Associative matrix algebras with unit of a general form

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Abstract

We consider matrix Lie algebras and groups with associative and distributive matrix product, without assuming that the unit matrix has the standard form. The axiom of associative and distributive matrix product can be realized in different ways implying accordingly different forms of the unit matrix obeying the axiom of unit element. Such algebras were first studied by Santilli (Lie-Santilli algebras). We investigate general properties of such groups and algebras, and present examples which are of interest in physics.

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Contents

1 Introduction

It is wellknown that unit in $n \times n$ matrix algebras over field of complex numbers $\mathbb{C}$ is defined due to axiom of unit element. This axiom uses matrix product which is axiomatically defined as an associative and distributive one [1, 2]. Usually, the unit is taken as diagonal $n \times n$ matrix of the standard form $I = \mathrm{diag}(1, 1, \ldots, 1)$. However, this specific form of unit matrix does not follow directly from the axioms of matrix product and unit element.

In this paper, we consider matrix Lie algebras and groups with associative and distributive matrix product, without assuming that the unit matrix has the above standard form. The main idea is that the axiom of associative and distributive matrix product can be realized in different ways implying accordingly different forms of the unit matrix obeying the axiom of unit element. Such algebras were first studied and developed by Santilli [3].

In Sec. 2, we study admissible realizations of the axiom of matrix product. We conjecture that the most general form of the associative and distributive product of two $n \times n$ matrices M and N is of the form $M\hat{T}N$, where $\hat{T}$ is fixed $n \times n$ matrix, with the underlying product between M, $\hat{T}$, and N being the standard matrix one. This form of product in algebras was introduced and studied by Santilli [3], with $\hat{T}$ being called isotopic element. The axiom of unit element then implies that the unit matrix $\hat{I}$, corresponding to this product, is of the form $\hat{I} = \hat{T}^{-1}$, where the inverse matrix $\hat{T}^{-1}$ is defined due to $\hat{I}\hat{T} = \hat{T}\hat{I} = I$. In the particular case of $\hat{I} = \hat{T}$, the standard unit matrix and standard matrix product are recovered, $\hat{I} = I = \hat{T}$. In the general case, which is of primary interest in the present paper, $\hat{I} \neq \hat{T}$. We introduce definition of the dual matrix algebra, which is defined due to the interchange $\hat{I} \leftrightarrow \hat{T}$. Also, we consider metrics and associated coordinate systems in the matrix space. We study the conditions of reducing the unit matrix $\hat{I}$ to the standard one, and introduce the notion of generating matrix, which is of particular relevance for applications in physics.

Throughout the paper, we use notation $M \hat{\times} N = M\hat{T}N$ adopted from ref. [4], to denote an associative and distributive matrix product. Matrix algebra of $n \times n$ matrices over field $\mathbb{C}$ equipped with an associative and distributive matrix product is denoted as $\mathrm{M}(n, \mathbb{C}, \hat{\times})$.

In Sec. 3, we study some classical matrix Lie groups and Lie algebras with the general from of unit, and derive restrictions on the form of unit matrix $\hat{I}$ from definitions of the Lie algebras and associated Lie groups. By collecting all the derived restrictions we arrive at the conclusion that the most simple admissible nonstandard form of unit matrix appears to be of the form of a positive definite diagonal matrix, $\hat{I} = \mathrm{diag}(q_1, q_2, \ldots, q_n)$, $q_i > 0$, i=1,. . . ,n. We consider action of the Lie groups on classical linear spaces, $\mathbb{R}^n$ and $\mathbb{C}^n$, and indicate that the associated matrix Lie groups, in which matrix product is $M\hat{T}N$ and unit is $\hat{I}$, conserve metrics which are, in general, *not conformally equivalent* to the Euclidean one. Also, we briefly review the infinite dimensional case, and indicate the relevance of the unit $\hat{I}$ in mathematical and physical context.

In Sec. 4, we consider in detail $\mathrm{SO}(3, \mathbb{R}, \hat{\times})$, $\mathrm{SO}(2, \mathbb{R}, \hat{\times})$, $\mathrm{SO}(1, 1, \mathbb{R}, \hat{\times})$, and $\mathrm{U}(1, \mathbb{C}, \hat{\times})$ matrix Lie groups and $\mathrm{M}(2, \mathbb{C}, \hat{\times})$ matrix Lie algebra, as examples which are of interest in physics. To construct nontrivial realization of $\mathrm{SO}(3, \mathbb{R}, \hat{\times})$, we use properties of the dual algebra. We establish the relationship between $\mathrm{SO}(2, \mathbb{R}, \hat{\times})$ and $\mathrm{U}(1, \mathbb{C}, \hat{\times})$, and consider the action of $\mathrm{U}(1, \mathbb{C}, \hat{\times})$ on complex plane $\mathbb{C}$. It is remarkable to note that $\mathrm{U}(1, \mathbb{C}, \hat{\times})$ makes, in general, linear *non complex analytic* transformation of complex plane $\mathbb{C}$. This is in confirmation of emphasize made by Santilli [4] that theory with the nonstandard form of product (and unit) is related to the standard one by *non-unitary* transformation. We construct the realization map which connects $\mathrm{GL}(n, \mathbb{C}, \hat{\times})$ to $\mathrm{GL}(2n, \mathbb{R}, \hat{\times})$. One of the open problems is construction of nontrivial realization of the $\mathrm{SU}(2, \mathbb{C}, \hat{\times})$ Lie group, to which we give a detailed approach.

Detailed studies on associative algebras and groups with unit element different from the standard one, and its applications to physics, were made by Santilli [3, 4] since 1978; see also Sourlas and Tsagas [5]. We refer the reader to these papers for review and results of recent development of the Lie-Santilli algebras. The main motivating idea lying behind the present paper, as well as some of the results, are due to the recent study presented in ref. [4]. Particularly, there it has been emphasized that the physical theories which are based on various types of classical and operator *deformations* of the standard ones should be reformulated in order to provide their physical self-consistency and predictivity by preserving Lie character of the theories.

In the present paper, we develop the approach of ref. [4] giving self-consistent and detailed consideration of the matrix algebras, with some novel results being obtained. Particularly, we show that the associative and distributive product, $M \hat{\times} N$, has a unique representation $M \hat{\times} N = M\hat{T}N$, introduce and use the dual algebra $\mathrm{M}(n, \mathbb{C}, \hat{\times}^{-1})$, analyze the coordinate systems in the matrix space and homotopy class of unit, derive restrictions on the form of unit implied by self-consistent consideration of some classical Lie algebras, establish the relationship between $\mathrm{SO}(2,\mathbb{R},\hat{\times})$ and $\mathrm{U}(1,\mathbb{R},\hat{\times})$, and the realization map for the case $n>2$, and present nontrivial examples of some Lie groups and algebras with the nonstandard unit.

2 Matrix algebra $\mathbf{M}(n,\mathbb{C},\hat{\times})$

2.1 The $\hat{\times}$-product of matrices

The usual matrix algebra $\mathrm{M}(n,\mathbb{C})$ consisting of all $n\times n$ matrices over field of complex numbers $\mathbb{C}$ is a Lie algebra in respect to commutator

$$[M,N]=MN-NM,\quad M,N\in M(n,\mathbb{C}), \tag{2.1}$$

where underlying product is the usual product of matrices in the underlying associative algebra with *standard* unit $I=\mathrm{diag}(1,1,\ldots,1)$.

In the set of all $n\times n$ matrices over field of complex numbers, $\mathcal{M}(n,\mathbb{C})$, we define $\hat{\times}$-commutator

$$[M,N]_{\hat{\times}}=M\hat{\times}N-N\hat{\times}M,\quad M,N\in\mathcal{M}(n,\mathbb{C}), \tag{2.2}$$

in respect to the $\hat{\times}$-product

$$M\hat{\times}N=M\hat{T}N, \tag{2.3}$$

i.e.

$$(M\hat{\times}N)_{ij}=\sum_{k,l=1}^{n}M_{ik}\hat{T}^{kl}N_{lj}, \tag{2.4}$$

where

$$\hat{T}=\hat{I}^{-1},\quad \hat{T}\hat{I}=\hat{I}\hat{T}=I,\quad \hat{I}\in\mathcal{M}(n,\mathbb{C}), \tag{2.5}$$

Here, $\hat{I}$ is a *fixed* invertible matrix, which plays the role of (left and right) unit, instead of the usual unit matrix I, and we assume that in general $\hat{I}\neq\hat{T}$. Indeed, it can be immediately checked that $\hat{I}$ verifies axiom of the unit element,

$$M\hat{\times}\hat{I}=\hat{I}\hat{\times}M=M,\quad \forall M\in\mathcal{M}(n,\mathbb{C}). \tag{2.6}$$

