > \hat{A} =$$
$$= \hat{A} \times^{<} \hat{T}(\hat{t}, \hat{r}, \hat{p}, \hat{\psi}, \ldots.) \times \hat{H} - \hat{H} \times \hat{T}^{>}(\hat{t}, \hat{r}, \hat{p}, \hat{\psi}, \ldots) \times \hat{A}, \tag{4.4.6}$$

where there is no time arrow, since Heisenberg's equations are computed at a fixed time;

4) The equivalent *Schrödinger-Santilli genoequations*, first suggested in the original proposal [12] to build hadronic mechanics (see also Refs. [17,23,24]), formulated via conventional mathematics and in Refs. [14,18] via genomathematics, that can be written

$$\hat{i}^{>} > \frac{\hat{\partial}^{>}}{\hat{\partial}^{>}\hat{t}^{>}}|\hat{\psi}^{>} >= \hat{H}^{>} > |\hat{\psi}^{>} >=$$
$$= \hat{H}(\hat{r}, \hat{v}) \times \hat{T}^{>}(\hat{t}, \hat{r}, \hat{p}, \hat{\psi}, \hat{\partial}\hat{\psi} \ldots) \times |\hat{\psi}^{>} >= E^{>} > |\psi^{>} >, \tag{4.4.7}$$

where the time orderings in the second term are ignored for simplicity of notation;

5) The *forward genomomentum* that escaped identification for two decades and was finally identified thanks to the genodifferential calculus in Ref. [14] of 1996

$$\hat{p}_{k}^{>} > |\hat{\psi}^{>} >= -\hat{i}^{>} > \hat{\partial}_{k}^{>}|\hat{\psi}^{>} >= -i \times \hat{I}_{k}^{>\,i} \times \partial_i|\hat{\psi}^{>} >; \tag{4.4.8}$$

6) The *fundamental genocommutation rules* also first identified in Ref. [14],

$$(\hat{r}^{i}\hat{;}\hat{p}_{j}) = i \times \delta^{i}_{j} \times \hat{I}^{>}, \ \ (\hat{r}^{i}\hat{;}\hat{r}^{j}) = (\hat{p}_{i}\hat{;}\hat{p}_{j}) = 0; \tag{4.4.9}$$

7) The *genoexpectation values* of an observable for the forward motion $\hat{A}^{>}$ [14,19]

$$\frac{<^{<} \hat{\psi}| > \hat{A}^{>} > |\hat{\psi}^{>} >}{<^{<} \hat{\psi}| > |\hat{\psi}^{>} >} \times \hat{I}^{>} \in \hat{C}^{>}, \tag{4.4.10}$$

under which the genoexpectation values of the genounit recovers the conventional Planck's unit as in the isotopic case,

$$\frac{< \hat{\psi}| > \hat{I}^{>} > |\hat{\psi} >}{< \hat{\psi}| > |\hat{\psi} >} = I. \tag{4.4.11}$$

The following comments are now in order. Note first in the genoaction principle the crucial independence of isoaction $\hat{\mathcal{A}}^{>}$ in form the linear momentum, as expressed by the Hamilton-Jacobi-Santilli genoequations (4.3.25). Such independence assures that genoquantization yields a genowavefunction solely dependent on time and coordinates, $\hat{\psi}^{>} = \hat{\psi}^{>}(t, r)$.

Other geno-Hamiltonian mechanics studied previously [7] do not verify such a condition, thus implying genowavefunctions with an explicit dependence also on linear momenta, $\hat{\psi}^{>} = \hat{\psi}^{>}(t, r, p)$ that violate the abstract identity of quantum and hadronic mechanics whose treatment in any case is beyond our operator knowledge at this writing.

Note that *forward geno-Hermiticity coincides with conventional Hermiticity.* As a result, *all quantities that are observables for quantum mechanics remain observables for the above genomechanics.*

However, unlike quantum mechanics, physical quantities are generally *nonconserved,* as it must be the case for the energy,

$$\hat{i}^{>} > \frac{\hat{d}^{>}\hat{H}^{>}}{\hat{d}^{>}\hat{t}^{>}} = \hat{H} \times ({}^{<}\hat{T} - \hat{T}^{>}) \times \hat{H} \neq 0. \tag{4.4.12}$$

Therefore, *the genotopic branch of hadronic mechanics is the only known operator formulation permitting nonconserved quantities to be Hermitian as a necessary condition to be observable.*

Other formulation attempt to represent nonconservation, e.g., by adding an "imaginary potential" to the Hamiltonian, as it is often done in nuclear physics [25]. In this case the Hamiltonian is non-Hermitian and, consequently, the nonconservation of the energy cannot be an observable.

Besides, said "nonconservative models" with non-Hermitian Hamiltonians are nonunitary and are formulated on conventional spaces over conventional fields, thus suffering all the catastrophic inconsistencies of Theorem 1.3.

We should stress the representation of irreversibility and nonconservation beginning with the most primitive quantity, the unit and related product. *Closed irreversible systems* are characterized by the Lie-isotopic subcase in which

$$\hat{i}\hat{\times}\frac{\hat{d}\hat{A}}{\hat{d}\hat{t}} = [\hat{A}\hat{,}\hat{H}] = \hat{A} \times \hat{T}(t, \dots) \times \hat{H} - \hat{H} \times \hat{T}(t, \dots) \times \hat{A}, \tag{4.4.13a}$$

$${}^{<}\hat{T}(t, \dots) = \hat{T}^{>}(t, \dots) = \hat{T}(t, \dots) = \hat{T}^{\dagger}(t, \dots) \neq \hat{T}(-t, \dots), \tag{4.4.13b}$$

for which the Hamiltonian is manifestly conserved. Nevertheless the system is manifestly irreversible. Note also the first and only known observability of the Hamiltonian (due to its iso-Hermiticity) under irreversibility.

As one can see, brackets (A, B) of Eqs. (4.6) are jointly Lie- and Jordan-admissible.

Note also that finite genotransforms (4.4.5) verify the condition of genohermiticity, Eq. (4.4).

We should finally mention that, as it was the case for isotheories, *genotheories are also admitted by the abstract axioms of quantum mechanics, thus providing a broader realization.* This can be seen, e.g., from the invariance under a complex number C

$$< \psi|x|\psi > \times I =< \psi|xC^{-1} \times |\psi > \times (C \times I) =< \psi| > |\psi > \times I^{>}. \qquad (4.4.14)$$

Consequently, *genomechanics provide another explicit and concrete realization of "hidden variables" [26], thus constituting another "completion" of quantum mechanics in the E-P-R sense [27].* For the studies of these aspects we refer the interested reader to Ref. [28].

The above formulation must be completed with three additional Lie-admissible formulations, the backward formulation for matter under time reversal and the two additional isodual formulations for antimatter. Their study is left to the interested reader for brevity.

4.4.2 Simple Construction of Lie-Admissible Theories

As it was the case for the isotopies, a simple method has been identified in Ref. [44] for the construction of Lie-admissible (geno-) theories from any given conventional, classical or quantum formulation. It consists in *identifying the genounits as the product of two different nonunitary transforms,*

$$\hat{I}^{>} = ({}^{<}\hat{I})^{\dagger} = U \times W^{\dagger}, \quad {}^{<}\hat{I} = W \times U^{\dagger}, \qquad (4.4.15a)$$

$$U \times U^{\dagger} \neq 1, \quad W \times W^{\dagger} \neq 1, \quad U \times W^{\dagger} = \hat{I}^{>}, \qquad (4.4.15b)$$

and subjecting the totality of quantities and their operations of conventional models to said dual transforms,

$$I \rightarrow \hat{I}^{>} = U \times I \times W^{\dagger}, \quad I \rightarrow^{<} \hat{I} = W \times I \times U^{\dagger}, \qquad (4.4.16a)$$

$$a \rightarrow \hat{a}^{>} = U \times a \times W^{\dagger} = a \times \hat{I}^{>}, \qquad (4.4.16b)$$

$$a \rightarrow^{<} \hat{a} = W \times a \times U^{\dagger} =^{<} \hat{I} \times a, \qquad (4.4.16c)$$

$$a \times b \rightarrow \hat{a}^{>} > \hat{b}^{>} = U \times (a \times b) \times W^{>} =$$

$$= (U \times a \times W^{\dagger}) \times (U \times W^{\dagger})^{-1} \times (U \times b \times W^{\dagger}), \qquad (4.4.16d)$$

$$\partial/\partial x \rightarrow \hat{\partial}^{>}/\hat{\partial}^{>}\hat{x}^{>} = U \times (\partial/\partial x) \times W^{\dagger} = \hat{I}^{>} \times (\partial/\partial x), \qquad (4.4.16e)$$

$$< \psi| \times |\psi > \rightarrow <^{<} \psi| > |\psi^{>} >= U \times (< \psi| \times |\psi >) \times W^{\dagger}, \qquad (4.4.16f)$$

$$H \times |\psi > \rightarrow \hat{H}^{>} > |\psi^{>} >=$$

$$= (U \times H \times W^\dagger) \times (U \times W^\dagger)^{-1} \times (U \times \psi > W^\dagger), \text{ etc.} \tag{4.4.16g}$$

As a result, any given conventional, classical or quantum model can be easily lifted into the genotopic form.

Note that the above construction implies that *all conventional physical quantities acquire a well defined direction of time.* For instance, the correct genotopic formulation of energy, linear momentum, etc., is given by

$$\hat{H}^> = U \times H \times W^\dagger, \;\; \hat{p}^> = U \times p \times W^>, \text{ etc.} \tag{4.4.17}$$

In fact, under irreversibility, the value of a nonconserved energy at a given time t for motion forward in time is generally different than the corresponding value of the energy for $-t$ for motion backward in past times.

This explains the reason for having represented in this section energy, momentum and other quantities with their arrow of time $>$. Such an arrow can indeed be omitted for notational simplicity, but only after the understanding of its existence.

Note finally that a conventional, one dimensional, unitary Lie transformation group with Hermitian generator X and parameter w can be transformed into a covering Lie-admissible group via the following nonunitary transform

$$Q(w) \times Q^\dagger(w) = Q^\dagger(w) \times Q(w) = I, \; w \in R, \tag{4.4.18a}$$

$$U \times U^\dagger \neq I, \quad W \times W^\dagger \neq 1, \tag{4.4.18b}$$

$$\begin{gathered} A(w) = Q(w) \times A(0) \times Q^\dagger(w) = e^{X \times w \times i} \times A(0) \times e^{-i \times w \times X} \rightarrow \\ \rightarrow U \times (e^{X \times w \times i} \times A(0) \times e^{-i \times w \times X}) \times U^\dagger = \\ \equiv [U \times (e^{X \times w \times i}) \times W^\dagger \times (U \times W^\dagger)^{-1} \times A \times A(0) \times \\ \times U^\dagger \times (W \times U^\dagger)^{-1} \times [W \times (e^{-i \times w \times X}) \times U^\dagger] = \\ = (e^{i \times X \times X})^> > A(0) <^< (e^{-1 \times w \times X}) = \hat{U}^> > A(0) <^< \hat{U}, \end{gathered} \tag{4.4.18c}$$

which confirm the property of Section 4.2, namely, that under the necessary mathematics *the Lie-admissible theory is indeed admitted by the abstract Lie axioms, and it is a realization of the latter broader than the isotopic form.*

4.4.3 Invariance of Lie-Admissible Theories

Recall that a fundamental axiomatic feature of quantum mechanics is the invariance under time evolution of all numerical predictions and physical laws, which invariance is due to the *unitary structure* of the theory.

However, quantum mechanics is reversible and can only represent in a scientific way beyond academic beliefs reversible systems verifying total conservation laws due to the antisymmetric character of the brackets of the time evolution.

As indicated earlier, the representation of irreversibility and nonconservation requires theories with a *nonunitary structure*. However, the latter are afflicted by the catastrophic inconsistencies of Theorem 1.3.

The only resolution of such a basic impasse known to the author has been the achievement of invariance under nonunitarity and irreversibility via the use of genomathematics, provided that such genomathematics is applied to the *totality* of the formalism to avoid evident inconsistencies caused by mixing different mathematics for the selected physical problem.

Let us note that, due to decades of protracted use it is easy to predict that physicists and mathematicians may be tempted to treat the Lie-admissible branch of hadronic mechanics with conventional mathematics, whether in part or in full. Such a posture would be equivalent, for instance, to the elaboration of the spectral emission of the hydrogen atom with the genodifferential calculus, resulting in an evident nonscientific setting.

Such an invariance was first achieved by Santilli in Ref. [15] of 1997 and can be illustrated by reformulating any given nonunitary transform in the *genounitary form*

$$U = \hat{U} \times \hat{T}^{>1/2}, W = \hat{W} \times \hat{T}^{>1/2}, \tag{4.4.19a}$$

$$U \times W^{\dagger} = \hat{U} > \hat{W}^{\dagger} = \hat{W}^{\dagger} > \hat{U} = \hat{I}^{>} = 1/\hat{T}^{>}, \tag{4.4.19b}$$

and then showing that genounits, genoproducts, genoexponentiation, etc., are indeed invariant under the above genounitary transform in exactly the same way as conventional units, products, exponentiations, etc. are invariant under unitary transforms,

$$\hat{I}^{>} \rightarrow \hat{I}^{>\prime} = \hat{U} > \hat{I}^{>} > \hat{W}^{\dagger} = \hat{I}^{>}, \tag{4.4.20a}$$

$$\begin{gathered} \hat{A} > \hat{B} \rightarrow \hat{U} > (A > B) > \hat{W}^{\dagger} = \\ = (\hat{U} \times \hat{T}^{>} \times A \times T^{>} \times \hat{W}^{\dagger}) \times (\hat{T}^{>} \times W^{\dagger})^{-1} \times \hat{T}^{>} \times \\ \times (\hat{U} \times \hat{T}^{>})^{-1} \times (\hat{U} \times T^{>} \times \hat{A} \times T^{>} \times \hat{W}^{>}) = \\ = \hat{A}' \times (\hat{U} \times \hat{W}^{\dagger})^{-1} \times \hat{B} = \hat{A}' \times \hat{T}^{>} \times B' = \hat{A}' > \hat{B}', \text{ etc.,} \end{gathered} \tag{4.4.20b}$$

from which all remaining invariances follow, thus resolving the catastrophic inconsistencies of Theorem 1.3.

Note the *numerical invariances of the genounit* $\hat{I}^{>} \rightarrow \hat{I}^{>\prime} \equiv \hat{I}^{>}$, *of the genotopic element* $\hat{T}^{>} \rightarrow \hat{T}^{>\prime} \equiv \hat{T}^{>}$, *and of the genoproduct* $> \rightarrow >' \equiv >$ that are necessary to have invariant numerical predictions.