We denote the set $\mathcal{M}(n, \mathbb{C})$ equipped by the unit $\hat{I}$ and the associated $\hat{\times}$-product as $\mathrm{M}(n, \mathbb{C}, \hat{\times})$. Evidently, $\mathrm{M}(n, \mathbb{C}, \hat{\times})$ is a linear space,

$$M + N = N + M, \tag{2.7}$$

$$M + (N + P) = (M + N) + P, \tag{2.8}$$

$$M + 0 = M, \tag{2.9}$$

$$M + (-M) = 0, \tag{2.10}$$

$$\alpha(\beta M) = (\alpha\beta)M, \tag{2.11}$$

$$\alpha(M + N) = \alpha M + \alpha N, \quad (\alpha + \beta)M = \alpha M + \beta M, \tag{2.12}$$

$$1 \cdot M = M, \tag{2.13}$$

where α and β are complex numbers, and the $\hat{\times}$-product is associative and distributive,

$$(M \hat{\times} N) \hat{\times} P = M \hat{\times} (N \hat{\times} P), \tag{2.14}$$

$$M \hat{\times} (N + P) = M \hat{\times} N + M \hat{\times} P, \quad (M + N) \hat{\times} P = M \hat{\times} P + N \hat{\times} P. \tag{2.15}$$

Inverse of matrix in the algebra $\mathrm{M}(n, \mathbb{C}, \hat{\times})$ is defined as

$$M^{-\hat{1}} \hat{\times} M = M \hat{\times} M^{-\hat{1}} = \hat{I}, \tag{2.16}$$

and multiplication of matrix by complex number α is as usual,

$$\alpha M = (\alpha m_{ij}). \tag{2.17}$$

This means that $\mathrm{M}(1, \mathbb{C}, \hat{\times})$ is assumed to be isomorphic to $\mathrm{M}(1, \mathbb{C})$. In one-dimensional case, the $\hat{\times}$-product is $\alpha \hat{\times} \beta = \alpha \hat{T} \beta = \hat{T} \alpha \beta$, where α, β, and $\hat{T}$ are complex numbers, so that we can ignore overall fixed non-zero factor $\hat{T}$ in all the products. Indeed, there is an isomorphism between $\mathbb{C}$ and $\hat{T}\mathbb{C}$ provided by dilation. Nontriviality comes in *higher*-dimensional cases; see Secs. 3.2.3 and 4.2 for details. We denote nth power of matrix in $\mathrm{M}(n, \mathbb{C}, \hat{\times})$ by

$$M^{\hat{n}} = M \hat{\times} M \hat{\times} \cdots \hat{\times} M, \qquad (n \text{ times}) \tag{2.18}$$

and define $M^{\hat{0}} = \hat{I}$.

Note that $\mathrm{M}(n, \mathbb{C}, \hat{\times})$ is a Lie algebra in respect to the $\hat{\times}$-commutator. Indeed, $\hat{\times}$-commutator (2.2) is skew-symmetric, and Jacobi identity,

$$[M, [N, P]_{\hat{\times}}]_{\hat{\times}} + [P, [M, N]_{\hat{\times}}]_{\hat{\times}} + [N, [P, M]_{\hat{\times}}]_{\hat{\times}} = 0, \tag{2.19}$$

in respect to $\hat{\times}$-commutator is satisfied. Namely,

$$[M,[N,P]_{\hat{\times}}]_{\hat{\times}} = M\hat{\times}N\hat{\times}P - M\hat{\times}P\hat{\times}N - N\hat{\times}P\hat{\times}M + P\hat{\times}N\hat{\times}M,$$
$$[P,[M,N]_{\hat{\times}}]_{\hat{\times}} = P\hat{\times}M\hat{\times}N - P\hat{\times}N\hat{\times}M - M\hat{\times}N\hat{\times}P + N\hat{\times}M\hat{\times}P, \quad (2.20)$$
$$[N,[P,M]_{\hat{\times}}]_{\hat{\times}} = N\hat{\times}P\hat{\times}M - N\hat{\times}M\hat{\times}P - P\hat{\times}M\hat{\times}N + M\hat{\times}P\hat{\times}N,$$

and by summing up these three expressions we obtain identically zero.

In definition of usual Lie algebras $sl(n)$, $o(n)$, and $u(n)$, and associated Lie groups one uses the following operations with matrices: Trace, Transpose (M^t), and Complex Conjugate ($\bar{M}$). All these operations, and also Det, concern matrix elements and their definitions in algebra $\mathrm{M}(n,\mathbb{C},\hat{\times})$ remain the same as in algebra $\mathrm{M}(n,\mathbb{C})$.

We emphasize that both algebras $\mathrm{M}(n,\mathbb{C})$ and $\mathrm{M}(n,\mathbb{C},\hat{\times})$ obey the *same set of axioms* by construction. So, the product (2.3) is one of the *admissible realizations* of abstract definition of the product in matrix algebra accompanied by associated realization of the unit element $\hat{I}$. This realization of product is based on the usual matrix product and can be thought of as the simplest generalization of it. However, note that $\mathrm{M}(n,\mathbb{C},\hat{\times})$ is not generalization of abstract matrix algebra. Instead, $\mathrm{M}(n,\mathbb{C})$ and $\mathrm{M}(n,\mathbb{C},\hat{\times})$ are two *different realizations* of the abstract matrix algebra, with $\mathrm{M}(n,\mathbb{C})$ being a *simplest realization* while $\mathrm{M}(n,\mathbb{C},\hat{\times})$ is an example of *more general realization* of it. Perhaps, some other admissible realizations of the matrix product exist. In general, this means that the axioms of matrix algebra do not fix the form of matrix product and the form of unit matrix to be only the standard ones. The aim of this paper is to investigate implications of the form (2.3) of matrix product assuming that the unit matrix $\hat{I}$ is a matrix of general form.

As we will demonstrate in next sections, some severe restrictions on the form of unit matrix $\hat{I}$ naturally arise. Counterparts to the classical Lie groups associated to $\mathrm{M}(n,\mathbb{C},\hat{\times})$ reveal interesting properties. For example, the associated orthogonal group conserves the metrics defined by $\hat{I}$, which is, in general, *not conformally equivalent* to Euclidean metrics $I=(\delta^i_k)$. Also, the associated unitary groups make, in general, linear *non complex analytic* transformations of complex space.

We conjecture that the product (2.3) is the *most general* realization of the abstract associative and distributive product in matrix algebra with unit. In Appendix A, we sketch the proof.

2.2 The dual algebra $\mathbf{M}(n, \mathbb{C}, \hat{\times}^{-1})$

We start by noting that the transformation

$$\rho_0 : \; X \mapsto \hat{I} X \hat{I}, \quad X = M, N, \tag{2.21}$$

converts the $\hat{\times}$-commutator

$$M\hat{T}N - N\hat{T}M \tag{2.22}$$

to the commutator in respect to $\hat{I}$,

$$M\hat{I}N - N\hat{I}M. \tag{2.23}$$

Indeed, we have

$$[\rho_0(M), \rho_0(N)]_{\hat{\times}} = \hat{I}M\hat{I}\hat{T}\hat{I}N\hat{I} - \hat{I}N\hat{I}\hat{T}\hat{I}M\hat{I} \tag{2.24}$$
$$= \hat{I}(M\hat{I}N - N\hat{I}M)\hat{I} = \rho_0(M\hat{I}N - N\hat{I}M) \neq \rho_0[M, N]_{\hat{\times}},$$

that means that this is not endomorphism of Lie algebra $\mathrm{M}(n, \mathbb{C}, \hat{\times})$. Vice versa, the transformation $\rho'_0 : \; X \mapsto \hat{T} X \hat{T}$ converts the commutator (2.23) to the $\hat{\times}$-commutator.

In general, one can construct some algebra by replacing

$$\hat{I} \leftrightarrow \hat{T}. \tag{2.25}$$

We call this algebra as a *dual* to $M(n, \mathbb{C}, \hat{\times})$, and denote it by $\mathrm{M}(n, \mathbb{C}, \hat{\times}^{-1})$. In the dual algebra, $\hat{T}$ is a unit matrix while $\hat{I}$ is used in the definition of $\hat{\times}$-product. Accordingly, we call commutator (2.23) as a *dual* commutator, which defines Lie algebra dual to the one specified by $\hat{\times}$-commutator (2.2).

In the case $\hat{I} = I$, the two algebras, $\mathrm{M}(n, \mathbb{C}, \hat{\times})$ and $\mathrm{M}(n, \mathbb{C}, \hat{\times}^{-1})$, degenerate to one algebra, $\mathrm{M}(n, \mathbb{C})$, and this is the only way to obey the *self-duality* condition, $\mathrm{M}(n, \mathbb{C}, \hat{\times}) \simeq M(n, \mathbb{C}, \hat{\times}^{-1})$, which simply means that $\hat{I} = \hat{T}$. Thus, the usual matrix algebra $\mathrm{M}(n, \mathbb{C})$ with standard form of unit is picked up by the self-duality condition. Indeed, $\hat{I} = \hat{T} \equiv \hat{I}^{-1}$ has only one non-trivial solution, $\hat{I} = \mathrm{diag}(1, 1, \ldots, 1)$, for the unit element.

Also, we note that the transformation

$$\rho_1 : \; X \mapsto \hat{I} X \hat{T} \equiv \hat{I} X \hat{I}^{-1} \tag{2.26}$$

is an inner automorphism of $\mathrm{M}(n, \mathbb{C}, \hat{\times})$ in terms of standard product. Indeed, this is a homomorphism,

$$\rho_1(M) \hat{\times} \rho_1(N) = (\hat{I} M \hat{T}) \hat{T} (\hat{I} N \hat{T}) = \hat{I}(M \hat{\times} N) \hat{T} = \rho_1(M \hat{\times} N), \tag{2.27}$$

maps $\hat{I}$ to $\hat{I}$, and makes one-to-one correspondence, with the inverse transformation

$$\rho_1^{-1}: \ X \mapsto \hat{T} X \hat{I}. \tag{2.28}$$

Transformations ρ_1 and ρ_1^{-1} make one-to-one correspondence between element X and its *conjugate*. It should be emphasized that $\hat{I}$ in Eq.(2.26) is fixed so that $\hat{I} X \hat{I}^{-1}$ do not, of course, form a class of conjugated elements for X. Instead, $\hat{I} X \hat{I}^{-1}$ form a *similarity* class ($\hat{I}$ is fixed, X is arbitrary).

Also, the transformation

$$\rho_2: \ X \mapsto S \hat{\times} X \hat{\times} S^{-\hat{1}} \quad X, S \in M(n, \mathbb{C}, \hat{\times}), \tag{2.29}$$

is endomorphism of $\mathrm{M}(n, \mathbb{C}, \hat{\times})$,

$$\begin{aligned} \rho_2(M) \hat{\times} \rho_2(N) &= (S \hat{\times} M \hat{\times} S^{-\hat{1}}) \hat{T} (S \hat{\times} N \hat{\times} S^{-\hat{1}}) \\ = S \hat{\times} (M \hat{T} S^{-\hat{1}} \hat{T} S \hat{T} N) \hat{\times} S^{-\hat{1}} &= S \hat{\times} (M \hat{\times} N) \hat{\times} S^{-\hat{1}} = \rho_2(M \hat{\times} N), \end{aligned} \tag{2.30}$$

where we have used the definition (2.16).

2.3 Metrics and coordinates of $\mathbf{M}(n, \mathbb{C}, \hat{\times})$

The $\hat{\times}$-product (2.3) is a smooth function (polynomial) of matrix elements of multipliers M and N. We introduce metrics in $\mathrm{M}(n, \mathbb{C}, \hat{\times})$ as follows,

$$|M|^2 = \sum_{i,j} |m^i_k| \hat{T}^k_l |m^l_j|. \tag{2.31}$$

Here, we have denoted $M = (m^i_k)$ and $\hat{T} = (\hat{T}^k_l)$, and naturally require $\hat{T}$ to be a matrix of *positive definite* form. In the standard case [6], $\hat{T} = I \equiv (\delta^k_l)$, the metrics (2.31) is Euclidean, and simply computed as sum of all squared matrix elements, $|m^i_k|^2$, giving us in the result some real number; for example, $|I|^2 = n$ in $\mathrm{M}(n, \mathbb{C})$.

Then, for the metrics (2.31) we have, evidently,

$$|M + N| \leq |M| + |N|, \tag{2.32}$$

and also it can be easily proved that

$$|M\hat{\times}N| \leq |M|\hat{\times}|N|. \tag{2.33}$$

Indeed, we have for the inner product $\langle , \rangle$ with positive definite form $\hat{T}$

$$\langle x, \hat{T}x\rangle\langle y, \hat{T}y\rangle - \langle x, \hat{T}y\rangle^2 = \frac{1}{2}(\langle x, \hat{T}y\rangle - \langle y, \hat{T}x\rangle), \tag{2.34}$$

from which (2.33) follows.

Let us introduce local coordinates in the space of matrices, in the *vicinity* of $\hat{I}$,

$$|M - \hat{I}| \leq 1. \tag{2.35}$$

Coordinate $x(M)$ of matrix M in $\mathrm{M}(n, \mathbb{C}, \hat{\times})$ is defined as

$$x^i_j(M) = m^i_j - \hat{I}^i_j, \quad x^i_j(\hat{I}) = 0. \tag{2.36}$$

If we multiply all the matrices, in the vicinity of $\hat{I}$, to $\hat{T} = \hat{I}^{-1}$ then we can introduce the following coordinate, $y(M)$, of matrix M:

$$y^i_j(M) = m^k_j \hat{T}^i_k - \delta^i_j, \quad y^i_j(\hat{I}) = 0, \tag{2.37}$$

which is coordinate of M in the vicinity of $I = (\delta^i_j)$. This coordinate system can be used for matrices M such that

$$|M - \hat{I}| \leq |\hat{I}|. \tag{2.38}$$

Thus, we have coordinate system $x(M)$ in the vicinity of $\hat{I}$, which is related to coordinate system $y(M)$ in the vicinity of I. The two coordinate systems coincide if $\hat{I} = I$.

In addition to the coordinate system $y(M)$, we can introduce the alternative one,

$$z^i_j(M) = \hat{T}^i_k m^k_j - \delta^i_j, \quad z^i_j(\hat{I}) = 0. \tag{2.39}$$

This coordinate system is not equivalent to $y(M)$ since in general $M\hat{T} \neq \hat{T}M$.

The following remarks are in order.

(a) The coordinate system $x(M)$ can be introduced in the vicinity of *any* matrix $\hat{I}$.

(b) The coordinate system $y(M)$ can be introduced for any *invertible* $\hat{I}$. The procedure of moving the vicinity, $x(M) \mapsto y(M)$, described above is formal and means that one can introduce coordinate system in the vicinity of any invertible element of the matrix space which is "equivalent" in some sense to the standard coordinate system in the vicinity of $I = (\delta^i_j)$.

(c) The choice of the center of coordinate system is a matter of convenience. Natural preference is made to the usual unit matrix $I = (\delta^i_j)$ as a center for the coordinate system. However, when one uses $\hat{I}$ as unit in the matrix algebra, as it is the case for $\mathrm{M}(n, \mathbb{C}, \hat{\times})$, it becomes natural to choose $\hat{I}$ as a center of coordinate system to have a consistent picture. However, even in this case one can move the vicinity of $\hat{I}$ to the vicinity of $I = (\delta^i_j)$ by using coordinate system $y(M)$ or $z(M)$ because $\hat{I}$ is an invertible matrix, and $\hat{T}$ in Eqs.(2.37) and (2.39) is always well defined. Note however that the center is still $\hat{I}$ due to second equation in (2.37).

(d) It is remarkable to note that the sizes of the vicinities are different; see Eqs.(2.35) and (2.38).

(e) Also, we emphasize here that while the center of coordinate system in, e.g., Euclidean space $\mathbb{R}^n$ is indeed of no importance in accordance to its homogeneity, the choice of the center in matrix spaces, which are not in general homogeneous and commutative, is of some importance. This is reflected partially by the fact that we have two-fold way to rich vicinity of standard I, namely, coordinate systems $y(M)$ and $z(M)$.

2.4 Homotopy class of unit

Let us specify the form of unit $\hat{I}$ by picking up diagonal form

$$\hat{I} = \mathrm{diag}(q_1, q_2, \ldots, q_n), \tag{2.40}$$

where parameters q_i satisfy the following conditions:

$$q_i \in \mathbb{R}, \quad q_i \neq 0 \ \ (i = 1, 2, \ldots, n), \quad \sum q_i \neq 0, \tag{2.41}$$

that is

$$\hat{I} \in M(n, \mathbb{R}), \quad \mathrm{Det}\, \hat{I} \neq 0, \quad \mathrm{Trace}\, \hat{I} \neq 0, \tag{2.42}$$

This specific form of $\hat{I}$ obeys the conditions (3.76) requiring that $\hat{I}$ should be real, symmetric, and non-traceless matrix, and the *diagonal* form of $\hat{I}$ appears to be important in universal definitions of algebras of pseudo-unitary and pseudo-orthogonal groups; see Sec. 3.3. Complete list of the requirements on the form of $\hat{I}$ is presented in Sec. 3.4.