4.5 APPLICATIONS

4.5.1 Lie-admissible Treatment of Particles with Dissipative Forces

In this section we present a variety of classical and operator representations of nonconservative systems by omitting hereon for simplicity of notation all "hats" on quantities (denoting isotopies not considered in this section), omitting the symbol $\times$ to denote the conventional (associative) multiplication, but preserving the forward (backward) symbols $>$ ($<$) denoting forward (backward) motion in time for quantities and products. The content of this section was presented for the first time by the author in memoir [32].

Let us begin with a classical and operator representation of the simplest possible dissipative system, a massive particle moving within a physical medium, and being subjected to a linear, velocity-dependent resistive force

$$m\frac{dv}{dt} = F^{NSA} = -kv, \tag{4.5.1}$$

for which we have the familiar *variation (dissipation) of the energy*

$$\frac{d}{dt}(\frac{1}{2}mv^2) = -kv^2. \tag{4.5.2}$$

Progressively more complex examples will be considered below.

The representations of system (5.1) via the *Newton-Santilli genoequations* (3.12) is given by

$$m^> > \frac{d^>v^>}{d^>t^>} = 0. \tag{4.5.3}$$

As indicated in Section 3, the representation requires the selection of *three* generally different genounits, $I_t^>, I_r^>, I_v^>$. Due to the simplicity of the case and the velocity dependence of the applied force, the simplest possible solution is given by

$$I_t^> = I_r^> = 1, \quad I_v^>(t) = e^{\frac{k\times t}{m}} = 1/T_v^>(t) > 0, \tag{4.5.4a}$$

$$m^> > \frac{d^>v^>}{d^>t^>} = m\frac{d(vI_v^>)}{dt} = m\frac{dv}{dt}I^> + kv\frac{dI_v^>}{dt} = 0. \tag{4.5.4b}$$

The representation with *Hamilton-Santilli genoequations* (3.22) is also straightforward and can be written in disjoint $r^>$ and $p^>$ notations

$$H^> = \frac{p^{>2>}}{2^> > m^>} = \frac{p^2}{2m}I_p^>, \tag{4.5.5a}$$

$$v^> = \frac{\partial^> H^>}{\partial^> p^>} = \frac{p^>}{m}, \quad \frac{d^>p^>}{d^>t^>} = -\frac{\partial^> H^>}{\partial^> r^>} = 0. \tag{4.5.5b}$$

The last equation then reproduces equation of motion (5.1) identically under assumptions (5.4a).

The above case is instructive because the representation is achieved via the genoderivatives (Section 2.2). However, the representation exhibits no algebra in the time evolution. Therefore, we seek an alternative representation in which the dissipation is characterized by the Lie-admissible algebra, rather by the differential calculus.

This alternative representation is provided by the Hamilton-Santilli genoequations (3.22) in the unified notation $a^> = (r^{>k}, p^>_k)$ that become for the case at hand

$$\frac{da^{>\mu}}{dt} = \begin{pmatrix} dr^>/dt \\ dp^>/dt \end{pmatrix} = S^{>\mu\nu}\frac{\partial^> H^>}{\partial^> a^{>\nu}} = \begin{pmatrix} 0 & -1 \\ 1 & \frac{-kv}{(\partial H/\partial p)} \end{pmatrix} \begin{pmatrix} \partial^> H^>/\partial^> r^> \\ \partial^> H^>/\partial^> p^> \end{pmatrix}, \tag{4.5.6}$$

under which we have the genoequations

$$\frac{dr^>}{dt} = \frac{\partial^> H^>}{\partial^> p^>} = \frac{p^>}{m}, \quad \frac{dp^>}{dt} = -kv, \tag{4.5.7}$$

where one should note that the derivative can be assumed to be conventional, since the system is represented by the mutation of the Lie structure.

To achieve a representation of system (5.1) suitable for operator image, we need the following *classical, finite, Lie-admissible transformation genogroup*

$$A(t) = (e^{-t\frac{\partial H}{\partial a^\mu}S^{>\mu\nu}\frac{\partial}{\partial a^\nu}})A(0)(e^{\frac{\partial}{\partial a^\nu}{}^<S^{\nu\mu}\frac{\partial H}{\partial a^\mu}t}), \tag{4.5.8}$$

defined in the 12−dimensional *bimodular genophasespace* ${}^<T^*M \times T^*M^>$, with *infinitesimal Lie-admissible time evolution*

$$\frac{dA}{dt} = \frac{\partial A}{\partial a^\mu}({}^<S^{\mu\nu} - S^{>\mu\nu})\frac{\partial H}{\partial a^\nu} =$$
$$= (\frac{\partial A}{\partial r^k}\frac{\partial H}{\partial p_k} - \frac{\partial H}{\partial r^k}\frac{\partial A}{\partial p_k}) - (\frac{kv}{(\partial H/\partial p)})\frac{\partial H}{\partial p}\frac{\partial A}{\partial p}) =$$
$$= [A, H] - kv\frac{\partial A}{\partial p}, \tag{4.5.9}$$

where we have dropped the forward arrow for notational convenience, and $\omega^{\mu\nu}$ is the canonical Lie tensor, thus proving the Lie-admissibility of the S-tensors. In fact, the attached antisymmetric brackets $[A, H]$ are the conventional Poisson brackets, while $\{A, H\}$ are indeed symmetric brackets (as requested by Lie-admissibility), but they do not characterize a Jordan algebra (Section 4.1.3).

It is easy to see that the time evolution of the Hamiltonian is given by

$$\frac{dH}{dt} = -kv\frac{\partial H}{\partial p} = -kv^2, \tag{4.5.10}$$

thus correctly reproducing behavior (5.2).

The operator image of the above dissipative system is straightforward. Physically, we are also referring to a first approximation of a massive and stable elementary particle, such as an electron, penetrating within hadronic matter (such as a nucleus). Being stable, the particle is not expected to "disappear" at the initiation of the dissipative force and be converted into "virtual states" due to the inability of represent such a force, but more realistically the particle is expected to experience a rapid dissipation of its kinetic energy and perhaps after that participate in conventional processes.

Alternatively, we can say that an electron orbiting in an atomic structure does indeed evolve in time with conserved energy, and the system is indeed Hamiltonian. By the idea that the same electron when in the core of a star also evolves with conserved energy is repugnant to reason. Rather than adapting nature to manifestly limited Hamiltonian theories, we seek their covering for the treatment of systems for which said theories were not intended for.

The problem is to identify forward and backward genounits and related genotopic elements $I^> = 1/T^>, ^<I = 1/^<T$ for which the following *operator Lie-admissible genogroup* now defined on a genomodule $^<\mathcal{H} \times \mathcal{H}^>$

$$A(t) = (e^{iHT^> t})A(0)(e^{-it^<TH}), \tag{4.5.11}$$

and related infinitesimal form, the *Heisenberg-Santilli genoequations*

$$i\frac{dA}{dt} = A < H - H > A = A^<TH - HT^>A, \tag{4.5.12}$$

correctly represent the considered dissipative system.

By noting that the Lie-brackets in Eqs. (4.5.9) are conventional, we seek a realization of the genotopic elements for which the Lie brackets attached to the Lie-admissible brackets (5.12) are conventional and the symmetric brackets are Jordan-isotopic. A solution is then given by [32]

$$T^> = 1 - \Gamma, \quad ^<T = 1 + \Gamma, \tag{4.5.13}$$

for which Eq. (5.12) becomes

$$i\frac{dA}{dt} = (AH - HA) - (A\Gamma H + H\Gamma A) =$$

$$= [A, H] - \{A\hat{,}H\}, \tag{4.5.14}$$

where $[A, H]$ are a conventional Lie brackets as desired, and $\{A\hat{,}H\}$ are Jordan-isotopic brackets. The desired representation then occurs for

$$I^> = e^{(k/m)H^{-1}} = 1/T^>, \quad ^<I = e^{-H^{-1}(k/m)} = 1/^<T, \tag{4.5.15a}$$

$$i\frac{dH}{dt} = -\frac{kp^2}{m^2} = -kv^2. \tag{4.5.15b}$$

Note that the achievement of the above operator form of system (5.1) without the Lie-admissible structure would have been impossible, to our knowledge.

Despite its elementary character, the above illustration has deep implications. In fact, the above example constitutes the only known operator formulation of a dissipative system in which the *nonconserved* energy is represented by a *Hermitian* operator H, thus being an *observable* despite its nonconservative character. In all other cases existing in the literature the Hamiltonian is generally *non-Hermitian,* thus *non-observable.*

The latter occurrence may illustrate the reason for the absence of a consistent operator formulation of nonconservative systems throughout the 20-th century until the advent of the Lie-admissible formulations.

4.5.2 Direct Universality of Lie-Admissible Representations for Nonconservative Systems

We now show that the Lie-admissible formulations are "directly universal," namely, they provide a classical and operator representation of all infinitely possible (well behaved) nonconservative systems of N particles (universality)

$$m_n \frac{dv_{nk}}{dt} + \frac{\partial V}{\partial r_n^k} = F_{nk}^{NSA}(t, r, p, \dot{p}, ...), \quad n = 1, 2, 3, ..., N, \quad k = 1, 2, 3, \tag{4.5.16}$$

directly in the frame of the observer, i.e., without transformations of the coordinates of the experimenter to mathematical frames (direct universality).

An illustration is given by a massive object moving at high speed within a resistive medium, such as a missile moving in our atmosphere. In this case the resistive force is approximated by a power series expansion in the velocity truncated up to the 10-th power for the high speeds of contemporary missiles

$$m \frac{dv}{dt} = \Sigma_{\alpha=1,2,...,10} k_\alpha v^\alpha, \tag{4.5.17}$$

for which any dream of conventional Hamiltonian representation is beyond the boundary of science.

The direct universality of the Hamilton-Santilli genomechanics was proved in Section 3.3. The representation in geno-phase-space is characterized by the conventional Hamiltonian representing the physical total energy, and the genounit for forward motion in time representing the NSA forces, according to the equations

$$H = \Sigma_{n,k} \frac{p_{nk}^2}{2m_n} + V(r), \quad I^> = \begin{pmatrix} 1 & \frac{F^{NSA}}{(\partial H/\partial p)} \\ 1 & 0 \end{pmatrix} \tag{4.5.18}$$

under which we have the equations of motion (for $\mu, \nu = 1, 2, 3, ...6N$) [32]

$$\frac{da^{>\mu}}{dt} = \begin{pmatrix} dr_n^{>k}/dt \\ dp_{nk}^{>}/dt \end{pmatrix} = S^{>\mu\nu} \frac{\partial^> H^>}{\partial^> a^{>\nu}} = \begin{pmatrix} 0 & -1 \\ 1 & \frac{F^{NSA}}{(\partial H/\partial p)} \end{pmatrix} \begin{pmatrix} \partial^> H^> / \partial^> r_n^{>k} \\ \partial^> H^> / \partial^> p_{nk}^{>} \end{pmatrix}, \tag{4.5.19}$$

the classical, finite, Lie-admissible genosgenogroup

$$A(t) = \exp\left(-t\frac{\partial H}{\partial a^\mu}S^{>\mu\nu}\frac{\partial}{\partial a^\nu}\right)A(0)\exp\left(\frac{\partial}{\partial a^\nu}{}^{<}S^{\nu\mu}\frac{\partial H}{\partial a^\mu}t\right), \qquad (4.5.20)$$

with infinitesimal time evolution

$$\frac{dA}{dt} = \frac{\partial A}{\partial a^\mu}({}^{<}S^{\mu\nu} - S^{>\mu\nu})\frac{\partial H}{\partial a^\nu} =$$
$$= (\frac{\partial A}{\partial r^k_n}\frac{\partial H}{\partial p_{nk}} - \frac{\partial H}{\partial r^k_n}\frac{\partial A}{\partial p_{nk}}) - (\frac{km}{(\partial H/\partial p)})^{nk}\frac{\partial A}{\partial p_{nk}}\frac{\partial H}{\partial p_{nk}}) =$$
$$= [A, H] + \{A, H\}, \qquad (4.5.21)$$

yielding the correct *nonconservation of the energy*

$$\frac{dH}{dt} = v^k F^{NSA}_k. \qquad (4.5.22)$$

The operator image can be characterized by the genounits and related genotopic elements

$$I^> = e^\Gamma = 1/T^>, \quad {}^{<}I = e^{-\Gamma} = 1/{}^{<}T, \quad \Gamma = H^{-1}(v^k_n F^{NSA}_{nk})H^{-1}, \qquad (4.5.23)$$

with finite Lie-admissible time evolution

$$A(t) = \exp(iHe^{-\Gamma}t)A(0)\exp(-ite^{+\Gamma}H) \qquad (4.5.24)$$

and related Heisenberg-Santilli genoequations

$$i\frac{dA}{dt} = A < H - H > A = [A, H] + \{A \hat{,} H\} =$$
$$= (AH - HA) + (A\Gamma H + H\Gamma A), \qquad (4.5.25)$$

that correctly represent the time rate of variation of the nonconserved energy,

$$i\frac{dH}{dt} = v^k_n F^{NSA}_{nk}. \qquad (4.5.26)$$

The uninitiated reader should be incidentally aware that generally different genounits may be requested for different generators, as identified since Ref. [11].

In the latter operator case we are referring to an extended, massive and stable particle, such as a proton, penetrating at high energy within a nucleus, in which case the rapid decay of the kinetic energy is caused by contact, resistive, integrodifferential forces of nonlocal type, e.g., because occurring over the volume of the particle.

The advantages of the Lie-admissible formulations over pre-existing representation of nonconservative systems should be pointed out. Again, a primary advantage of the Lie-admissible treatment is the characterization of the *nonconserved* Hamiltonian with a *Hermitian,* thus *observable* quantity, a feature generally absent in other treatments.

Moreover, the "direct universality" of Lie-admissible representations requires the following comments. Recall that coordinates transformations have indeed been used in the representation of nonconservative systems because, under sufficient continuity and regularity, the Lie-Koening theorem assures the existence of coordinate transformations $(r,p) \rightarrow (r'(r,p),\ p'(r,p))$ under which a system that is non-Hamiltonian in the original coordinates becomes Hamiltonian in the new coordinates (see Ref. [6] for details). However, the needed transformations are necessarily nonlinear with serious physical consequences, such as:

1) Quantities with direct physical meaning in the coordinates of the experimenter, such as the Hamiltonian $H(r,p) = \frac{p^2}{2m} + V(r)$, are transformed into quantities that, in the new coordinates, have a purely mathematical meaning, such as $H'(r',p') = Nexp(Mr'^2/p'^3), N, M \in R$, thus preventing any physically meaningful operator treatment;

2) There is the loss of any meaningful experimental verifications, since it is impossible to place any measurement apparatus in mathematical coordinates such as $r' = KlogLr^3, p' = Pexp(Qrp), K, L, P, Q \in R$;

3) There is the loss of Galileo's and Einstein's special relativity, trivially, because the new coordinates (r', p') characterize a highly *noninertial* image of the original inertial system of the experimenter.

All the above, and other insufficiencies are resolved by the Lie-admissible treatment of nonconservative systems.

4.5.3 Pauli-Santilli Lie-Admissible Matrices

Following the study of the nonconservation of the energy, the next important topic is to study the behavior of the conventional quantum spin under contact nonconservative forces, a topic studied for the first time in memoir [32]. For this objective, it is most convenience to use the method of Suctions 4.4.2 and 4.4.3, namely, subject the conventional Pauli's matrices to two different nonunitary transforms. To avoid un-necessary complexity, we select the following two matrices

$$A = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}, \quad AA^\dagger \neq I, \ \ ]\, BB^\dagger \neq I, \qquad (4.6.1)$$

where a and b are non-null real numbers, under which we have the following forward and backward genounits and related genotopic elements

$$I^> = AB^\dagger = \begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix}, \quad T^> = \frac{1}{(1-ab)} \begin{pmatrix} 1 & -b \\ -a & 1 \end{pmatrix}, \tag{4.6.2a}$$

$$^<I = BA^\dagger = \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}, \quad ^<T = \frac{1}{(1-ab} \begin{pmatrix} 1 & -a \\ -b & 1 \end{pmatrix}. \tag{4.6.2b}$$

The *forward and backward Pauli-Santilli genomatrices* are then given respectively by

$$\sigma_1^> = A\sigma_1 B^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & (a+b) \end{pmatrix}, \quad \sigma_2^> = A\sigma_2 B^\dagger = \begin{pmatrix} 0 & -i \\ i & (a+b) \end{pmatrix}, \tag{4.6.3a}$$

$$\sigma_3^> = A\sigma_3 B^\dagger = \begin{pmatrix} 1 & b \\ a & -1 \end{pmatrix}, \quad ^<\sigma_1 = B\sigma_1 A^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & (a+b) \end{pmatrix}, \tag{4.6.3b}$$

$$^<\sigma_2 = B\sigma_2 A^\dagger = \begin{pmatrix} 0 & -i \\ i & (a+b) \end{pmatrix}, \quad ^<\sigma_3 = A\sigma_3 B^\dagger = \begin{pmatrix} 1 & a \\ b & -1 \end{pmatrix}, \tag{4.6.3c}$$

in which the direction of time is embedded in the structure of the matrices.

It is an instructive exercise for the interested reader to verify that *conventional commutation rules and eigenvalues of Pauli's matrices are preserved under forward and backward genotopies,*

$$\sigma_i^> > \sigma_j^> - \sigma_j^> > \sigma_i^> = 2i\epsilon_{ijk}\sigma_k^>, \tag{4.6.4a}$$

$$\sigma_3^> > | >= \pm 1| >, \quad \sigma^{>2>} > | >= 2(2+1)| >, \tag{4.6.4b}$$

$$^<\sigma_i >^< \sigma_j -^< \sigma_j >^< \sigma_i = 2i\epsilon^<_{ijk}\sigma_k, \tag{4.6.4c}$$

$$< | <^< \sigma_3 =< | \pm 1, \; ; < | <^< \sigma^{2>} =< |(2(2+1). \tag{4.6.4d}$$

We can, therefore, conclude by stating that *Pauli's matrices can indeed be lifted in such an irreversible form to represent the direction of time in their very structure.* However, in so doing the conventional notion of spin is lost in favor of a covering notion in which the spin becomes a locally varying quantity, as expected to a proton in the core of a star.

Consequently, the Lie-admissible formulation of Pauli matrices confirms the very title of memoir [12] proposing the construction of hadronic mechanics.

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The argument is that, while special relativity and Pauli exclusion principe are unquestionably valid for the conditions of their original conception, particles at large mutual distances under action-at-a-distance interactions (such as for a point-like proton in a particle accelerator under long range electromagnetic interactions), by no means the same doctrines have to be necessarily valid for *one* hadronic constituent when considering all other constituents as external.[6]

The above analysis focuses the attention in an apparent fundamental structural difference between electromagnetic and strong interactions. Irrespective of whether considered part of the system (closed system) or external (open system), *electromagnetic interactions do verify Pauli principle,* as well known. The best example is given by Dirac's equation for the hydrogen atom that, as known to experts to qualify as such, represents one electron under the *external* electromagnetic field of the proton. The origin of the preservation of Pauli principle is that, whether electromagnetic interactions are closed or open, they are Hamiltonian. Lie's theory then applies with the conventional notion of spin, and Pauli principle follows.

By comparison, strong interactions are non-Hamiltonian for the numerous reasons indicated during tour analysis. Consequently, the conventional notion of spin cannot be preserved, and Pauli principle is inapplicable in favor of broader vistas. It is intriguing to note that the representation of a proton via isomechanics allows indeed a representation of its extended, nonspherical and deformable shape. Nevertheless, Pauli's principle is preserved under isotopies, as indicated in Chapter 3. Hence, the inapplicability of Pauli's principle is here referred to, specifically and solely, for open irreversible conditions at short mutual distances, exactly according to the original proposal to build hadronic mechanics [12].

The above distinction between electromagnetic and strong interactions is the conceptual foundation of monographs [40,41] suggesting the characterization of the hadronic constituents via *Lie-admissible,* rather than Lie or Lie-isotopic algebras, with the consequential inapplicability of the conventional. notion of spin. These basic issues will be studied in detail in Volume II in connection with explicit structure models of hadrons with physical constituents, that is, constituents that can be produced free in spontaneous decays while being compatible with the SU(3)-color Mendeleev-type *classification* of hadrons.

To conclude, *not only special relativity, but also Pauli principle is inapplicable (rather than violated) for a hadron under external strong interactions.* Needless to say, when a particle with the open nonconservative spin under consideration here is "completed" with the inclusion of all remaining strong interacting particles here considered as external, Pauli principle is recovered in full for the center of mass of the ensemble as a whole because the "completion" is treated via isomechanics.

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[6]The reader should always keep in mind that, even though not stated in the technical literature for evident political reasons, quantum mechanics can only represent the proton as a dimensionless point.

4.5.4 Minkowski-Santilli Irreversible Genospacetime

One of the fundamental axiomatic principles of hadronic mechanics is that irreversibility can be directly represented with the background geometry and, more specifically, with the metric of the selected geometry. This requires the necessary transition from the conventional *symmetric* metrics used in the 20-th century to covering *nonsymmetric* genometric.

To show this structure, we study in this section the genotopy of the conventional Minkowskian spacetime and related geometry with the conventional metric $\eta = Diag.(1,1,1,-1)$ and related spacetime elements $x^2 = x^\mu \eta_{\mu\nu} x^\nu$, $x = (x^1, x^2, x^3, x^4)$, $x^4 = ct$, $c = 1$. For this purpose, we introduce the following four-dimensional non-Hermitian, nonsingular and real-valued forward and backward genounits

$$I^> = CD^\dagger = 1/T^>, \quad {}^<I = DC^\dagger = 1/{}^<T, \quad CC^\dagger \neq I, \quad DD^\dagger \neq I, \tag{4.6.5}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p & 0 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ q & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.6.6}$$

where $p \neq q$ are non-null real numbers, under which we have the following forward and backward genotopy of the Minkowskian line element

$$x^2 \rightarrow x^{>2>} = Cx^2D^\dagger = C(x^t\eta x)D^\dagger =$$

$$= (C^tx^tD^{t\dagger})(CD^\dagger)^{-1}(C\eta D^\dagger)(CD^\dagger)^{-1}(CxD^\dagger) =$$

$$= (x^tI^>)T^>\eta^>T^>(I^>x) = x^\mu\eta^>_{\mu\nu}x^\nu =$$

$$= (x^1x^1 + x^1qx^3 + x^2x^2 + x^3x^3 + x^1px^4 - x^4x^4), \tag{4.6.7a}$$

$$Dx^2C^\dagger = D(x^t\eta x)C^\dagger =$$

$$= (x^{t<}I)^<T^<\eta^<T(^<Ix) = x^{\mu<}\eta_{\mu\nu}x^\nu =$$

$$= (x^1x^1 + x^1px^3 + x^2x^2 + x^3x^3 + x^1qx^4 - x^4x^4), \tag{4.6.7b}$$

resulting in the forward and backward nonsymmetric genometrics

$$\eta^> = \begin{pmatrix} 1 & 0 & q & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ q & 0 & 0 & 1 \end{pmatrix}, \quad {}^<\eta = \begin{pmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & 0 \\ q & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.6.8}$$

exactly as desired.

Note that irreversibility selects a mutation of the line elements along a preselected direction of space and time.

Note also that the quantities p and q can be functions of the local spacetime variables, in which case the resulting *Minkowskian genogeometry* can be equipped by a suitable lifting of the machinery of the Riemannian geometry (see Ref. [16] for the isotopic case and Chapter 3).

Note finally that the above genospacetime includes, as particular case, *an irreversible formulation of the Riemannian geometry,* where irreversibility is represented at the ultimate geometric foundations, the basic unit and the metric.

It should be indicated that the above irreversible formulation of spacetime has intriguing implications for the mathematical model known as *geometric locomotion* studied in detail in monograph [73] via the isotopies of the Minkowskian geometry. In fact, a main unresolved problems is the directional deformation of the geometry as needed to permit the geometric locomotion in one preferred direction of space. An inspection of the mutated line elements (4.6.7) clearly shows that the genotopies are preferable over the isotopies for the geometric locomotion, as well as, more generally, for a more realistic geometric characterization of irreversible processes.

The construction of the *Lorentz-Santilli genotransformations* is elementary, due to their formal identify with the isotopic case of Chapter 3, and its explicit construction left as an instructive exercise for the interested reader.

4.5.5 Dirac-Santilli Irreversible Genoequation

To complete the illustrations in particle physics, we now outline the simplest possible genotopy of Dirac's equation via the genotopies of the preceding two suctions, one for the spin content of Dirac's equation and the other for its spacetime structure. Also, we shall use Dirac's equation in its isodual re-interpretation representing a direct product of one electron and one positron, the latter without any need of second quantization (see monograph [73] for detail). In turn, the latter re-interpretation requires the use of the *isodual transform* $A \to A^d = -A^\dagger$) as being distinct from Hermitian conjugation. Under the above clarifications, the *forward Dirac genoequation* here referred to can be written

$$\eta^{>\mu\nu}\gamma^>_\mu T^> p^>_\nu - im)T^>|\psi^> >= 0 \tag{4.6.9a}$$

$$p^>_\nu T^>|\psi^> >= -i\frac{\partial^>}{\partial^> x^{>\nu}}|\psi^> >= -iI^>\frac{\partial}{\partial x^>}|\psi^> >, \tag{4.6.9b}$$

with *forward genogamma matrices*

$$\gamma^>_4 = \begin{pmatrix} A & 0 \\ 0 & B^d \end{pmatrix}\begin{pmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{pmatrix}\begin{pmatrix} A^d & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} AA^d & 0 \\ 0 & -B^d B \end{pmatrix}, \tag{4.6.10a}$$

$$\gamma^>_k = \begin{pmatrix} A & 0 \\ 0 & B^d \end{pmatrix}\begin{pmatrix} 0 & \sigma_k \\ \sigma^d_k & 0 \end{pmatrix}\begin{pmatrix} A^d & 0 \\ 0 & B \end{pmatrix} = \tag{4.6.10b}$$

$$= \begin{pmatrix} 0 & A\sigma_k B^\dagger \\ B\sigma_k^d A^d & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_k \\ \sigma_k^d & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_k^> \\ {}^<\sigma_k^d & 0 \end{pmatrix}, \qquad (4.6.10c)$$

$$\{\gamma_\mu^> \hat{,} \gamma_\nu^>\} = \gamma_\mu^> T^> \gamma_\nu^> + \gamma_\nu^> T^> \gamma_\mu^> = 2\eta_{\mu\nu}^>, \qquad (4.6.10d)$$

where $\eta_{\mu\nu}^>$ is given by the same genotopy of Eqs. (4.6.10a).

Interested readers can then construct the backward genoequation. They will discover in this way a new fundamental symmetry of Dirac's equation that remained undiscovered throughout the 20-th century, its *isoselfduality* (invariance under isoduality.) This new symmetry is now playing an increasing role for realistic cosmologies, those inclusive of antimatter, or for serious unified theories that must also include antimatter to avoid catastrophic inconsistencies [73] (see Volume II).

It is an instructive exercise for the interested reader to verify a feature indicated earlier, the inapplicability of the conventional notion of spin and, consequently, of Pauli principle for the Dirac-Santilli genoequation. As we shall see in Volume II, the conventional Dirac equation represents the electron in the structure of the hydrogen atom. By comparison, the Dirac-Santilli genoequation represents the same electron when totally immersed in the hyperdense medium inside a proton, thus characterizing the structure of the neutron according to hadronic mechanics..

Note that, while the electron is moving forward, the positron is moving backward in time although referred to a negative unit of time, as a necessary condition to avoid the inconsistencies for negative energies that requested the conjecture of the "hole theory" (see monograph xxx for brevity).

4.5.6 Dunning-Davies Lie-Admissible Thermodynamics

A scientific imbalance of the 20-th century has been the lack of interconnections between thermodynamics, on one side, and classical and quantum mechanics, on the other side. This is due to the fact that the very notion of entropy, indexEntropy let alone all thermodynamical laws, are centrally dependent on irreversibility, while classical and quantum Hamiltonian mechanics are structurally reversible (since all known potentials are reversible in time).

As recalled in Section 4.1, said lack of interconnection was justified in the 20-th century on the belief that the nonconservative forces responsible for irreversibility according to Lagrange and Hamilton, are "fictitious" in the sense that they only exist at the classical level and they "disappear" when passing to elementary particles, since the latter were believed to be completely reversible. In this way, thermodynamics itself was turned into a sort of "fictitious" discipline.

This imbalance has been resolved by hadronic mechanics beginning from its inception. In fact, Theorems 1.3.3 has established that, far from being "fictitious," nonconservative forces originate at the ultimate level of nature, that of elementary particles in conditions of mutual penetration causing contact nonpotential (NSA) interactions. The insufficiency rested in the inability by quantum mechanics to represent nonconservative forces, rather than in nature. In fact, hadronic mechanics was proposed and developed precisely to reach an operator representation of the nonconservative forces originating irreversibility along the legacy of Lagrange and Hamilton.

As a result of the efforts presented in this chapter, we now possess not only classical and operator theories, but more particularly we have a *new mathematics,* the genomathematics, whose basic axioms are not invariant under time reversal beginning from the basic units, numbers and differentials.

Consequently, hadronic mechanics does indeed permit quantitative studies of the expected interplay between thermodynamics and classical as well as operator mechanics. These studies were pioneered by J. Dunning Davies [30] who introduced the first known study of thermodynamics via methods as structurally irreversible as their basic laws, resulting in a formulation we hereon call *Dunning-Davies Lie-admissible thermodynamics.* This section is dedicated to a review of Dunning-Davies studies.