The matrix $\hat{T}$ is then given by

$$\hat{T} = \mathrm{diag}(1/q_1, 1/q_2, \ldots, 1/q_n). \tag{2.43}$$

The norm of the unit $\hat{I}$ defined by (2.31) is

$$|\hat{I}| = \sqrt{\sum q_i} = \sqrt{\mathrm{Trace}\ \hat{I}}, \tag{2.44}$$

and not equal to zero due to (2.42). Then, to have real positive norm we must put

$$\mathrm{Trace}\ \hat{I} > 0. \tag{2.45}$$

Below, we restrict consideration on the unit $\hat{I}$ of the form (2.40), which defines *n-parametric family of algebras* $\mathrm{M}(n, \mathbb{C}, \hat{\times})$.

We note that in the case

$$\hat{I} \to I, \tag{2.46}$$

if such a limit exists, we recover the original algebra $\mathrm{M}(n, \mathbb{C})$. The limit does not always exists since $\hat{I}$ is a deformation of I by n real parameters $q_1, q_2, \ldots, q_n$, which can have both negative and positive values, whereas $\hat{I}$ should always be invertible by definition. So, $\hat{I}$ and I should be *homotopically equivalent*, i.e. it must exist a smooth path in space $\mathrm{M}(n, \mathbb{C})$ connecting $\hat{I}$ and I. This is possible if and only if q_i's are positive numbers,

$$q_i > 0, \quad i = 1, 2, \ldots, n. \tag{2.47}$$

Indeed, for negative value of some q_i, the path should go through the point $q_i = 0$, in which $\hat{I}$ is not an invertible matrix (Det $\hat{I} = 0$ at $q_i = 0$), and $\hat{T}$ blows up as $q_i \to 0$; see Eq.(2.43). Also, condition (2.47) follows from the requirement that $\hat{I}$ must be positive definite matrix; see Eq.(2.31).

So, in general we must restrict consideration to the homotopy class of matrices to which standard unit matrix I belongs. Thus, the inverse map $I \mapsto \hat{I}$ is a smooth deformation, along with n-parametric path in space $\mathrm{M}(n, \mathbb{C})$. The parameterization is given simply by diagonal matrix $W = \mathrm{diag}(w_1, w_2, \ldots, w_n)$, with the parameters w_i running from 1 to q_i.

In view of the conditions (2.41) and (2.47), we can represent the matrix $\hat{I}$ as follows:

$$\hat{I} = \mathrm{diag}(1 + r_1, 1 + r_2, \ldots, 1 + r_n) \equiv I + R, \quad r_i > -1, \tag{2.48}$$

where $R = \mathrm{diag}(r_1, r_2, \ldots, r_n)$.

Whereas I commute with any element of $\mathrm{M}(n, \mathbb{C})$, $[I, M] = 0$, one observes that $\hat{I}$ does not in general commute with any element of algebra $\mathrm{M}(n, \mathbb{C})$,

$$[\hat{I}, M] \neq 0, \tag{2.49}$$

that is, $\hat{I}$ is not in the center of $\mathrm{M}(n, \mathbb{C})$. However, it is in the center of the algebra $\mathrm{M}(n, \mathbb{C}, \hat{\times})$,

$$[\hat{I}, M]_{\hat{\times}} = 0, \quad \forall M \in M(n, \mathbb{C}, \hat{\times}). \tag{2.50}$$

To see more details on the connection between I and $\hat{I}$ and to provide an example, let us consider usual $\mathrm{M}(2, \mathbb{C})$ case,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.51}$$

and the matrix $\hat{I} \in M(2, \mathbb{C})$ of the form

$$\hat{I} = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}, \tag{2.52}$$

where

$$q_1 \neq 1, \quad q_2 \neq 1, \quad q_1 \neq q_2. \tag{2.53}$$

We emphasize here that I is unit matrix, and matrix product is an ordinary one; $\hat{I}$ is not unit matrix in $\mathrm{M}(2, \mathbb{C})$.

We observe that both matrices are *Hermitean*,

$$I^\dagger \equiv \bar{I}^t = I, \quad \hat{I}^\dagger = \hat{I}, \tag{2.54}$$

and therefore *normal*, i.e.,

$$I^\dagger I = I I^\dagger, \quad \hat{I}^\dagger \hat{I} = \hat{I} \hat{I}^\dagger, \tag{2.55}$$

and therefore they are *simple*, i.e. multiplicity of each eigenvalue of the matrices is equal to its geometrical multiplicity.

While I is *unitary* matrix in $\mathrm{M}(2, \mathbb{C})$,

$$I^\dagger I = I, \tag{2.56}$$

the matrix $\hat{I}$ is *not unitary*,

$$\hat{I}^{\dagger}\hat{I} = \hat{I}\hat{I} = \mathrm{diag}(q_1^2, q_2^2) \neq I. \tag{2.57}$$

It is remarkable to note, however, that in the algebra $\mathrm{M}(2, \mathbb{C}, \hat{\times})$ we have

$$\hat{I}^{\dagger}\hat{\times}\hat{I} = \hat{I}\hat{\times}\hat{I} = \hat{I}, \tag{2.58}$$

so that $\hat{I}$ is unitary matrix in $\mathrm{M}(2, \mathbb{C}, \hat{\times})$ while I is not unitary in $\mathrm{M}(2, \mathbb{C}, \hat{\times})$.

Matrices I and $\hat{I}$ have different spectra and therefore they are not *unitary similar* to each other, i.e.,

$$\hat{I} \neq U^{-1}IU, \tag{2.59}$$

for any unitary matrix $U \in M(2, \mathbb{C})$. Moreover, they are not even simply *similar* to each other, i.e.,

$$\hat{I} \neq S^{-1}IS, \tag{2.60}$$

for any matrix $S \in M(2, \mathbb{C})$. Indeed, $S^{-1}IS = S^{-1}S = I$ by definition, and thus this can not be equal to $\hat{I}$. So, vice versa,

$$I \neq V^{-1}\hat{I}V, \tag{2.61}$$

for any $V \in M(2, \mathbb{C})$ (see Appendix B for strong proof).

The same properties are valid for higher dimensional cases, $n > 2$.

Let us consider ordinary eigenavlue problem,

$$(qI - Q)x = 0, \tag{2.62}$$

where $Q \in M(n, \mathbb{C})$. In the case Q is positive definite Hermitean matrix, we have set of positive real eigenvalues, q_1, q_2, $\ldots$, q_n, so that we can rewrite the above equation as

$$(\hat{I} - Q)x = 0, \tag{2.63}$$

where $\hat{I} = \mathrm{diag}(q_1, q_2, \ldots, q_n)$, $q_i > 0$. This means that Q and $\hat{I}$ are unitary similar to each other, $Q = U\hat{I}U^{\dagger}$, and $\hat{I}$ is positive definite Hermitean matrix. We note that it is exactly the matrix we have as a general form of unit in $\mathrm{M}(n, \mathbb{C}, \hat{\times})$; see also Sec. 3.4. Particularly, scalar matrix, $\hat{I} = \lambda I$, corresponds to fully degenerate spectrum of Q, in which case Q has necessarily the form $Q = \lambda I$.

We call positive definite Hermitean matrix Q satisfying Eq.(2.63) as a *generating matrix* for unit $\hat{I}$, $\hat{I} \in M(n, \mathbb{C}, \hat{\times})$. Clearly, all generating matrices for a given fixed $\hat{I}$ are unitary similar to each other, and they are not necessarily of a diagonal form.

If we relax the condition of diagonality of unit $\hat{I}$ (see Sec. 3.4) we can take generating matrix Q as a unit in algebra $\mathrm{M}(n, \mathbb{C}, \hat{\times})$ provided that in some basis, unitary related to the original one, Q has a diagonal form $\hat{I}$.

3 Lie groups and Lie algebras

3.1 Lie groups

3.1.1 Group GL$(n, \mathbb{C}, \hat{\times})$

We denote subgroup of $\mathrm{M}(n, \mathbb{C}, \hat{\times})$ consisting of matrices obeying the condition

$$\text{Det } M \neq 0, \quad M \in M(n, \mathbb{C}, \hat{\times}), \tag{3.1}$$

as $\mathrm{GL}(n, \mathbb{C}, \hat{\times})$.