Let us use conventional thermodynamical symbols, a classical form of thermodynamics, and the simple construction of irreversible formulations via two different complex valued quantities A and B. Then, the first law of thermodynamics can be lifted from its conventional formulation, that via reversible mathematics, into the form permitted by genomathematics

$$Q \rightarrow Q^> = AQB^\dagger = QI^>, \quad U \rightarrow U^> = AUB^\dagger = UI^>, \quad \text{etc.}, \qquad (4.6.11a)$$

$$dQ = dU + pdV \quad \rightarrow \quad d^>Q^> = d^>U^> + p^> > d^>V^>, \qquad (4.6.11b)$$

where, in the absence of operator forms, Hermitian conjugation is complex conjugation. For the second law we have

$$dQ = TdS \quad \rightarrow \quad d^>Q^> = T^> > d^>S^>, \qquad (4.6.12)$$

thus implying that

$$TdS = dU + pdV \quad \rightarrow \quad T^> > d^>S^> = d^>U^> + p^> > d^>V^>. \qquad (4.6.13)$$

As one can see, genomathematics permits the *first known formulation of entropy with a time arrow,* the only causal form being that forward in time. When the genounit does not depend on the local variables, the above genoformulation reduces to the conventional one identically, e.g.,

$$T^> > d^>S^> = (TI^>)I^{>-1}[I^{>-1}d(SI^>) = TdS =$$

$$= I^{>-1}d(VI^>) + (pI^>)I^{>-1}d(VI^>) = dU + pdV. \qquad (4.6.14)$$

This confirms that genomathematics is indeed compatible with thermodynamical laws.

However, new vistas in thermodynamics are permitted when the genounit is dependent on local variables, in which case reduction (4.6.13) is no longer possible. An important case occurs when the genounit is explicitly dependent on the entropy. In this case the l.h.s. of Eq. (4.6.13) becomes

$$TdS + TS(I^{>-1}dI^{>}) = dU + pdV. \tag{4.6.15}$$

We then have new thermodynamical models of the type

$$I^{>} = e^{f(S)}, \;\; T^{>} > d^{>}S^{>} = T\left(1 + S\frac{\partial f(S)}{\partial S}\right)dS = dU + pdV, \tag{4.6.16}$$

permitting thermodynamical formulations of the behavior of anomalous gases (such as magnegases [21]) via a suitable selection of the $f(S)$ function and its fit to experimental data. Needless to say, equivalent models can be constructed for an explicit dependence of the genounit from the other variables. For these and other aspects we have to refer the interested reader to Volume II.

4.5.7 Ongoing Applications to New Clean Energies

A primary objective of Volume II is to study industrial applications of hadronic mechanics to new clean energies that are under development at the time of writing this first volume (2002). Hence, we close this chapter with the following preliminary remarks.

The societal, let alone scientific implications of the proper treatment of irreversibility are rather serious. Our planet is afflicted by increasingly catastrophic climactic events mandating the search for basically new, environmentally acceptable energies, for which scope the studies reported in these monographs were initiated.

All known energy sources, from the combustion of carbon dating to prehistoric times to the nuclear energy, are based on irreversible processes. By comparison, all established doctrines of the 20-th century, such as quantum mechanics and special relativity, are reversible, as recalled in Section 4.1.

It is then easy to see that *the serious search for basically new energies requires basically new theories that are as structurally irreversible as the process they are expected to describe.* At any rate, all possible energies and fuels that could be predicted by quantum mechanics and special relativity were discovered by the middle of the 20-th century. Hence, the insistence in continuing to restrict new energies to verify preferred reversible doctrines may cause a condemnation by posterity due to the environmental implications.

An effective way to illustrate the need for new irreversible theories is given by nuclear fusions. All efforts to date in the field, whether for the "cold fusion" or the "hot fusion," have been mainly restricted to verify quantum mechanics and special relativity. However, *whether "hot" or "cold," all fusion processes are irreversible, while quantum mechanics and special relativity are reversible.*

It has been shown in Ref. [31] that the failure to date by both the "cold" and the "hot" fusions to achieve industrial value is primarily due to the treatment of irreversible nuclear fusions with reversible mathematical and physical methods.

In the event of residual doubt due to protracted use of preferred theories, it is sufficient to compute the quantum mechanical probability for two nuclei to "fuse" into a third one, and then compute its time reversal image. In this way the serious scholar will see that special relativity and quantum mechanics may predict a fully causal *spontaneous disintegration of nuclei following their fusion,* namely, a prediction outside the boundary of science.

The inclusion of irreversibility in quantitative studies of new energies suggests the development, already partially achieved at the industrial level (see Chapter 8 of Ref. [20]), of the new, controlled "intermediate fusion" of light nuclei [31], that is, a fusion occurring at minimal threshold energies needed: 1) To verify conservation laws; 2) To expose nuclei as a pre-requisite for their fusion (a feature absent in the "cold fusion" due to insufficient energies), and 3) To prevent uncontrollable instabilities (as occurring at the very high energies of the "hot fusion").

It is hoped that serious scholars will participate with independent studies on the irreversible treatment of new energies, as well as on numerous other open problems, because in the final analysis lack of participation in basic advances is a gift of scientific priorities to others.

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Chapter 5

HYPERSTRUCTURAL BRANCH OF HADRONIC MECHANICS AND ITS ISODUAL

5.1 The Scientific Imbalance in Biology

In our view, the biggest scientific imbalance of the 20-th century has been the treatment of biological systems (herein denoting DNA, cells, organisms, etc.) via conventional mathematics, physics and chemistry because of various reasons studied in detail in Chapter 1.1.

We here limit ourselves to recall that biological events, such as the growth of an organism, are irreversible over time, while the mathematics of the 20-th century and related formulations are structurally reversible, that is, reversible for all possible Hamiltonians. Therefore, any treatment of biological systems via reversible mathematics, physical and chemical formulations can indeed receive temporary academic acceptance, but cannot pass the test of time.

Quantum mechanics is ideally suited for the treatment of the structure of the hydrogen atom or of crystals, namely, systems that are fully reversible. These systems are represented by quantum mechanics as being ageless. Recall also that quantum mechanics is unable to treat deformations because of incompatibilities with basic axioms, such as that of the rotational symmetry.

Therefore, *the strict application to biological systems of the mathematics underlying quantum mechanics and chemistry implies that all organisms from cells to humans are perfectly reversible, totally rigid and fully eternal.*

5.2 The Need in Biology of Irreversible Multi-Valued Formulations

It is possible to see that, despite their generality, the invariant irreversible genoformulations studied in the preceding chapter are insufficient for in depth treatments of biological systems.

In fact, recent studies conducted by Illert [1] have pointed out that the *shape* of sea shells can certainly be represented via conventional mathematics, such as the Euclidean geometry.

However, the latter conventional geometries are inapplicable for a representation of the *growth over time* of sea shells. Computer simulations have shown that the imposition to sea shell growth of conventional geometric axioms causes the lack of proper growth, such as deformations and cracks, as expected, because said geometries are strictly reversible over time, while the growth of sea shells is strictly irreversible.

The same studies by Illert [1] have indicated the need of a mathematics that is not only structurally irreversible, but also *multi-dimensional.* As an example, Illert achieved a satisfactory representation of sea shells growth via the *doubling of the Euclidean reference axes,* namely, via a geometry appearing to be six-dimensional.

A basic problem in accepting such a view is the lack of compatibility with our sensory perception. When holding sea shells in our hands, we can fully perceive their shape as well as their growth with our three Eustachian tubes. Hence, any representation of sea shells growth with more than three dimensions is incompatible with our perception of reality.

Similarly, our sensory perception can indeed detect curvature. Thus, any representation of sea shell growth with the Riemannian geometry would equally be incompatible with our sensory perception. At any rate, any attempt at the use of the Riemannian geometry for sea shell growth would be faced with fatal inconsistencies, such as the inability to represent bifurcations and other aspects since such representations would be prohibited by curvature.

These occurrences pose a rather challenging problem, the construction of yet another *new mathematics* that is

(1) Structurally irreversible over time (as that of the preceding section);

(2) Capable to represent deformations;

(3) Invariant under the time evolution in the sense of predicting the same number under the same conditions but at different times;

(4) Multi-dimensional; and, last but not least,

(5) Compatible with our sensory perception.

The only solution known to the author is that of building an irreversible *multi-valued* (rather than multi-dimensional) new mathematics, in the sense that the basic axioms of the space representation can remain three-dimensional to achieve compatibility with our sensory perception, but each axis can have more than one value, thus being multi-valued.

A search in the mathematical literature soon revealed that a mathematics verifying all the above requirements did not exist and had to be constructed.

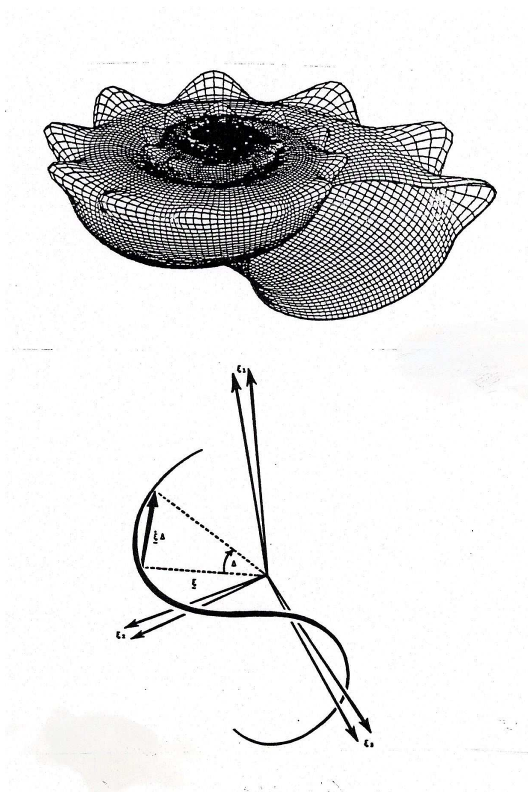

Figure 5.1. A schematic view of Illert [1] has shown that a representation of the growth over time of a seashell can be effectively done by doubling the number of reference axes. However, seashell growth is perceived by our sensory perception as occurring in three-dimensional space. The multi-valued hyperstructural branch of hadronic mechanics studied in this chapter provides a solution of these seemingly discordant requirements because, on side, it is as multi-valued as desired while, on the other side, remains three-dimensional at the abstract, realization-free level.

As an example, in their current formulations, *hyperstructures* (see, e.g., Ref. [2]) lack a well defined left and right unit thus lacking the applicability to the measurements; they do not have conventional operations, but rather the so-called *weak operations*, thus lacking applicability to experiments; they are not structurally irreversible; and they lack invariance. Consequently, conventional hyperstructures are not suitable for applications in biology.

5.3 Rudiments of Santilli Hyper-Mathematics and Hypermechanics

After a number of trials and errors, a yet broader mathematics verifying the above five conditions was identified by R. M. Santilli in monographs [3] of 1995 and in works [4,5], and subsequently studied by R. M. Santilli and the mathematician T. Vougiouklis in paper [6] of 1996 (see also mathematical study [7]). These studies resulted in a formulation today known as *Santilli hypermathematics.*

For an in depth study, including the all crucial *Lie-Santilli hypertheory,* we refer the reader to the mathematical treatments [4–7]. By assuming an in depth knowledge of genomathematics of the preceding chapter, we here limit ourselves to indicate that the selected hypermathematics is based on the assumption that the single-valued forward and backward genounits of the preceding chapter although replaced with the following *multi-valued hyperunits*

$$\hat{I}^{>}(t,x,v,\psi,\partial_x\psi,...) = Diag.(\hat{I}^{>}_1,\hat{I}^{>}_2,\hat{I}^{>}_3) =$$

$$= Diag.\Big[(\hat{I}^{>}_{11},\hat{I}^{>}_{12},...,\hat{I}^{>}_{1m}),(\hat{I}^{>}_{21},\hat{I}^{>}_{22},...,\hat{I}^{>}_{2m}),(\hat{I}^{>}_{31},\hat{I}^{>}_{32},...,\hat{I}^{>}_{3m})\Big], \tag{5.1a}$$

$$^{<}\hat{I}(t,x,v,\psi,...) = Diag.(^{<}\hat{I}_1,^{<}\hat{I}_2,^{<}\hat{I}_3) =$$

$$= Diag.\Big[(^{<}\hat{I}_{11},^{<}\hat{I}_{12},...,^{<}\hat{I}_{1m}),(^{<}\hat{I}_{21},^{<}\hat{I}_{22},...,^{<}\hat{I}_{2m}),$$

$$(^{<}\hat{I}_{31},^{<}\hat{I}_{32},...,^{<}\hat{I}_{3m})\Big], \tag{5.1b}$$

with corresponding *ordered hyperproducts to the right and to the left*

$$A > B = A \times \hat{T}^{>} \times B, A < B = A \times^{<} \hat{T} \times B, \tag{5.2a}$$

$$\hat{I}^{>} > A = A > \hat{I}^{>} = A, \quad ^{<}\hat{I} < AA <^{<} \hat{I} = A, \tag{5.2b}$$

$$\hat{I}^{>} = (^{<}\hat{I})^{\dagger} = 1/\hat{T}^{>}. \tag{5.2c}$$

Following the hyperlifting of the methods of the preceding chapter, we reach the following basic equations of the *multi-valued hyperstructural branch of hadronic mechanics,* first proposed by Santilli in monographs [3] of 1995 (see also the mathematical works [4–6], here written in the finite and infinitesimal forms

$$i\ dA/dt = A \lhd H - H \rhd A, \tag{5.3a}$$

$$A(t) = \hat{e}^{i\times H\times t} \lhd A(0) \rhd \hat{e}^{-i\times t\times H}, \tag{5.3b}$$

quoted in Footnote 15 of Chapter 1, where the multivalued character of all quantities and their operations is assumed.

In the above expressions the reader should recognize the diagonal elements of the genounits of the preceding chapter and then identify the multi-valued character for each diagonal element. Consequently, the above mathematics *is not* $3m$-dimensional, but rather it is 3-dimensional and m-multi-valued, namely, each axis in three-dimensional space can assume m different values.

Such a feature permits the increase of the reference axes, e.g., for $m = 2$ we have six axes as used by Illert [1], while achieving compatibility with our sensory perception because at the abstract, realization-free level hypermathematics characterized by hyperunit is indeed 3-dimensional.

It is instructive for readers interested in learning the new mathematics to prove the following

LEMMA 5.1 [3]: All rings of elements $a \times \hat{I}^>$ (${}^<\hat{I} \times a$), where a is an ordinary (real, complex or quaternionic) number and $\hat{I}^>$ (${}^<\hat{I}$) is the forward (backward) multivalued hyperunit, when equipped with the forward (backward) hyperproduct, verify all axioms of a field.

A good understanding of the above property can be reached by comparison with the preceding studies. The discovery of isofields [8] studied in Chapter 3 was made possible by the observation that *the axioms of a field are insensitive to the value of the unit.* As a result of which we have isoproducts of the type

$$\hat{I} = 1/3 = 1/\hat{T}, \quad 2 \hat{\times} 3 = 2 \times \hat{T} \times 3 = 18. \tag{5.4}$$

The discovery of genofields also in Ref. [8] was due to the observation that *the axioms of a field are additionally insensitive to the ordering of a product to the right or to the left, provided that all operations are restricted to one selected order.* This lead to *two* inequivalent multiplications, one to the right and one to the left, as necessary to represent irreversibility, such as

$$\hat{I}^> = 1/3 = \hat{T}, \;\; 2 > 3 = 18, {}^<\hat{I} = 3, \;\; 2 < 3 = 2. \tag{5.5}$$

Lemma 5.1 essentially reflects the additional property according to which *the axioms of a field are also insensitive as to whether, in addition to the selection of an ordering as per genofields, the units and (ordered) products are multivalued,* e.g.,

$$\hat{I}^> = \{1/3, 1/5\}, \;\; 2 \hat{>} 3 = \{18, 30\}, \;\; {}^<\hat{I} = \{3, 2\}, \;\; 2 \hat{<} 3 = \{2, 3\}, \tag{5.6}$$

where the results of the hypermultiplications should be interpreted as an ordered set.

Once the notion of hyperfield is understood, the construction of all remaining aspects of hypermathematics can be conducted via simple compatibility arguments, thus leading in this way to *hyperspaces, hyperfunctional analysis, hyperdifferential calculus, hyperalgebras, etc.*

Note that the resulting hyperformulations are invariant as it is the case for genomathematics. The proof of such an invariance is here omitted for brevity, but recommended to readers interested in a serious study of the field.

The above features serve to indicate that the biological world has a complexity simply beyond our imagination, and that studies of biological problems conducted in the 20-th century, such as attempting an understanding the DNA code via numbers dating back to biblical times, are manifestly insufficient.

The above features appear to be necessary for the representation of biological systems. As an example, consider the association of two atoms in a DNA producing an organ composed by a very large number of atoms, such as a liver. A quantitative treatment of this complex event is given by representing the two atoms with α and β and by representing their association in a DNA with the hyperproduct. The resulting large number of atoms γ_k in the organ is then represented by the ordered multi-valued character of the hyperproduct, such as

$$\alpha \hat{>} \beta = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \ldots, \gamma_n, \}. \tag{5.7}$$

The above attempt at decybrings the DNA code is another illustration of our view that the complexity of biological systems is simply beyond our comprehension at this time. A mathematical representation will eventually be achieved in due time. However, any attempt at its "understanding" would face the same difficulties of attempting to understand infinite-dimensional Hilbert space in quantum mechanics, only the difficulties are exponentially increased for biological structures.

5.4 Rudiments of Santilli Isodual Hypermathematics

The *isodual hypermathematics* can be constructed via the isodual map of Chapter 2 here expressed for an arbitrary operator $\hat{A}$,

$$\hat{A}(\hat{t}, \hat{r}, \hat{p}, \hat{\psi}, \ldots) \rightarrow -\hat{A}^\dagger(-\hat{t}^\dagger, -\hat{r}^\dagger, -\hat{p}^\dagger, -\hat{\psi}^\dagger, \ldots) = \hat{A}^d(\hat{t}^d, \hat{r}^d, \hat{p}^d, \hat{\psi}^d, \ldots), \tag{5.8}$$

applied to the *totality* of hypermathematics, including its operations, with no exception (to avoid inconsistencies), thus yielding *isodual hyperunits, isodual hypernumbers, isodual hyperspaces, etc.*

Consequently, the formulations here considered have *four* different hyperunits, the forward and backward hyperunits and their isoduals,

$$\hat{I}^>,\ {}^<\hat{I},\ \hat{I}^{>d},\ {}^{<d}\hat{I}, \tag{5.9}$$

that, in turn, have to be specialized into forward and backward *space and time hyperunits* and their isoduals.

Consequently, the formulations herein considered have *four* different *hypercoordinates*

$$\hat{x}^>,\ {}^<\hat{x},\ \hat{x}^{>d},\ {}^<\hat{x}^d, \tag{5.10}$$

and *four* different *hypertimes,*

$$\hat{t}^>,\ {}^<\hat{t},\ \hat{t}^{>d},\ {}^<\hat{t}^d. \tag{5.11}$$

In chapter 2 (see also Figure 2.2) we have studied the need for four different times. We now have the four different hypertimes for: 1) Motion forward to future times characterized by $\hat{t}^>$; 2) Motion backward to past time characterized by $^<\hat{t}$; 3) Motion backward from future times characterized by $\hat{t}^{>d}$, and 4) motion forward from past times characterized by $^{<d}\hat{t}$. The main difference between the four times of Chapter 2 and the four hypertimes of this chapter is that the former are single-valued while the latter are multi-valued.

Note again the *necessity of the isodual map to represent all four possible time evolutions.* In fact, the conventional mathematics, such as that underlying special relativity, can only represent two our four possible time evolutions, motion forward to future time and motion backward to past time, the latter reached via the conventional time reversal operation.

The following intriguing and far reaching aspect emerges in biology. Until now we have strictly used isodual theories for the sole representation of antimatter. However, Illert [1] has shown that *the representation of the bifurcations in sea shells requires the use of all four directions of time.*

The latter aspect is an additional illustration of the complexity of biological system. In fact, the occurrence implies that the "intrinsic time" of a seashell, that is, the time perceived by a sea shells as a living organism, is so complex to be beyond our comprehension at this writing. Alternatively, we can say that the complexity of hypertimes is intended to reflect the complexity of biological systems.

In conclusion, the achievement of invariant representations of biological structures and their behavior can be one of the most productive frontiers of science, with far reaching implications for other branches, including mathematics, physics and chemistry.

As an illustration, a mathematically consistent representation of the non-Newtonian propulsion of sap in trees, all the way up to big heights, automatically provides a model of *geometric propulsion* studied in Volume II, namely propulsion caused by the alteration of the local geometry without any external applied force.

5.5 Santilli Hyperrelativity and Its Isodual

All preceding formulations can be embodied into one single axiomatic structure submitted in monographs [3,5] and today known as *Santilli hyperrelativity and its isodual,* that are characterized by:

1) The *irreversible, multi-valued, forward and backward, Minkowski-Santilli hyperspace* with the following *forward and backward spacetime hypercoordinates* and *forward and backward hyperintervals* over *forward and backward hyperfields*, and their isoduals

$$\hat{M}^>(\hat{x}^>, \hat{\eta}^>, \hat{R}^>), \hat{x}^{>2} = \hat{x}^{>\mu} \hat{>} \hat{\eta}^>_{\mu\nu} \hat{>} \hat{x}^{>\nu} \in \hat{R}^>, \tag{5,12a}$$

$$^<\hat{M}(^<\hat{x}, ^<\hat{\eta}, ^<\hat{R}), ^<\hat{x}^2 = ^< \hat{x}^{>\mu} \hat{<} ^<\hat{\eta}_{\mu\nu} \hat{<} ^<x^\nu \in ^< \hat{R}, \tag{5.12b}$$

$$\hat{m}^{>d}(\hat{x}^{>d}, \hat{\eta}^{>d}, \hat{R}^{>d}, ^<\hat{m}^d(^<\hat{x}^d, ^<\hat{\eta}^d, ^<\hat{R}^d; \tag{5.12c}$$

2) The corresponding *irreversible, multi-valued, forward and backward Poincaré-Santilli hypersymmetry* and their isoduals here written via the Kronecker product

$$\hat{P}^{>}(3,1)time^{<}\hat{P}(3.1)\times\hat{P}^{>d}(3,1)time^{<}\hat{P}^{d}(3.1), \tag{5.13}$$

essentially given by the Poincaré-Santilli genosymmetry of the preceding chapter under a multi-valued realization of the local coordinates and their operations;

3) The corresponding *forward and backward hyperaxioms* and their isoduals:

FORWARD HYPERAXIOM I. The projection in our spacetime of the maximal causal invariant speed on forward Minkowski-Santilli hyperspace in $(3,4)$*-dimensions is given by:*

$$\hat{V}_{Max}=c_o\times\frac{b_4^{>}}{b_3^{>}}=c_o\times\frac{n_3^{>}}{n_4^{>}}=\hat{c}^{>}/\hat{b}_3^{>},\quad \hat{c}^{>}=c_o\times b_4^{>}=\frac{c_o}{\hat{n}_4^{>}}, \tag{5.14}$$

FORWARD HYPERAXIOM II. The projection in our spacetime of the hyperrelativistic addition of speeds within MULTI-VALUED physical media represented by the forward Minkowski-Santilli hyperspace is given by:

$$\hat{V}_{tot}^{>}=\frac{\hat{v}_1^{>}+\hat{v}_2^{>}}{\hat{1}^{>}+\frac{\hat{v}_1^{>}\times b_3^{>2}\times\hat{v}_2^{>}}{c_o\times b_4^{>2}\times c_o}}=\frac{\hat{v}_1^{>}+\hat{v}_2^{>}}{\hat{1}^{>}+\frac{\hat{v}_1^{>}\times \hat{n}_4^{>2}\times\hat{v}_2^{>}}{c_o\times \hat{n}_3^{>2}\times c_o}}. \tag{5.15}$$

FORWARD HYPERAXIOM III. The projection in our spacetime of the forward hyperdilation of forward hypertime, forward hypercontraction of forward hyperlength and the variation of forward hypermass with the forward hyperspeed are given respectively by

$$\hat{t}^{>}=\hat{\gamma}^{>}\times\hat{t}_{\circ}^{>}, \tag{5.16a}$$

$$\hat{\ell}_{\circ}^{>}=\hat{\gamma}^{>}\times\hat{\ell}, \tag{5.16b}$$

$$\hat{m}^{>}=\hat{\gamma}^{>}\times\hat{m}_{\circ}^{>}. \tag{5.16c}$$

FORWARD HYPERAXIOM IV. The projection in our spacetime of the Doppler-Santilli forward hyperlaw is given by the expression (here formulated for simplicity for 90° *angle of aberration):*

$$\hat{\omega}^{>}=\hat{\gamma}\times^{>}\hat{\omega}_{\circ}^{>}. \tag{5.17}$$

ISOAXIOM V. The projection in our spacetime of the hyperrelativistic law of equivalence of forward hypermass and the forward hyperenergy is given by:

$$\hat{E}^{>}=\hat{m}^{>}\times\hat{V}_{\max}^{2>}=\hat{m}^{>}\times c_o^2\times\frac{\hat{b}_4^{>\,2}}{\hat{b}_3^{>\,2}}=\hat{m}^{>}\times c_o^2\times\frac{\hat{n}_3^{>\,2}}{\hat{n}_4^{>\,2}}. \tag{5.18}$$

Figure 5.2. Samples of sliced seashells showing the complexity of their structure. Illert [1] has shown that a mathematical representation of their four-lobes bifurcations requires all four directions of times, namely, the knowledge by the seashell of motions forward in future and past times as well as motions backward from future and in past times. The need for multi-valued methods, plus these four different time arrows then identify our hyperstructures and their isoduals quite uniquely. Whatever the appropriate theory, it can be safely stated that the complexity of the "intrinsic time" of biological structure (that perceived by said structures rather than by us) can be safely stated to be beyond our comprehension at this writing.

In the above expressions we have used the following notations: *hypergamma and hyperbeta* are given by

$$\hat{\gamma}^{>} = (1-\hat{\beta}^{2>})^{-1/2}, \quad \hat{\beta}^{>} = \hat{v}^{2>} \times \hat{n}_4^{>\ 2}/c_o^2 \times \hat{n}_3^{>\ 2} = \hat{v}^{2>} \times \hat{b}_3^{>\ 2}/c_o^2 \times \hat{b}_4^{>\ 2}; \quad (5.19)$$

the upper symbol $^>$ denotes motion forward to future times; the upper symbol $\hat{x}$, etc., denotes multivalued character; and all multiplications are conventional (rather than being hyperproducts) since the hyperaxioms are expressed in their projection in our spacetime to avoid excessive complexity.

The study of the backward and isodual hyperaxioms is left to the interested reader.

A few comments are now inn order:

i) Hyperrelativity and its isodual are the most general forms of relativities known at this writing that can be formulated on numbers verifying the axioms of a field, thus admitting a well defined left and right unit with consequential applicability to measurements;

ii) Hyperrelativity and its isodual are invariant under their respective time hyperevolutions, thus predicting the same numerical results at different time, and being applicable to experiments;

iii) Hyperrelativity and its isodual are multi-valued rather than multi-dimensional, namely, they permit the representation of multi-universes in a form compatible with our sensory perception of spacetime;

iv) The speed of light in vacuum c_o has been assumed to remain unchanged under hyperlifting, thus meaning that the speed of light is the same for all vacuum foliations of spacetime.

v) Like all other quantities, hyperspeeds in general and, in particular, the hyperspeed of light must necessarily be multi-valued for consistency, namely, assume different values for different foliations of spacetime.

Note the covering character of hyperrelativity in the sense of admitting as particular cases the genorelativity of Chapter 4, the isorelativity of Chapter 3 and the conventional special relativity whenever all units return to have the value 1 dating back to biblical times.

As we shall see in Volume II, hyperrelativity and its isodual, with particular reference to the 44-multi-valued hyperdimensional hypersymmetry (5.13)[1], will allow the formulation of the most general known, thus the most complex known, cosmology that includes, for the first time, biological structure as a condition for the appropriate use of the word "cosmology" in its Greek sense.

[1]The reader should recall that the Poincaré symmetry is *eleven*-dimensional and not ten dimensional as popularly believed because of the discovery permitted by isomathematics of the additional, 11-th dimensional isoscalar isoinvariance studied in Section 3.5.

Appendix 5.A
Vougiouklis Studies on the Lie-Santilli Hyper-Theory

5.A.1 Foreword

In this appendix we report *ad litteram* the studies on the Lie-Santilli hypertheory conducted by Thomas Vougiouklis, Democritus University of Thrace, School of Education, 681 00 Alexandroupolis, Greece, email tvougiou@eled.duth.gr. These studies are fundamental for the proper mathematical formulation of the class of hyperstructures necessary for hadronic mechanics, that with a well defined left and right hyperunit.

5.A.2 Introduction

The hyperstructures were introduced by F. Marty in 1934 when he first defined the hypergoup as a set equipped with an associative and reproductive hyperoperation. The motivating example was the quotient of a group by any, not necessarily normal, subgroup. M. Koskas in 1970 was introduced the fundamental relation β^*, which it turns to be the main tool in the study of hyperstructures. T. Vougiouklis in 1990 was introduced the H_v-structures, by defining the weak axioms. The motivating example of those hyperstructures is the quotient of any group by any partition. Therefore the class of H_v-structures is the largest class of hyperstructures.