3.1.2 Unitary group U$(n, \mathbb{C}, \hat{\times})$

The group $\mathrm{U}(n, \mathbb{C}, \hat{\times})$ is a subgroup of $\mathrm{GL}(n, \mathbb{C}, \hat{\times})$ defined by the following unitarity condition:

$$U \hat{\times} \hat{I} \hat{\times} U^\dagger \equiv U \hat{\times} U^\dagger \equiv U \hat{T} U^\dagger = \hat{I}, \tag{3.2}$$

where we have denoted for Hermitean conjugation

$$U^\dagger \equiv \bar{U}^t. \tag{3.3}$$

From the unitarity condition (3.2), it follows that

$$\begin{aligned} \text{Det } (U \hat{\times} U^\dagger) \equiv \text{Det } (U \hat{T} U^\dagger) &= (\text{Det } \hat{T})(\text{Det } U)(\overline{\text{Det}}\ U) \\ &= (\text{Det } \hat{T})|\text{Det } U|^2 = \text{Det } \hat{I}, \end{aligned} \tag{3.4}$$

that is,

$$|\text{Det } U|^2 = (\text{Det } \hat{I})^2, \quad U \in U(n, \mathbb{C}, \hat{\times}). \tag{3.5}$$

Apart from the usual case, determinant of unitary matrices in $\mathrm{M}(n, \mathbb{C}, \hat{\times})$ is not, in general, equal to ± 1.

We define the subgroup $\mathrm{SU}(n, \mathbb{C}, \hat{\times})$ by the condition

$$|\mathrm{Det}\ U| = \mathrm{Det}\ \hat{I}, \quad U \in SU(n, \mathbb{C}, \hat{\times}), \tag{3.6}$$

i.e., determinant of special unitary matrices in $\mathrm{M}(n, \mathbb{C}, \hat{\times})$ is equal to Det $\hat{I}$.

3.1.3 Orthogonal group $\mathbf{O}(n, \mathbb{R}, \hat{\times})$

Accordingly, for orthogonal group $\mathrm{O}(n, \mathbb{R}, \hat{\times})$ we have

$$O \hat{\times} \hat{I} \hat{\times} O^t \equiv O \hat{\times} O^t \equiv O \hat{T} O^t = \hat{I}, \quad O \in O(n, \mathbb{R}, \hat{\times}), \tag{3.7}$$

and for $\mathrm{SO}(n, \mathbb{R}, \hat{\times})$ we have additionally,

$$\mathrm{Det}\ O = \mathrm{Det}\ \hat{I}. \tag{3.8}$$

3.1.4 Group $\mathbf{SL}(n, \mathbb{C}, \hat{\times})$

The subgroup $\mathrm{SL}(n, \mathbb{C}, \hat{\times})$ of $\mathrm{GL}(n, \mathbb{C}, \hat{\times})$ is defined due to condition

$$\mathrm{Det}\ M = \mathrm{Det}\ \hat{I}, \quad M \in SL(n, \mathbb{C}, \hat{\times}). \tag{3.9}$$

Note that for the usual case, $\hat{I} = I$, we have Det $\hat{I} = 1$ while for $\hat{I}$ of the form (2.40) we have

$$\mathrm{Det}\ \hat{I} = q_1 q_2 \cdots q_n. \tag{3.10}$$

3.2 Action of the groups on classical linear spaces

3.2.1 Eigenvalue problem

We define natural (left) action of the group $\mathrm{GL}(n, \mathbb{C}, \hat{\times})$ on complex space $\mathbb{C}^n$ as

$$z \mapsto M \hat{\times} z = M \hat{T} z, \quad M \in GL(n, \mathbb{C}, \hat{\times}), \quad z \in \mathbb{C}^n. \tag{3.11}$$

This definition is consistent with the action of unit $\hat{I}$,

$$\hat{I} \hat{\times} z = z, \tag{3.12}$$

which we identify with identity transformation of $\mathbb{C}^n$. Also, for two consequential actions, we have

$$z' = M\hat{\times}z, \quad z'' = N\hat{\times}z' = N\hat{\times}(M\hat{\times}z) = (N\hat{\times}M)\hat{\times}z = Q\hat{\times}z, \qquad M, N, Q \in GL(n, \mathbb{C}, \hat{\times}), \tag{3.13}$$

that means that this action is consistent with the algebra $\mathrm{GL}(n, \mathbb{C}, \hat{\times})$.

The eigenvalue problem is then defined by the following equation:

$$M\hat{\times}z = \lambda z, \tag{3.14}$$

where $\lambda \in \mathbb{C}$, or, equivalently,

$$(\lambda\hat{I} - M)\hat{\times}z = 0. \tag{3.15}$$

This equation can be identically rewritten as

$$(\lambda I - M\hat{T})z = 0. \tag{3.16}$$

Thus, in the algebra $\mathrm{M}(n, \mathbb{C}, \hat{\times})$ the characteristic polynomial of the matrix M is

$$c(\lambda) = \mathrm{Det}\ (\lambda I - M\hat{T}). \tag{3.17}$$

3.2.2 Action of unitary group

Let us identify (Hermitean) scalar product in $\mathbb{C}^n$,

$$\langle z_1, z_2\rangle_{\mathbb{C}} = \sum_{i,j=1}^{n} z_1^i g_{ij} \bar{z}_2^j, \tag{3.18}$$

where $z_{1,2} \in \mathbb{C}^n$, such that it is conserved by unitary matrix $U \in U(n, \mathbb{C}, \hat{\times})$,

$$\langle U\hat{\times}z_1, U\hat{\times}z_2\rangle = \langle z_1, z_2\rangle. \tag{3.19}$$

We have from this equation

$$\sum U_k^i \hat{T}_m^k z^m g_{ij} \bar{z}^n \hat{T}_n^l \bar{U}_l^j = \sum z^m g_{mn} \bar{z}^n, \tag{3.20}$$

i.e.

$$U_k^i \hat{T}_m^k g_{ij} \hat{T}_n^l \bar{U}_l^j = g_{mn}, \tag{3.21}$$

or

$$U \hat{\times} g \hat{\times} U^{\dagger} = g. \tag{3.22}$$

So, we put $g = \hat{I}$ to achieve consistency with the definition (3.2). Obviously, this does not mean that $\hat{I}$ is conserved by the unitary matrix in the sense $U\hat{I}U^{\dagger} = \hat{I}$. Instead, the matrix

$$\tilde{U} = U\hat{T}, \tag{3.23}$$

with $\tilde{U}^{\dagger} = \hat{T}U^{\dagger}$, plays such a role, namely, we have from Eq.(3.22)

$$\tilde{U}\hat{I}\tilde{U}^{\dagger} = \hat{I}, \tag{3.24}$$

which is consistent with the definition (3.2).

Note that the scalar product (3.18), with $g = \hat{I}$, is Hermitean since $\hat{I}$ is real symmetric positive definite matrix (Hermitean, $\hat{I}^{\dagger} = I$).

3.2.3 Action of orthogonal group

Similarly, group $\mathrm{O}(n, \mathbb{C}, \hat{\times})$ conserves the following scalar product:

$$\langle x_1, x_2 \rangle_{\mathbb{R}} = \sum_{i,j=1}^{n} \hat{I}_{ij} x_1^i x_2^j, \tag{3.25}$$

where $x_{1,2} \in \mathbb{R}^n$, or explicitly,

$$\langle x_1, x_2 \rangle_{\mathbb{R}} = q_1 x_1^1 x_2^1 + q_2 x_1^2 x_2^2 + \cdots + q_n x_1^n x_2^n. \tag{3.26}$$

This scalar product defines Euclidean space $\mathbb{R}^n$ having the metrics $\hat{I}$,

$$\langle x, x \rangle = \sum_{i=1}^{n} q_i (x^i)^2, \tag{3.27}$$

i.e. metric tensor is

$$g_{ij} = q_i \delta_{ij}. \tag{3.28}$$

We denote Euclidean space $\mathbb{R}^n$ equipped by the metrics (3.28) by $\mathbb{R}^n_q$. The matrix conserving the above scalar product is

$$\tilde{O} = O\hat{T}, \tag{3.29}$$

i.e.

$$\tilde{O}\hat{I}\tilde{O}^t = \hat{I}. \tag{3.30}$$

Note that

$$\langle z, z\rangle_{\mathbb{C}} = \langle x, x\rangle_{\mathbb{R}}, \tag{3.31}$$

as in the usual case.

In terms of the algebra $\mathrm{M}(n, \mathbb{C}, \hat{\times})$, the above unitarity and orthogonality definitions mean that the matrix $\hat{I}$ is an invariant. In terms of the usual algebra $\mathrm{M}(n, \mathbb{C})$ and geometry, they mean that the metric tensor $\hat{T}$ is *transformed* to metric tensor $\hat{I}$; see Eqs.(3.2) and (3.7). In the limiting case $\hat{I} = I = \hat{T}$, we have *conservation* of metric tensor δ_{ij}. This situation can be readily understood in terms of duality property (2.25) of algebra $\mathrm{M}(n, \mathbb{C}, \hat{\times})$. Namely, the definitions relate *dual spaces*, the one equipped by the metric (3.28) and the other equipped by the metric $\hat{T}$,

$$g_{ij}^{dual} = \frac{1}{q_i}\delta_{ij}, \tag{3.32}$$

which are simply inverse to each other, and coincide when $\hat{I} = I = \hat{T}$.