In 1996 R. M. Santilli and T. Vougiouklis, point out that in physics the most interesting hyperstructures are the one called e-hyperstructures. These hyperstructures contain a unique left ant right scalar unit, which is the most important tool in Lie-Santilli theory. In what follows we present the related hyperstructure theory mainly from the paper [6], enriched with some new results on the related hyperstructures. However one can see the books by T. Vougiouklis [27] and by P. Corsini–V. Leoreanu [13], for more definitions as well as the site: aha.eled.duth.gr, for an extensive bibliography on the concept. Moreover, in this site one can see the Vougiouklis's point of view on the birth and the history of H_v-structures in the above site: *An H_v-interview, i.e. weak, with Th. Vougiouklis,* Interviewer N. Lygeros.

5.A.3 Basic definitions

In a set H is called *hyperoperation* or *multivalued operation*, any map from $H \times H$ to the power set of H. Therefore, in a hyperoperation

$$\cdot\colon H \times H \to \quad \wp(H)\colon (x, y) \to x \cdot y \subset H$$

the result is a subset of H, instead of an element as we have in usually operations.

In a set H equipped with a hyperoperation $\cdot: H \times H \rightarrow \wp(H) - \{\varnothing\}$, we abbreviate by

WASS the *weak associativity*: $(xy)z \cap x(yz) \neq \varnothing$, $\forall x, y, z \in H$ and by

COW the *weak commutativity*: $xy \cap yx \neq \varnothing$, $\forall x, y \in H$.

The hyperstructure $(H, \cdot)$ is called *H_v-semigroup* if it is *WASS* and it is called *H_v-group* if it is reproductive H_v-semigroup, i.e. $xH = Hx = H$, $\forall x \in H$. The hyperstructure $(R, +, \cdot)$ is called *H_v-ring* if $(+)$ and $(\cdot)$ are *WASS*, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* with respect to $(+)$:

$$x(y+z) \cap (xy+xz) \neq \varnothing, \quad (x+y)z \cap (xz+yz) \neq \varnothing, \quad \forall x, y, z \in R.$$

An extreme class of hyperstructures is the following [23]: An H_v-structure is called *very thin* iff all hyperoperations are operations except one, which has all hyperproducts singletons except one, which is a subset of cardinality more than one.

A H_v-group is called *cyclic* [27], if there is an element, called *generator*, which the powers have union the underline set. The minimal power with the above property is called *period* of the generator. Moreover if there exist an element and a special power, the minimum one, is the underline set, then the H_v-group is called *single-power cyclic*.

The main tool to study all hyperstructures are the fundamental relations β^*, γ^* and ε^*, which are defined, in H_v-groups, H_v-rings and H_v-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector space, respectively [27]. A way to find the fundamental classes is given by analogous theorems to the following [24, 27, 28]:

THEOREM: Let $(H, \cdot)$ be a H_v-group and U be all finite products of elements of H. Define the relation β by setting $x\beta y$ iff $\{x, y\} \subset u$, $u \in U$. Then β^* is the transitive closure of β.

Analogous theorems for the relations γ^* in H_v-rings and ε^* in H_v-modules and H_v-vector spaces, are also proved.

An element is called *single* if its fundamental class is singleton.

Fundamental relations are used for general definitions. Thus, in the definition of the H_v-field the γ^* is used: A H_v-ring $(R, +, \cdot)$ is called *H_v-field* if R/γ^* is a field.

Let $(H, \cdot)$, $(H, {}^*)$ be H_v-semigroups defined on the same set H. $(\cdot)$ is called *smaller* than $({}^*)$, and $({}^*)$ *greater* than $(\cdot)$, iff there exists an

$$f \in \mathrm{Aut}(H, {}^*) \text{ such that } xy \subset f(x^*y), \ \ \forall x, y \in H.$$

Then we write $\cdot \leqslant^*$ and we say that $(H, {}^*)$ *contains* $(H, \cdot)$. If $(H, \cdot)$ is a structure then it is called *basic structure* and $(H, {}^*)$ is called *H_b-structure*.

THEOREM: Greater hyperoperations than the ones which are WASS or COW, are also WASS or COW, respectively.

The definition of the H_v-field introduced a new class of hyperstructures [39]:

DEFINITION: The H_v-semigroup $(H,\cdot)$ is called h/v-group if the quotient H/β^* is a group.

The h/v-groups are a generalization of the H_v-groups because in h/v-groups the reproductivity is not necessarily valid. However, sometimes a kind of *reproductivity of classes* is valid. This leads the quotient to be reproductive. In a similar way the *h/v-rings, h/v-fields, h/v-modulus, h/v-vector spaces* etc., are defined.

Definitions [33, 36, 37]. Let $(H,\cdot)$ be hypergroupoid. We *remove* h∈H, if we consider the restriction of $(\cdot)$ in the set $H-\{h\}$. $\underline{h}\in H$ *absorbs* $h\in H$ if we replace h by $\underline{h}$ and h does not appear in the structure. $\underline{h}\in H$ *merges* with $h\in H$, if we take as product of any $x\in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$, and consider h and $\underline{h}$ as one class with representative $\underline{h}$, therefore, h does not appear in the hyperstructure.

Hyperoperations on any type of matrices can be defined:

DEFINITION [42]: Let $A=(a_{ij})\in\mathbf{M}_{m\times n}$ be matrix and $s,t\in N$, with $1\leqslant s\leqslant m$, $1\leqslant t\leqslant n$. Then helix-projection is a map $\underline{st}$: $\mathbf{M}_{m\times n}\to\mathbf{M}_{s\times t}$: $A\to A\underline{st}=(\underline{a}_{ij})$, where $A\underline{st}$ has entries

$$\underline{a}_{ij}=\{a_{i+\kappa s,j+\lambda t}\;\big|\;1\leqslant i\leqslant s,\ 1\leqslant j\leqslant t \text{ and } \kappa,\,\lambda\in N,\ i+\kappa s\leqslant m, j+\lambda t\leqslant n\}.$$

Let $A=(a_{ij})\in\mathbf{M}_{m\times n}$, $B=(b_{ij})\in\mathbf{M}_{u\times v}$ be matrices and $s=\min(m,u)$, $t=\min(n,v)$. We define a hyper-addition, called *helix-addition*, by

$$\begin{aligned}\oplus:\mathbf{M}_{m\times n}\times\mathbf{M}_{u\times v}\to P(\mathbf{M}_{s\times t}):(A,B)&\to A\oplus B\\ &=A\underline{st}+B\underline{st}=(\underline{a}_{ij})+(\underline{b}_{ij})\subset\mathbf{M}_{s\times t},\end{aligned}$$

where $(\underline{a}_{ij})+(\underline{b}_{ij})=\{(c_{ij})=(a_{ij}+b_{ij})\;\big|\;a_{ij}\in\underline{a}_{ij}\text{ and } b_{ij}\in\underline{b}_{ij})\}$. Let $A=(a_{ij})\in\mathbf{M}_{m\times n}$, $B=(b_{ij})\in\mathbf{M}_{u\times v}$ and $s=\min(n,u)$. We define the *helix-multiplication*, by

$$\begin{aligned}\otimes:\mathbf{M}_{m\times n}\times\mathbf{M}_{u\times v}\to P(\mathbf{M}_{m\times v}):(A,B)&\to A\otimes B\\ &=A\underline{ms}\cdot B\underline{sv}=(\underline{a}_{ij})\cdot(\underline{b}_{ij})\subset\mathbf{M}_{m\times v},\end{aligned}$$

where $(\underline{a}_{ij})\cdot(\underline{b}_{ij})=\{(c_{ij})=(\sum a_{it}b_{tj})\;\big|\;a_{ij}\in\underline{a}_{ij}\text{ and } b_{ij}\in\underline{b}_{ij})\}$.

The helix-addition is commutative, WASS but not associative. The helix-multiplication is WASS, not associative and it is not distributive, not even weak, to the helix-addition. For all matrices of the same type, the inclusion distributivity, is valid.

5.A.4 Small sets

The problem of enumeration and classification of H_v-structures, was started from the beginning [23, 22]. However, the problem becomes more complicate in

H_v-structures because we have very great numbers in this case. The partial order in H_v-structures [24], transfers and restrict the problem in finding the minimal, *up to isomorphisms*, H_v-structures. In this direction we have results by Bayon & Lygeros [10, 11]:

Let $H = \{a, b\}$ a set of two elements. There are 20 H_v-groups, up to isomorphism.

Suppose in $H = \{e, a, b\}$, a hyperoperation is defined and there exists a scalar unit, then, there are 13 *minimal* H_v-groups. The number of all H_v-groups with three elements, up to isomorphism, which have a scalar unit, is 292.

In a set with three elements there are, exactly 6.494 minimal H_v-groups. 137 are abelians and the 6.357 are non-abelians; the 6.152 are cyclic and the 342 are not cyclic.

The number of H_v-groups with three elements, up to isomorphism, is 1.026.462. More precisely, there are 7.926 abelians and 1.018.536 non-abelians; the 1.013.598 are cyclic and the 12.864 are not cyclic. The 16 of them are very thin.

The number of all H_v-groups with four elements, up to isomorphism, which have a scalar unit, is 631.609. There are 10.614.362 abelian hypergroups from which the 10.607.666 are cyclic and the 6.696 are not. There are 8.028.299.905 abelian H_v-groups from which the 7.995.884.377 are cyclic and the 32.415.528 are not.

5.A.5 Uniting elements

The *uniting elements* method was introduced by Corsini-Vougiouklis [14] in 1989. With this method one puts in the same class, two or more elements. This leads, through hyperstructures, to structures satisfying additional properties.

The *uniting elements* method is the following: Let G be algebraic structure and let d be a property, which is not valid and it is described by a set of equations; then, consider the partition in G for which it is put together, in the same partition class, every pair of elements that causes the non-validity of the property d. The quotient by this partition G/d is an H_v-structure. Then, quotient out the H_v-structure G/d by the fundamental relation β^*, a stricter structure $(G/d)\beta^*$ for which the property d is valid, is obtained.

An interesting application of the uniting elements is when more than one properties are desired. The reason for this is that some of the properties lead straighter to the classes than others. So, it is better to apply the straightforward classes followed by the more complicated ones. The commutativity and reproductivity are easy applicable properties. One can do this because the following is valid.

THEOREM [27]: Let $(G, \cdot)$ be a groupoid, and $F = \{f_1, \ldots, f_m, f_{m+1}, \ldots, f_{m+n}\}$ be a system of equations on G consisting of two subsystems $F_m = \{f_1, \ldots, f_m\}$ and $F_n = \{f_{m+1}, \ldots, f_{m+n}\}$. Let σ, σ_m be the equivalence relations defined by the uniting elements procedure using the systems F and F_m resp., and *let σ_n be the equivalence relation defined using the induced equations of F_n on the grupoid $G_m = (G/\sigma_m)/\beta^*$. Then $(G/\sigma)/\beta^* \cong (G_m/\sigma_n)/\beta^*$.*

5.A.6 Theta (∂) hyperoperations

In [40] a hyperoperation, in a groupoid with a map on it, called *theta* ∂, is introduced.

DEFINITION: Let H be a set equipped with n operations (or hyperoperations) $\otimes_1, \otimes_2, \ldots, \otimes_n$ and a map (or multivalued map) $f : H \to H$ (or $f : H \to P(H)$, resp.), then n hyperoperations ∂_1, ∂_2, ..., ∂_n on H can be defined, called theta-operations by putting

$$x\partial_i y = \{f(x) \otimes_i y,\, x \otimes_i f(y)\},\ \forall x, y \in H \text{ and } i \in \{1, 2, \ldots, n\}$$

or, in case where $\otimes_i$ is hyperoperation or f is multivalued map, we have

$$x\partial_i y = (f(x) \otimes_i y) \cup (x \otimes_i f(y)),\ \forall x, y \in H \text{ and } i \in \{1, 2, \ldots, n\}.$$

If $\otimes_i$ is associative then ∂_i is WASS.

DEFINITIONS: Let $(G, \cdot)$ be a groupoid and $f_i : G \to G$, $i \in I$, be a set of maps on G. Take the map $f_\cup : G \to P(G)$ such that $f_\cup(x) = \{f_i(x) \mid i \in I\}$ and we call it the union of the $f_i(x)$. We call union theta-operation (∂), on G if we consider the map $f_\cup(x)$. A special case is to take the union of f with the identity, i.e. $\underline{f} \equiv f \cup (id)$, so $\underline{f}(x) = \{x, f(x)\}$, $\forall x \in G$, which is called b-theta-operation. We denote by ($\underline{\partial}$) the b-theta-operation, so

$$x\underline{\partial}y = \{xy, f(x) \cdot y, x \cdot f(y)\},\ \forall x, y \in G.$$

Remark that this hyperoperation is a b-operation. If $f : G \to P(G)$ then the b-theta-operation is defined by using the map $\underline{f}(x) = \{x\} \cup f(x)$, $\forall x \in G$.

Motivation for the definition of the theta-operation is the map *derivative* where only the multiplication of functions can be used. Therefore, in these terms, for two functions $s(x)$, $t(x)$, we have $s\partial t = \{s't, st'\}$, where ($'$) denotes the derivative.

Example. Taking the application on the derivative, consider all polynomials of first degree $g_i(x) = a_i x + b_i$. We have

$$g_1\partial g_2 = \{a_1a_2x + a_1b_2,\ a_1a_2x + b_1a_2\},$$

so this is a hyperoperation in the set of the first degree polynomials. Moreover all polynomials x+c, where c be a constant, are units.

Properties [40, 41]. If $(G, \cdot)$ is a semigroup then:

For every f, the hyperoperation (∂) is *WASS*, and the b-theta-operation (∂) is *WASS*.

If f is homomorphism and projection, then (∂) is associative.

Reproductivity. For the reproductivity we must have

$$x\partial G = \cup_{g\in G}\{f(x)\cdot g,\, x\cdot f(g)\} = G \text{ and } G\partial x = \cup_{g\in G}\{f(g)\cdot x,\, g\cdot f(x)\} = G.$$

Thus, if $(\cdot)$ is reproductive then (∂) is also reproductive.

Commutativity. If $(\cdot)$ is commutative then (∂) is commutative. If f is into the centre of G, then (∂) is commutative. If $(\cdot)$ is COW then, (∂) is COW.

Unit elements. u is a right unit element if $x\partial u = \{f(x)\cdot u,\, x\cdot f(u)\} \ni x$. So $f(u) = e$, where e be a unit in $(G,\cdot)$. The elements of the kernel of f, are the units of (G,∂).

Inverse elements. Let $(G,\cdot)$ be a monoid with unit e and u be a unit in (G,∂), then $f(u) = e$. For given x, the element x' is an inverse with respect to u, if

$$x\partial x' = \{f(x)\cdot x',\, x\cdot f(x')\} \ni u \text{ and } x'\partial x = \{f(x')\cdot x,\, x'\cdot f(x)\} \ni u.$$

So, $x' = (f(x))^{-1}u$ and $x' = u(f(x))^{-1}$, are the right and left inverses, respectively. We have two-sided inverses iff $f(x)u = uf(x)$.