Note that the above space $\mathbb{R}^n_q$ with metrics (3.27) can not be obtained from Euclidean space $\mathbb{R}^n$ by dilation, $x \mapsto \lambda x$, except for one-dimensional case. Instead we have transformation

$$x^i \mapsto \hat{x}^i = x^i/\sqrt{q_i}, \tag{3.33}$$

which we call *inhomogeneous dilation*, $\hat{x} \in \mathbb{R}^n_q$. So the map $I \mapsto \hat{I}$ does not correspond in general to any linear conformal transformation of $\mathbb{R}^n$. Only when $q_1 = q_2 = \cdots = q_n$ this is the case. The transformation (3.33) can be thought of as that it gives the coordinates x^i different *weights*.

Accordingly, using of the general form (2.40) of unit $\hat{I}$ assumes, in general, *different weights* of the coordinates in contrast to *equal weights* provided by the standard unit I.

The equations $g_{ij}x^ix^j = (\mathrm{Det}\ \hat{I})^2$ and $g_{ij}^{dual}x^ix^j = (\mathrm{Det}\ \hat{T})^2$ define *fundamental ellipsoids*,

$$q_1(x^1)^2 + q_2(x^2)^2 + \cdots + q_n(x^n)^2 = (q_1q_2\cdots q_n)^2, \tag{3.34}$$

$$\frac{1}{q_1}(x^1)^2 + \frac{1}{q_2}(x^2)^2 + \cdots + \frac{1}{q_n}(x^n)^2 = \frac{1}{(q_1q_2\cdots q_n)^2}, \tag{3.35}$$

corresponding to $\hat{I}$ and $\hat{T}$, respectively. They are regular $(n-1)$-hypersurfaces in $\mathbb{R}^n$. In usual terms, definitions of the unitarity and orthogonality are such that corresponding matrices transform the second ellipsoid to the first one. The sphere $\sum(x^i)^2 = 1$ lies between the ellipsoids, and is a limiting case of both the ellipsoids.

The following remarks are in order.

(a) In usual geometrical terms, these ellipsoids are not conserving under the orthogonality group $\mathrm{O}(n, \mathbb{C}, \hat{\times})$. So, none of which is a homogeneous space of this group, and group $\mathrm{O}(n, \mathbb{C}, \hat{\times})$ does not act on it transitively in a usual sense. Indeed, varying matrix O in Eq.(3.7), we observe that they act on $\hat{T}\hat{I}\hat{T} = \hat{T}$, which is a fixed matrix, and the result is another fixed matrix $\hat{I}$. However, in terms of the group $\mathrm{O}(n, \mathbb{C}, \hat{\times})$, the ellipsoid (3.34) defined by $\hat{I}$ is *conserved* due to Eq.(3.7). So this ellipsoid is a homogeneous space of group $\mathrm{O}(n, \mathbb{C}, \hat{\times})$ under the action of this group on it, and every two points of the ellipsoid can be connected by some $O \in O(n, \mathbb{C}, \hat{\times})$ (transitivity).

(b) Also, we see from the above considerations that the matrix of the form

$$\tilde{M} = M\hat{T} \tag{3.36}$$

is of frequent use. Note that according to Eq.(2.49), we have in general $M\hat{T} \neq \hat{T}M$. We shall see in Sec. 3.3 that matrices of the form $M\hat{T}$ is also of use in the Lie algebras. From Eq.(2.37), we see that such matrices correspond to those described in the vicinity of standard unit $I = (\delta_{ij})$.

(c) Changing of the definitions of unitarity (3.2) and orthogonality (3.7) to

$$U^{\dagger}\hat{T}U = \hat{I}, \quad O^t\hat{T}O = \hat{I}, \tag{3.37}$$

respectively, yields the same set up as above, with the matrix $M\hat{T}$ replaced by $\hat{T}M$. Note that this corresponds to choosing of the coordinate system (2.39) instead of (2.37).

3.2.4 Action of pseudo-unitary and pseudo-orthogonal groups

Definitions given in Sec. 3.1 can be extended to the case of pseudo-Euclidean spaces, with accordingly defined pseudo-unitary group $\mathrm{U}(m, k, \mathbb{C}, \hat{\times})$ and pseudo-orthogonal group $\mathrm{O}(m, k, \mathbb{R}, \hat{\times})$.

Let us define the metrics

$$\hat{G}_r = G\hat{I}, \tag{3.38}$$

where

$$G = \mathrm{diag}(1, 1, \ldots, 1, -1, -1, \ldots, -1) \tag{3.39}$$

is metrics of pseudo-Euclidean space $\mathbb{R}^{m,k}$. Then, definitions of pseudo-unitary group $U_r(m, k, \mathbb{C}, \hat{\times})$ and pseudo-orthogonal group $O_r(m, k, \mathbb{R}, \hat{\times})$ are, respectively,

$$U \hat{\times} \hat{G}_r \hat{\times} U^\dagger = \hat{G}_r, \quad O \hat{\times} \hat{G}_r \hat{\times} O^t = \hat{G}_r. \tag{3.40}$$

These groups conserve the metrics $\hat{G}_r$. The other possible definition of metrics,

$$\hat{G}_l = \hat{I} G, \tag{3.41}$$

leads to the definitions of other groups, $U_l(m, k, \mathbb{C}, \hat{\times})$ and $O_l(m, k, \mathbb{R}, \hat{\times})$,

$$U \hat{\times} \hat{G}_l \hat{\times} U^\dagger = \hat{G}_l, \quad O \hat{\times} \hat{G}_l \hat{\times} O^t = \hat{G}_l, \tag{3.42}$$

since in general $G\hat{I} \neq \hat{I}G$. Also, note that

$$[\hat{G}_r, \hat{G}_l] = G\hat{I}\hat{I}G - \hat{I}GG\hat{I} \neq 0 \tag{3.43}$$

and

$$[\hat{G}_r, \hat{G}_l]_{\hat{\times}} = G\hat{I}G - \hat{I}G\hat{T}G\hat{I} \neq 0. \tag{3.44}$$

The groups $U_l(m, k, \mathbb{C}, \hat{\times})$ and $O_l(m, k, \mathbb{R}, \hat{\times})$ conserve the metrics $\hat{G}_l$. Evidently, these definitions of the groups are directly equivalent to that in respect to (3.38) if and only if pseudo-Minkowskian metrics G and unit $\hat{I}$ commute, $G\hat{I} = \hat{I}G$, so that $\hat{G}_r$ and $\hat{G}_l$ coincide,

$$\hat{G}_r = \hat{G}_l = \hat{G}, \tag{3.45}$$

and we can put

$$U \hat{\times} \hat{G} \hat{\times} U^\dagger = \hat{G}, \quad O \hat{\times} \hat{G} \hat{\times} O^t = \hat{G}, \tag{3.46}$$

for definitions of pseudo-unitary group $\mathrm{U}(m, k, \mathbb{C}, \hat{\times})$ and pseudo-orthogonal group $\mathrm{O}(m, k, \mathbb{R}, \hat{\times})$, respectively. This is the case only for *diagonal* form of the unit $\hat{I}$ because in general only diagonal matrices commute with pseudo-Minkowskian metrics G.

Note that due to the inner automorphism (2.26), $\rho_1 : \hat{G}_r \mapsto \hat{G}_l$, namely, $\hat{G}_l = \hat{I}\hat{G}_r\hat{T}$, so that $\hat{G}_r$ and $\hat{G}_l$ are elements conjugated to each other in $\mathrm{GL}(n, \mathbb{C}, \hat{\times})$, and the groups $U_r(m, k, \mathbb{C}, \hat{\times})$ ($O_r(m, k, \mathbb{R}, \hat{\times})$) and $U_l(m, k, \mathbb{C}, \hat{\times})$ ($O_l(m, k, \mathbb{R}, \hat{\times})$) are conjugated to each other, as subgroups of $\mathrm{GL}(n, \mathbb{C}, \hat{\times})$.

3.3 Matrix exponent and Lie algebras

Tangent spaces in the vicinity of the unit $\hat{I}$ for the above groups are corresponding Lie algebras, $gl(n, \mathbb{C}, \hat{\times})$, $sl(n, \mathbb{C}, \hat{\times})$, $u(n, \mathbb{C}, \hat{\times})$, and $o(n, \mathbb{R}, \hat{\times})$, which are well defined Lie algebras, as in the usual case.

The map from the tangent spaces to the groups is achieved by matrix exponent. The matrix exponent is defined, as usually, due to its formal series expansion. In $\mathrm{M}(n, \mathbb{C}, \hat{\times})$, we define

$$\hat{e}^M = \sum_{n=0}^{\infty} \frac{M^{\hat{n}}}{n!}, \tag{3.47}$$

where the $\hat{n}$-power of matrix M is defined in accord to Eq.(2.18), and we put

$$\hat{e}^0 = \hat{I}. \tag{3.48}$$

Explicitly,

$$\hat{e}^M = \hat{I} + M + \frac{1}{2!} M \hat{\times} M + \cdots . \tag{3.49}$$

This series expansion converges due to Eqs.(2.32) and (2.33). Then, one can easily prove using Eqs. (3.47) and (3.48) that

$$\hat{e}^{M+N} = \hat{e}^M \hat{\times} \hat{e}^N, \quad \text{for } \hat{\times}\text{-commuting matrices } M \text{ and } N, \tag{3.50}$$

$$\text{If } M = \hat{e}^X, \text{ then exists } \ M^{-1} = \hat{e}^{-X}, \tag{3.51}$$

$$\hat{e}^{X^t} = (\hat{e}^X)^t. \tag{3.52}$$

The above definition of matrix exponent defines local coordinates in the tangent space of group in the vicinity of unit element $\hat{I}$ of the group which have the following explicit form:

$$x^i_j(M) = (\ln M)^i_j = (M - \hat{I})^i_j - \cdots , \tag{3.53}$$

where M is a group element. This map is one-to-one correspondence in some vicinity of the point $x^i_j(M) = 0$.

The matrix exponent in $\mathrm{M}(n, \mathbb{C}, \hat{\times})$ is simply related to the usual matrix exponent by

$$\hat{e}^M = \hat{I} e^{\hat{T} M}, \quad M \in GL(n, \mathbb{C}, \hat{\times}), \tag{3.54}$$

with $e^0 = I$. Indeed, by using power expansion we have

$$\hat{e}^M = \hat{I} + M + \frac{1}{2!} M\hat{T}M + \cdots = \hat{I}(I + \hat{T}M + \frac{1}{2!}\hat{T}M\hat{T}M + \cdots) = \hat{I}e^{\hat{T}M}. \quad (3.55)$$

The following remarks are in order. In fact, we need only in the vicinity of unit $\hat{I}$ when dealing with Lie algebras. In general, matrix exponent is not one-to-one correspondence when it is extended to the whole group; well known example is usual $\mathrm{SL}(2, \mathbb{R})$.

Note that there is an alternative relation,

$$\hat{e}^M = e^{M\hat{T}}\hat{I}, \quad (3.56)$$

between the matrix exponents. This relation is equivalent to (3.55), in the algebra $\mathrm{M}(n, \mathbb{C}, \hat{\times})$. Indeed, let us check that the r.h.s. of (3.54) $\hat{\times}$-commute with the r.h.s. of (3.56), in the vicinity of unit,

$$\begin{aligned} [e^{M\hat{T}}\hat{I}, \hat{I}e^{\hat{T}M}]_{\hat{\times}} &= e^{M\hat{T}}\hat{I}e^{\hat{T}M} - \hat{I}e^{\hat{T}M}\hat{T}e^{M\hat{T}}\hat{I} \quad (3.57) \\ &\simeq (I + M\hat{T})\hat{I}(I + \hat{T}M) - \hat{I}(I + \hat{T}M)\hat{T}(I + M\hat{T})\hat{I} \\ &= (I + 2M + M\hat{T}M) - (I + 2M + M\hat{T}M) = 0, \end{aligned}$$

where we have dropped higher order terms.

Using the matrix exponent (3.47) we can prove that if $M \in SL(n, \mathbb{C}, \hat{\times})$, i.e. Det $M =$ Det $\hat{I}$, then algebra $sl(n, \mathbb{C}, \hat{\times})$, as a tangent space of the group in the vicinity of unit $\hat{I}$, consists of matrices X such that

$$\mathrm{Trace}\ X\hat{T} = 0, \quad (3.58)$$

and vice versa.

Indeed, let Trace $X\hat{T} = 0$. For $\mathrm{M}(t) = \hat{e}^{tX}$ we have

$$M(t_1 + t_2) = M(t_1)\hat{\times}M(t_2), \quad (3.59)$$

where we used the fact that $\mathrm{M}(t_1) = \hat{e}^{t_1 X}$ and $\mathrm{M}(t_2) = \hat{e}^{t_2 X}$ $\hat{\times}$-commute. Therefore,

$$\mathrm{Det}\ M(t_1 + t_2) = \mathrm{Det}\ M(t_1)\mathrm{Det}\ \hat{T}\mathrm{Det}\ M(t_2). \quad (3.60)$$

Solution for this equation is given by $F(t) \equiv \mathrm{Det}\ M(t) = c_1 e^{c_2 t}$, where $c_{1,2}$ are constants. Evidently, $c_1 = (\mathrm{Det}\ \hat{T})^{-1} = \mathrm{Det}\ \hat{I}$. On the other hand,

$$\begin{aligned} F(t) &= \mathrm{Det}\ \hat{e}^{tX} = \mathrm{Det}\ (\hat{I} + tX + o(t)) = \mathrm{Det}\ (I + t\hat{T}X + o(t))\hat{I} \quad (3.61) \\ &= (t\ \mathrm{Trace}\ X\hat{T} + o(t))(\mathrm{Det}\ \hat{I}). \end{aligned}$$

So, if Trace $X\hat{T} = 0$ then

$$c_2 = \frac{1}{c_1}\frac{dF}{dt}_{|t=0} = \text{Trace } X\hat{T} = 0. \tag{3.62}$$

Thus we have, finally, $F(t) = \text{Det } \hat{I}$, i.e. $\text{Det } M = \text{Det } \hat{I}$. It can be easily shown that, vice versa, if $\text{Det } M = \text{Det } \hat{I}$ then Trace $X\hat{T} = 0$.

Comparing this to the usual relation, Trace $X = 0$, for $sl(n, \mathbb{C})$ we see some modification. Let us denote

$$\hat{\text{Trace}}\ M = \text{Trace } M\hat{T}. \tag{3.63}$$

One can verify that Trace M is not conserved under unitary transformation while $\hat{\text{Trace}}$ M does. Indeed, let us make unitary transformation,

$$M \mapsto M' = U\hat{\times}M\hat{\times}U^\dagger, \tag{3.64}$$

where unitary matrix U is given by (3.2). Then,

$$\hat{\text{Trace}}\ M' = \sum M'_{ij}\hat{T}^{ji} = \sum(U_{ik}\hat{T}^{km}M_{mn}\hat{T}^{nl}U^\dagger_{lj})\hat{T}^{ji} \tag{3.65}$$
$$= \sum \hat{I}_{kl}\hat{T}^{km}M_{mn}\hat{T}^{nl} = \sum \delta^m_l M_{mn}\hat{T}^{nl} = \sum M_{mn}\hat{T}^{nm} = \hat{\text{Trace}}\ M,$$

where we used the unitarity condition $\sum U^\dagger_{lj}\hat{T}^{ji}U_{ik} = \hat{I}_{lk}$, and the fact that $\sum \hat{I}_{kl}\hat{T}^{km} = \sum \hat{I}_{lk}\hat{T}^{km} = \delta^m_l$.

Below, we investigate explicitly relations between the groups $\text{U}(n, \mathbb{C}, \hat{\times})$, $\text{O}(n, \mathbb{R}, \hat{\times})$ and their tangent spaces.

Let us consider vicinity of the unit $\hat{I}$ of the group $\text{U}(n, \mathbb{C}, \hat{\times})$. Let $\text{U}(t) \in U(n, \mathbb{C}, \hat{\times})$ and $\text{U}(0) = \hat{I}$, where t is parameter. Then, we have

$$U(t)\hat{\times}U^\dagger(t) = \hat{I}, \quad \frac{dU}{dt}|_{t=0} = X, \tag{3.66}$$

where X belongs to tangent space of $\text{U}(n, \mathbb{C}, \hat{\times})$ in the vicinity of $\hat{I}$. Differentiating first equation of (3.66), we have

$$\frac{d}{dt}(U\hat{T}U^\dagger)|_{t=0} = [\frac{dU}{dt}\hat{T}U^\dagger + U\hat{T}\frac{dU^\dagger}{dt}]|_{t=0} \tag{3.67}$$
$$= X\hat{T}\hat{I} + \hat{I}\hat{T}X^\dagger = X + X^\dagger = 0.$$

So, $u(n, \mathbb{C}, \hat{\times})$ consists of the skew-Hermitean matrices,

$$X = -X^\dagger. \tag{3.68}$$

Similarly, it can be shown that if $O \in O(n, \mathbb{R}, \hat{\times})$ is orthogonal matrix then matrices from the tangent space in the vicinity of unit of this group are skew-symmetric, and vice versa. Indeed, for X such that

$$X = -X^t, \tag{3.69}$$

we have

$$O \hat{\times} O^t = \hat{e}^X \hat{\times} (\hat{e}^X)^t = \hat{e}^X \hat{\times} \hat{e}^{X^t} = \hat{e}^{X+X^t} = \hat{I}, \tag{3.70}$$

where we used Eqs.(3.50) and (3.52), and the fact that X and X^t are $\hat{\times}$-commuting matrices.