PROPOSITION: Let $(G,\cdot)$ be a group then, for all $f : G \to G$, the (G,∂) is a H_v-group.

For several results one can see [20–22, 6, 23].

In order to see a connection of the merge with the ∂-operation, consider the map f such that $f(\underline{h}) = h$ and $f(x) = x$ in the rest cases.

Example. P-hyperoperations. Let $(G,\cdot)$ be commutative semigroup and $P \subset G$. Consider the multivalued map f such that $f(x) = P\cdot x$, $\forall x \in G$. Then we have

$$x\partial y = x\cdot y\cdot P,\ \forall x, y \in G.$$

So the ∂-operation coincides with the well known class of P-hyperoperations [14].

One can define theta-operations on rings and other more complicate structures, where more than one theta-operations can be defined.

DEFINITION: Let $(R,+,\cdot)$ be a ring and $f : R \to R$, $g : R \to R$ be two maps. We define two hyperoperations (∂_+) and $(\partial\cdot)$, called both theta-operations, on R as follows

$$x\partial_+ y = \{f(x) + y,\, x + f(y)\} \text{ and } x\partial\cdot y = \{g(x)\cdot y,\, x\cdot g(y)\},\ \ \forall x, y \in G.$$

A hyperstructure $(R,+,\cdot)$, where $(+)$, $(\cdot)$ be hyperoperations which satisfy all H_v-ring axioms, except the weak distributivity, will be called *H_v-near-ring.*

PROPOSITION: Let $(R,+,\cdot)$ ring and $f : R \to R$, $g : R \to R$ maps. The hyperstructure $(R,\partial_+,\,\partial\cdot)$, called theta, is a H_v-near-ring. Moreover $(+)$ is commutative.

PROPOSITION: Let $(R,+,\cdot)$ ring and $f: R\to R$, $g: R\to R$ maps, then $(R,\partial_+,\partial\cdot)$, is H_v-ring.

Properties. The theta hyperstructure $(R,\partial_+,\partial\cdot)$ takes new form in special cases:

(a) If $f(x)\equiv g(x)$, $\forall x\in R$, i.e. the two maps coincide, then we have

$$x\partial\cdot(y\partial_+ z)\cap(x\partial\cdot y)\partial_+(x\partial\cdot z)=\varnothing.$$

If f is homomorphism and projection, then $(R,\partial_+,\partial\cdot)$ is H_v-ring.

(b) If $f(x)=x$, $\forall x\in R$, then $(R,+,\partial\cdot)$ becomes a multiplicative H_v-ring:

$$x\partial\cdot(y+z)\cap(x\partial\cdot y)+(x\partial\cdot z)=\{g(x)y+g(x)z\}\neq\varnothing.$$

5.A.7 The H_v-vector spaces and H_v-Lie algebras

DEFINITION [27, 34]: Let $(F,+,\cdot)$ be a H_v-field, $(V,+)$ be a COW H_v-group and there exists an external hyperoperation

$$\cdot: F\times V\to\mathcal{P}(V):(a,x)\to ax$$

such that, for all a,b in F and x,y in V we have

$$a(x+y)\cap(ax+ay)\neq\varnothing,\quad (a+b)x\cap(ax+bx)\neq\varnothing,\quad (ab)x\cap a(bx)\neq\varnothing,$$

then V is called a H_v-vector space over F.

In the case of a H_v-ring instead of H_v-field then the H_v-modulo is defined.

In the above cases the fundamental relation ε^* is the smallest equivalence relation such that the quotient V/ε^* is a vector space over the fundamental field F/γ^*.

The general definition of a H_v-Lie algebra over a field F is the following [27, 34]:

DEFINITION. Let $(L,+)$ be a H_v-vector space over the field $(F,+,\cdot)$, $\varphi: F\to F/\gamma^*$, the canonical map and $\omega_F=\{x\in F:\varphi(x)=0\}$, where 0 is the zero of the fundamental field F/γ^*. Similarly, let ω_L be the core of the canonical map $\varphi': L\to L/\varepsilon^*$ and denote by the same symbol 0 the zero of L/ε^*. Consider the bracket (commutator) hyperoperation:

$$[,]: L\times L\to\mathcal{P}(L):(x,y)\to[x,y],$$

then L is a H_v-Lie algebra over F if the following axioms are satisfied:

(L1) The bracket hyperoperation is bilinear, i.e.

$$[\lambda_1x_1+\lambda_2x_2,y]\cap(\lambda_1[x_1,y]+\lambda_2[x_2,y])\neq\varnothing,$$
$$[x,\lambda_1y_1+\lambda_2y]\cap(\lambda_1[x,y_1]+\lambda_2[x,y_2])\neq\varnothing$$

for all $x, x_1, x_2, y, y_1, y_2 \in L$ and λ_1, λ_2 in F;

(L2) $[x, x] \cap \omega_L \neq \varnothing$ for all x in L;

(L3) $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \varnothing$ for all x, y in L.

We remark that this is a very general definition therefore one can use special cases in order to face several problems in applied sciences. Moreover, from this definition we can see how the weak properties can be defined as the above weak linearity (L1), anti-commutativity (L2) and the Jacobi identity (L3).

We present here a direction to obtain results from special cases by applying ∂-operations on more complicated structures, in the sense that they have more than one operation.

THEOREM: Consider the group of integers $(Z, +)$ *and let* $n \neq 0$ *be a natural number. Take the map* f *such that* $f(0) = n$ *and* $f(x) = x$, $\forall x \in Z - \{0\}$. *Then* $(Z, \partial)/\beta^* \cong (Z_n, +)$.

THEOREM: Consider the ring of integers $(Z, +, \cdot)$ *and let* $n \neq 0$. *Consider the map* f *such that* $f(0) = n$ *and* $f(x) = x$, $\forall x \in Z - \{0\}$. *Then* $(Z, \partial_+, \partial\cdot)$ *is a* H_v*-near-ring, with* $(Z, \partial_+, \partial\cdot)/\gamma^* \cong Z_n$.

PROPOSITION: Let $(V, +, \cdot)$ *be an algebra over the field* $(F, +, \cdot)$ *and* $f : V \to V$ *be a map. Consider the* ∂*-operation defined only on the multiplication of the vectors* $(\cdot)$, *then* $(V, +, \partial)$ *is a* H_v*-algebra over* F, *where the related properties are weak. If, moreover* f *is linear then we have more strong properties.*

DEFINITION: Let L be a Lie algebra, defined on an algebra $(V, +, \cdot)$ *over the field* $(F, +, \cdot)$ *where the Lie bracket* $[x, y] = xy - yx$. *Consider any map* $f : L \to L$, *then the* ∂*-operation is defined as follows*

$$x\partial y = \{f(x)y - f(y)x,\ f(x)y - yf(x),\ xf(y) - f(y)x,\ xf(y) - yf(x)\}.$$

PROPOSITION: Let $(V, +, \cdot)$ *be an algebra over the field* $(F, +, \cdot)$ *and* $f : V \to V$ *be a linear map. Consider the* ∂*-operation defined only on the multiplication of the vectors* $(\cdot)$, *then* $(V, +, \partial)$ *is a* H_v*-algebra over* F, *with respect to Lie bracket, where the weak anti-commutatinity and the inclusion linearity is valid.*

We can see that the weak linearity is valid, more precisely, the inclusion linearity is valid: $[\lambda_1 x_1 + \lambda_2 x_2, y] \subset \lambda_1 [x_1, y] + \lambda_2 [x_2, y]$.

Remark that one can face the weak Jacobi identity in analogous to the above propositions as well. One can use well known maps as constants or linear.

5.A.8 Adding elements

In [33] the 'enlarged' hyperstructures were examined in the sense that an extra element, outside the underlying set, appears in one result. In both directions, enlargement or reduction, most useful in representation theory, are those H_v-structures with the same fundamental structure: Suppose we have a structure and one element, outside of the structure, then we can attach this element in order to have a hyperstructure which becomes h/v-structure. Moreover we have the opposite problem: How one can remove at least one element of an H_v-structure or a classical structure?

The Attach Construction [33, 36, 37]. Let $(H, \cdot)$ be an H_V-semigroup and $v \notin H$. We extend the $(\cdot)$ into the set $\underline{H} = H \cup \{v\}$ as follows: $x \cdot v = v \cdot x = v$, $\forall x \in H$, and $v \cdot v = H$.

The $(\underline{H}, \cdot)$ is a h/v-group where $(\underline{H}, \cdot)/\beta^* \cong Z_2$ and v is a single element.

We call the hyperstructure $(\underline{H}, \cdot)$ the attach h/v-group of $(H, \cdot)$.

Remarks. The core of $(\underline{H}, \cdot)$ is the set H. All scalar elements of $(H, \cdot)$ are also scalars in $(\underline{H}, \cdot)$ and any unit element of $(H, \cdot)$ is also a unit of $(\underline{H}, \cdot)$. Finally, if $(H, \cdot)$ is COW (resp. commutative) then $(\underline{H}, \cdot)$ is also COW (resp. commutative).

The motivation of the attach construction is the first kind very thin H_v-groups [23].

In the representation theory of H_v-groups by H_v-matrices one needs H_v-rings or H_v-fields which have non-degenerate fundamental structures in addition with only few of hypersums and hyperproducts to have cardinals greater than one.

THEOREM: Let $(G, \cdot)$ be semigroup and $v \notin G$ be an element appearing in a product ab, where $a, b \in G$, thus the result becomes a hyperproduct $a \otimes b = \{ab, v\}$. Then the minimal hyperoperation $(\otimes)$ extended in $G' = G \cup \{v\}$ such that $(\otimes)$ contains $(\cdot)$ in the restriction on G, and such that $(G', \otimes)$ is a minimal H_V-semigroup which has fundamental structure isomorphic to $(G, \cdot)$, is defined as follows:

$$a \otimes b = \{ab, v\},\ x \otimes y = xy,\ \forall (x, y) \in G^2 - \{(a, b)\},$$

$$v \otimes v = abab,\ x \otimes v = xab \ \text{ and } \ v \otimes x = abx,\ \forall x \in G.$$

Therefore $(G', \otimes)$ is a very thin H_V-semigroup.

If $(G, \cdot)$ is commutative then the $(G', \otimes)$ becomes strongly commutative.

5.A.9 Representations

Representations (we abbreviate here by *rep*) of H_v-groups, can be considered either by generalized permutations [25] or by H_v-matrices [26]. Here we present the matrix reps.

H_v-matrix (or *h/v-matrix*) is called a matrix with entries elements of a H_v-ring or H_v-field (or h/v-ring or h/v-field). The hyperproduct of H_v-matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$, of type $m \times n$ and $n \times r$, respectively, is a set of $m \times r$ H_v-matrices, defined in a usual manner:

$$\mathbf{A} \cdot \mathbf{B} = (a_{ij}) \cdot (b_{ij}) = \{\mathbf{C} = (c_{ij}) \bigm| c_{ij} \in \oplus \sum a_{ik} \cdot b_{kj},$$

where $(\oplus)$ denotes the *n-ary circle hyperoperation* on the hyperaddition [27]: that is the sum of products of elements of the H_v-ring is considered to be the union of the sets obtained with all possible parentheses. However, in the case of 2×2 H_v-matrices the 2-ary circle hyperoperation which coincides with the hyperaddition in the H_v-ring. Notice that the hyperproduct of H_v-matrices does not nessesarily satisfy *WASS*.

The rep problem by H_v-matrices is the following:

H_v-matrix is called a matrix if has entries from a H_v-ring.

DEFINITION: Let $(H, \cdot)$ be H_v-group, $(R, +, \cdot)$ be H_v-ring and $\mathbf{M}_R = \{(a_{ij}) \mid a_{ij} \in R\}$, then any

$$\mathbf{T} : H \to \mathbf{M}_R : h \to \mathbf{T}(h) \text{ with } \mathbf{T}(h_1h_2) \cap \mathbf{T}(h_1)\mathbf{T}(h_2) \neq \varnothing, \ \forall h_1, h_2 \in H,$$

is called H_v-matrix rep. If $\mathbf{T}(h_1h_2) \subset \mathbf{T}(h_1)\mathbf{T}(h_2)$, then $\mathbf{T}$ is an inclusion rep, if $\mathbf{T}(h_1h_2) = \mathbf{T}(h_1)\mathbf{T}(h_2)$, then $\mathbf{T}$ is a good rep.

In reps of H_v-groups by H_v-matrices, there are two difficulties: To find a H_v-ring and an appropriate set of H_v-matrices.

The problem of reps is very complicated mainly because the cardinality of the product of two H_v-matrices is normally very big. The problem can be simplified in several special cases such as the following:

(a) The H_v-matrices are over H_v-rings with 0 and 1 and if these are scalars. Thus the e-hyperstructures are interesting in the rep theory.

(b) The H_v-matrices are over *very thin* H_v-rings.

(c) The case of 2×2 H_v-matrices, since the 2-ary circle hyperoperation coincides with the hyperaddition in H_v-rings. This is the lowest dimensional, non degenerate, rep.

(d) The case of H_v-rings in which the strong associativity in hyperaddition is valid.

(e) The case of H_v-rings which contains singles, then these act as absorbings.

The main theorem of reps on h/v-structures, which has a completely analogous on H_v-structures [27], is the following:

THEOREM: A necessary condition in order to have an inclusion rep T of an h/v-group $(H, \cdot)$ by $n \times n$ h/v-matrices over the h/v-ring $(R, +, \cdot)$ is the following:

For all classes $\beta^*(x)$, $x \in H$ there must exist elements $a_{ij} \in H$, $i, j \in \{1, \ldots, n\}$ such that

$$T(\beta^*(a)) \subset \{A = (a'_{ij}) \mid a'_{ij} \in \gamma^*(a_{ij}), \ i, j \in \{1, \ldots, n\}\}.$$

Therefore, every inclusion rep $T : H \to M_R : a \mapsto T(a) = (a_{ij})$ induces a homomorphic rep T^* of the group H/β^* over the ring R/γ^* by setting $T^*(\beta^*(a)) = [\gamma^*(a_{ij})]$, $\forall \beta^*(a) \in H/\beta^*$, where the element $\gamma^*(a_{ij}) \in R/\gamma^*$ is the ij entry of the matrix $T^*(\beta^*(a))$. Then T^* is called *fundamental induced rep* of T.

In analogous way other concepts of the rep theory can be transferred for h/v structures. Thus, let T be a rep of an h/v-group H by h/v-matrices over the

h/v-ring R. Denote $\mathrm{tr}_\varphi(T(x)) = \gamma^*(T(x_{ii}))$ the fundamental trace, then the mapping

$$X_T : H \to R/\gamma^* : x \mapsto X_T(x) = \mathrm{tr}_\varphi(T(x)) = \mathrm{tr}T^*(x)$$

is called *fundamental character*. There are several types of traces.

For an attached h/v-field $(\underline{H}_o, +, \cdot)$, in $\sum a_{ik} \cdot b_{kj}$ the terms $a_{ik} \cdot b_{kj}$ could be $0, v, x$ or H (where $x \in H$). But any sum is only 0 or v or H. Thus, for finite h/v-fields $(\underline{H}_o, +, \cdot)$, if the set H appears in t entries then the cardinality of the hyperproducts is $(\mathrm{card}H)^t$.

The main attached h/v-fields give to rep theory some hyperfields for reps where the cardinality of any two elements is small. The point is that 0 is absorbing.

5.A.10 The e-constructions

The Lie-Santilli *isotopies* born in 1970's to solve Hadronic Mechanics problems. Santilli [6], proposed a 'lifting'of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed in this theory correspond into the hyperstructures called *e-hyperfields*. The H_v-fields or h/v-fields can give e-hyperfields which can be used in the isotopy theory in applications as in physics or biology.

DEFINITION: Let $(\underline{H}_o, +, \cdot)$ *be the attached h/v-field of the* H_v*-semigroup* $(H, \cdot)$*. If* $(H, \cdot)$ *has a left and right scalar unit e then* $(\underline{H}_o, +, \cdot)$ *is e-hyperfield, the attached h/v-field of* $(H, \cdot)$*.*

Applications. (1) The above constructions, especially the ones in enlarging H_v-rings, or H_v-fields can be used as entries of H_v-matrices to represent H_v-groups for which the cardinality of all hypreproducts equals to 2^s, $s \in N$. This is so, since in the hyperproducts of H_v-matrices we can have one or two elements.

(2) The monomial matrix reps are based, on the ring Z_2. The enlargments of the above ring are the following hyperrings

(i) $0 \oplus 0 = 1 \oplus 1 = 1 \oplus v = v \oplus 1 = v \oplus v = 0,$
$0 \oplus 1 = 0 \oplus v = 1 \oplus 0 = v \oplus 0 = \{1, v\},$
$0 \otimes 0 = 0 \otimes 1 = 1 \otimes 0 = 0 \otimes v = v \otimes 0 = 0,$
$1 \otimes 1 = 1 \otimes v = v \otimes 1 = v \otimes v = 1,$

(ii) $0 \oplus 0 = 0 \oplus v = 1 \oplus 1 = v \oplus 0 = \{0, v\}, \;\; v \oplus v = 0,$
$0 \oplus 1 = 1 \oplus 0 = 1 \oplus v = v \oplus 1 = 1, \;\; 1 \otimes 1 = 1,$
and in the rest cases 0.

(iii) $0 \oplus 0 = 1 \oplus 1 = v \oplus v = \{0, v\}, \;\; 0 \oplus v = v \oplus 0 = 0,$

$0 \oplus 1 = 1 \oplus 0 = 1 \oplus v = v \oplus 1 = 1, \;\; 1 \otimes 1 = 1,$

and in the rest cases 0.

CONSTRUCTION I: Let $(H, \cdot)$ be H_v-group, then for every $(\oplus)$ such that $x \oplus y \supset \{x, y\}$, $\forall x, y \in H$, the $(H, \oplus, \cdot)$ is an H_v-ring. These H_v-rings are called *associated to* $(H, \cdot)$.

In the theory of reps of the hypergroups, in the sense of Marty, there are three types of associated hyperrings $(H, \oplus, \cdot)$ to the hypergroup $(H, \cdot)$. The hyperoperation $(\oplus)$ is defined respectively, for all x, y in H, as follows:

$$\text{type a}: \; x \oplus y = \{x, y\}, \;\; \text{type b}: \; x \oplus y = \beta^*(x) \cup \beta^*(y), \;\; \text{type c}: \; x \oplus y = H.$$

In the above types the strong associativity and strong or inclusion distributivity, is valid.

CONSTRUCTION II: Let $(H, +)$ be H_v-group. Then for every hyperoperation $(\otimes)$ such that $x \otimes y \supset \{x, y\}$, $\forall x, y \in H$, the hyperstructure $(H, +, \otimes)$ is an H_v-ring.

CONSTRUCTION III: Let $(H, +)$ be H_v-group with a scalar zero 0. Then for every $(\otimes)$ such that $x \otimes y \supset \{x, y\}$, $\forall x, y \in H - \{0\}$, $x \otimes 0 = 0 \otimes x = 0$, $\forall x \in H$, the $(H, +, \otimes)$ is an H_v-ring.

In this construction 0 is absorbing scalar but not single.

CONSTRUCTION IV: Let $(H, \cdot)$ be H_v-group. Take a $0 \notin H$ and set $H' = H \cup \{0\}$. We define the hyperoperation $(+)$ as follows: $0+0 = 0$, $0+x = H = x+0$, $x+y = 0$, $\forall x, y \in H$, and we extend $(\cdot)$ in H' by putting $0 \cdot 0 = 0$, $0 \cdot x = x \cdot 0 = 0$, $\forall x, y \in H$. Then $(H', +, \cdot)$ is a reproductive H_v-field with $H'/\gamma^* \cong Z_2$ where 0 is absorbing and single.

Appendix 5.B
Eric Trell's Hyperbiological Structures TO BE COMPLETED AND EDITED.

A new conception of biological systems providing a true advance over rather primitive prior conceptions, has been recently proposed by Erik Trell (see Ref. (164) and contributions quoted therein). It is based on representative blocks which appear in our space to be next to each other, thus forming a cell or an organism, while having in reality hypercorrelations, thus having the structure of hypernumbers, hypermathematics and hyperrelativity, with consequential descriptive capacities immensely beyond those of pre-existing, generally single-valued and reversible biological models. Regrettably, we cannot review Trell's new hyperbiological model to avoid an excessive length, and refer interested readers to the original literature (164).

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Postscript

In the history of science some basic advances in physics have been preceded by basic advances in mathematics, such as Newtons invention of calculus and general relativity relying on Riemannian geometry. In the case of quantum mechanics the scientific revolution presupposed the earlier invention of complex numbers. With new numbers and more powerful mathematics to its disposition, physics could be lifted to explain broader and more complex domains of physical reality.

The recent and ongoing revolution of physics, initiated by Prof. Ruggero Maria Santilli, lifting the discipline from quantum mechanics to hadronic mechanics, is consistent with this pattern, but in a more far-reaching and radical way than earlier liftings of physics made possible from extensions of mathematics.

Santilli realized at an early stage that basic advances in physics required invention of new classes of numbers and more adequate and powerful mathematics stemming from this. His efforts to develop such expansions of mathematics started already in 1967, and this enterprise went on for four decades. Its basic novelties, architecture and fruits are presented in the present volume. During this period a few dozen professional mathematicians world wide have made more or less significant contributions to fill in the new Santilli fields of mathematics, but the honor of discovering these vast new continents and work out their basic topology is Santillis and his alone. These new fields initiated by Santilli made possible realization of so-called Lie-admissible physics. For this achievement Santilli in 1990 received the honor from Estonia Academy of Science of being appointed as mathematician number seven after world war two considered a landmark in the history of algebra.

With regard to Sophus Lie it may be of some interest to note that the Norwegian examiners of his groundbreaking doctoral thesis in 1871 were not able to grasp his work, due to its high degree of novelty and unfamiliarity. However, due to Lie already being highly esteemed among influential contemporary mathematicians at the continent, it was not an option to dismiss his thesis. As in other disciplines, highly acknowledged after Thomas Kuhns publication of The structure of scientific revolutions in 1962, sufficiently novel mathematics implies some paradigmatic challenge. Therefore, it is not strange that some mathematicians and physicists have experienced difficulties taking the paradigmatic leap necessary to grasp the basics of hadronic mathematics or to acknowledge its far-reaching implications. Such a challenge is more demanding when scientific novelty implies a reconfiguration of conventional basic notions in the discipline. This is, as Kuhn noted, typically easier for younger and more emergent scientific minds.

Until Santilli the number 1 was silently taken for granted as the primary unit of mathematics. However, as noted by mathematical physicist Peter Rowlands at University of Liverpool, the number 1 is already loaded with assumptions, that can be worked out from a lifted and broader mathematical framework. A partial and rough analogy might be linguistics where it is obvious that a universal science of language must be worked out from a level of abstraction that is higher than having to assume the word for mother to be the first word.

Santilli detrivialized the choice of the unit, and invented isomathematics where the crux was the lifting of the conventional multiplicative unit (i.e. conservation of its topological properties) to a matrix isounit with additional arbitrary functional dependence on other needed variables. Then the conventional unit could be described as a projection and deformation from the isounit by the link provided by the so-called isotopic element inverse of the isounit. This represented the creation of a new branch of mathematics sophisticated and flexible enough to treat systems entailing sub-systems with different units, i.e. more complex systems of nature.

Isomathematics proved necessary for the lifting of quantum mechanics to hadronic mechanics. With this new mathematics it was possible to describe extended particles and abandon the point particle simplification of quantum mechanics. This proved highly successful in explaining the strong force by leaving behind the non-linear complexities involved in quantum mechanics struggle to describe the relation between the three baryon quarks in the proton. Isomathematics also provided the mathematical means to explain the neutron as a bound state of a proton and an electron as suggested by Rutherford. By means of isomathematics Santilli was also able to discover the fifth force of nature (in cooperation with Professor Animalu), the contact force inducing total overlap between the wave packets of the two touching electrons constituting the isoelectron. This was the key to understanding hadronic superconductivity which also can take place in fluids and gases, i.e. at really high temperatures. These advances from hadronic mechanics led to a corresponding lifting of quantum chemistry to hadronic chemistry and the discovery of the new chemical species of magnecules with non-valence bounds. Powerful industrial-ecological technology exploiting these theoretical insights was invented by Santilli himself from 1998 on.

Thus, the development of hadronic mathematics by Santilli was not only motivated by making advances in mathematics per se, but also of its potential to facilitate basic advances in physics and beyond. These advances have been shown to be highly successful already. Without the preceding advances in mathematics, the new hadronic technology would not have been around. The mere existence of this technology is sufficient to demonstrate the significance of hadronic mathematics. It is interesting to note that the directing of creative mathematics into this path was initiated by a mathematical physicist, not by a pure mathematician. In general this may indicate the particular potential for mathematical advances by relating the mathematics to unsolved basic problems in other disciplines, as well as to real life challenges.

In the history of mathematics it is not so easy to find parallels to the achievements made by Santilli, due to hadronic mathematics representing a radical and general lifting, relegating the previous mathematics to a subclass of isomathematics, in some analogy to taking the step from the Earth to the solar system. However, the universe also includes other solar systems as well as galaxies.

In addition to isonumbers Santilli invented the new and broader class of genonumbers with the possibility of asymmetric genounits for forward vs. backward genofields, and designed to describe and explain irreversibility, characteristic for more complex systems of nature. Quantum mechanical approaches to biological systems never achieved appreciable success, mainly due to being restricted by a basic symmetry and hence reversibility in connected mathematical axioms. It represented an outstanding achievement of theoretical biology when Chris Illert in the mid-1990s was able to find the universal algorithm for growth of sea shells by applying hadronic geometry. Such an achievement was argued not to be possible for more restricted hyperdimensional geometries as for example the Riemannian. This specialist study in conchology was the first striking illustration of the potency as well as necessity of iso- and genomathematics to explain irreversible systems in biology.

Following the lifting from isomathematics to genomathematics, Santilli also established one further lifting, by inventing the new and broader class of hyperstructural numbers or Santilli hypernumbers. Such hypernumbers are multivalued and suitable to describe and explain even more complex systems of nature than possible with genonumbers. Due to its irreversible multivalued structure hypermathematics seems highly promising for specialist advances in fields such as genetics, memetics and communication theory. By the lifting to hypermathematics hadronic mathematics as a whole may be interpreted as a remarkable step forward in the history of mathematics, in the sense of providing the essential and sufficiently advanced and adequate tools for mathematics to expand into disciplines such as anthropology, psychology and sociology. In this way it is possible to imagine some significant bridging between the two cultures of science: the hard and the soft disciplines, and thus amplifying a tendency already represented to some extent by complexity science.

The conventional view of natural scientists has been to regard mathematics as a convenient bag of tools to be applied for their specific purposes. Considering the architecture of hadronic mathematics, this appears more as only half of the truth or one side of the coin. Besides representing powerful new tools to study nature, hadronic mathematics also manifests with a more intimate and inherent connection to physics (and other disciplines), as well as to Nature itself. In this regard hadronic geometry may be of special interest as an illustration:

Isogeometry provided the new notions of a supra-Euclidean isospace as well as its anti-isomorphic isodual space, and the mathematics to describe projections and deformations of geometrical relations from isospace and its isodual into Euclidean space. However, these appear as more than mere mathematical constructs. Illert showed that the universal growth pattern of sea shells could be found only by looking for it as a trajectory in a hidden isospace, a trajectory which is projected into Euclidean space and thereby manifest as the deformed growth patterns humans observe by their senses. Further, the growth pattern of a certain class of sea shells (with bifurcations) could only be understood from the addition and recognition of four new, non-trivial time categories (predicted to be discovered by hadronic mechanics) which manifest as information jumps back and forth in Euclidean space. With regard to sea shell growth, one of this non-trivial time flows could only be explained as a projection from isodual spacetime. This result was consistent with the physics of hadronic mechanics, analyzing masses at both operator and classical level from considering matter and anti-matter (as well as positive and negative energy) to exist on an equal footing in our universe as a whole and hence with total mass (as well as energy and time) cancelling out as zero for the total universe. To establish a basic physical comprehension of Euclidean space constituted as a balanced combination of matter and antimatter, it was required to develop new mathematics with isonumbers and isodual numbers basically mirroring each other. Later, corresponding anti-isomorphies were achieved for genonumbers and hypernumbers with their respective isoduals.

Thus, there is a striking and intimate correspondence between the isodual architecture of hadronic mathematics and the isodual architecture of hadronic mechanics (as well as of hadronic chemistry and hadronic biology). Considering this, one might claim that the Santilli inventions of new number fields in mathematics represent more than mere inventions or constructs, namely discoveries and reconstructions of an ontological architecture being for real also outside the formal landscapes created by the imagination of mathematics and logic. This opens new horizons for treating profound issues in cosmology and ontology.

One might say that with the rise of hadronic mathematics the line between mathematics and other disciplines has turned more blurred or dotted. In some respect this represents a revisit to the Pythagorean and Platonic foundations of mathematics in the birth of western civilization. Hadronic mathematics has provided much new food for thought and further explorations for philosophers of science and mathematics.

If our civilization is to survive despite its current problems, it seems reasonable to expect Santilli to be honored in future history books not only as a giant in the general history of science, but also in the specific history of mathematics. Hadronic mathematics provided the necessary fuel for rising scientific revolutions in other hadronic sciences. This is mathematics that matters for the future of our world, and hopefully Santillis extraordinary contributions to mathematics will catch fire among talented and ambitious young mathematicians for further advances to be made. The present mellowed volume ought to serve as an excellent appetizer in this regard.

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