For the tangent space elements X of pseudo-unitary group $\mathrm{U}(m, k, \mathbb{C}, \hat{\times})$ and pseudo-orthogonal group $\mathrm{O}(m, k, \mathbb{C}, \hat{\times})$, it is an easy exercise to obtain from Eqs.(3.46) the usual relations,

$$XG + GX^\dagger = 0, \quad XG + GX^t = 0, \tag{3.71}$$

respectively, where G is matrix of pseudo-Euclidean metrics (3.39). For example, for the pseudo-unitary group $\mathrm{U}(m, k, \mathbb{C}, \hat{\times})$ we have

$$\frac{d}{dt}(U\hat{T}\hat{G}\hat{T}U^\dagger)|_{t=0} = [\frac{dU}{dt}\hat{T}\hat{G}\hat{T}U^\dagger + U\hat{T}\hat{G}\hat{T}\frac{dU^\dagger}{dt}]|_{t=0} \tag{3.72}$$
$$= X(\hat{T}G\hat{I}\hat{T})\hat{I} + \hat{I}(\hat{T}G\hat{I}\hat{T})X^\dagger = XG + GX^\dagger = 0,$$

where we have assumed that the matrices G and $\hat{I}$ commute and thus $\hat{I}$ is *diagonal* matrix; see remark below Eq.(3.46). In general, we have instead of (3.71),

$$X\hat{T}G\hat{I} + GX^\dagger = 0, \quad X\hat{T}G\hat{I} + GX^t = 0, \tag{3.73}$$

for the groups $U_r(m, k, \mathbb{C}, \hat{\times})$ and $O_r(m, k, \mathbb{R}, \hat{\times})$, respectively, where $\hat{G} = G\hat{I} \equiv \hat{G}_r$, and

$$XG + \hat{I}G\hat{T}X^\dagger = 0, \quad XG + \hat{I}G\hat{T}X^t = 0, \tag{3.74}$$

for the groups $U_l(m, k, \mathbb{C}, \hat{\times})$ and $O_l(m, k, \mathbb{R}, \hat{\times})$, respectively, where $\hat{G} = \hat{I}G \equiv \hat{G}_l$. The definitions (3.73) and (3.74) can be rewritten in a compact natural form,

$$X \hat{\times} \hat{G} + \hat{G} \hat{\times} X^\dagger = 0, \quad X \hat{\times} \hat{G} + \hat{G} \hat{\times} X^t = 0, \tag{3.75}$$

where $\hat{G} = \hat{G}_r$, or $\hat{G} = \hat{G}_l$.

To prove that the above tangent spaces indeed are Lie algebras one must show that the following properties hold:

1) If Trace $M = 0$ and Trace $N = 0$ then Trace $[M, N]_{\hat{\times}} = 0$.

2) If $M^t = -M$ and $N^t = -N$ then $[M, N]^t_{\hat{\times}} = -[M, N]_{\hat{\times}}$.

3) If $M^\dagger = -M$ and $N^\dagger = -N$ then $[M, N]^\dagger_{\hat{\times}} = -[M, N]_{\hat{\times}}$.

In fact, by this one shows that the spaces $sl(n, \mathbb{C}, \hat{\times})$, $o(n, \mathbb{R}, \hat{\times})$, and $u(n, \mathbb{C}, \hat{\times})$ are closed in respect to $\hat{\times}$-commutator.

Note that the unit $\hat{I}$ is subject to Trace, Transpose and Complex conjugate operations in the above 1)-3). Let us put the following restrictions on $\hat{I}$:

$$\text{Trace } \hat{I} \neq 0, \quad \hat{I}^t = \hat{I}, \quad \bar{\hat{I}} = \hat{I}, \tag{3.76}$$

i.e. $\hat{I}$ is real symmetric matrix with non-zero trace. Particularly, the form (2.40) obeys the requirements (3.76).

Then, it is easy to check that the properties 1)-3) hold for any $\hat{I}$ obeying (3.76). Namely, we have

$$\begin{aligned} 1) \quad \text{Trace } [M, N]_{\hat{\times}} &= \text{Trace } M\hat{T}N - \text{Trace } N\hat{T}M \qquad (3.77)\\ &= \text{Trace } M\hat{T}N - \text{Trace } M\hat{T}N = 0.\\ 2) \qquad [M, N]^t_{\hat{\times}} &= (M\hat{T}N)^t - (N\hat{T}M)^t = N^t\hat{T}^tM^t - M^t\hat{T}^tN^t \qquad (3.78)\\ &= N^t\hat{T}M^t - M^t\hat{T}N^t = N\hat{T}M - M\hat{T}N\\ &= -[M, N]_{\hat{\times}}. \end{aligned}$$

Similarly, for 3).

In the same manner, one can prove that tangent spaces of $\mathrm{U}(m, k, \mathbb{C}, \hat{\times})$ and $\mathrm{O}(m, k, \mathbb{R}, \hat{\times})$ are Lie algebras in respect to $\hat{\times}$-commutator.

3.4 Restrictions on the form of unit

We are now in a position to collect all the restrictions on the form of $\hat{I}$ stemming from consideration made in the present paper.

1) To have well defined inverse needed to set up $\hat{\times}$-product, $\hat{I}$ must be *non-degenerate*; see Sec. 2.1.

2) To define positive norm in space of matrices, it must be matrix of *positive definite symmetric* form; see Sec. 2.3.

3) To have positive norm, it must have *positive trace*; see Sec. 2.3.

4) To be in the homotopy class of I, it must have *positive values* of the diagonal elements, for diagonal form of $\hat{I}$; see Sec. 2.4.

5) For consistent definitions of algebras of orthogonal and unitary groups, it must be *symmetric* and *Hermitean*, respectively; see Sec. 3.3.

6) To have conventional definitions of algebras of pseudo-unitary and pseudo-orthogonal groups, it must commute with the matrix of *pseudo*-Euclidean metrics. This means that $\hat{I}$ must be of *diagonal form* since symmetricity of $\hat{I}$ is not sufficient here; see Sec. 3.3.

All these requirements taken together put strong limitation on the form of $\hat{I}$, confining us with the choice made in Sec. 2.4. Namely, the family of possible units consists of *diagonal $n \times n$ matrices with positive real elements*,

$$\hat{I} = \mathrm{diag}(q_1, q_2, \ldots, q_n), \quad q_i > 0. \tag{3.79}$$

3.5 Infinite dimensional case

Most of the definitions and properties of $\mathrm{M}(n, \mathbb{C}, \hat{\times})$ studied in previous sections can be readily extended to infinite dimensional case, $n = \infty$. Here, the unit is, evidently,

$$\hat{I} = \mathrm{diag}(q_1, q_2, \ldots, q_i, \ldots), \quad \hat{I} \in M(\infty, \mathbb{C}, \hat{\times}), \tag{3.80}$$

and the $\hat{\times}$-product is as usual; see Eq.(2.3). Further, in the continuous limit we have the unit

$$\hat{I}_{p'p} = \hat{I}(p')\delta(p' - p), \tag{3.81}$$

and the product,

$$(M \hat{\times} N)_{p'p} = \int dp'' dp'''\ M_{p'p''} \hat{T}_{p''p'''} N_{p'''p}, \tag{3.82}$$

where

$$\hat{T}_{p'p} = \hat{I}^{-1}(p')\delta(p' - p). \tag{3.83}$$

One of the applications of $\mathrm{M}(\infty, \mathbb{C}, \hat{\times})$ and its continuous limit, which would be of interest to investigate is quantum mechanics. It is well known that quantum mechanical Hermitean operators in *any* representation can be given in the form of infinite dimensional matrices.

Considering action of Lie groups on classical spaces, in Sec. 3.2, we have seen that the coordinates x^i are given with *different* weights $\sqrt{q_i}$ by the general form of unit $\hat{I}$, in contrast to equal weights, $q_i = 1$, ascribed to the coordinates by standard unit I.

Such a property is quite natural in quantum mechanics when one deals with quantum mechanical ensemble of pure states realized with different probabilities, i.e. the pure states are given with *different* weights, and form *mixed state*. This formalism concerns von Neumann's density matrix and canonical ensembles. Let us see on the standard quantum mechanical definition of the density matrix,

$$\rho_{mn} = \sum P_k \bar{c}_{mk} c_{kn}, \tag{3.84}$$

where P_k are the weights, $P_k > 0$, $\sum P_k = 1$, and c_{kn} are amplitudes, and compare it with the $\hat{\times}$-product (2.3). We see that the density matrix ρ is obtained by $\hat{\times}$-product,

$$\rho = \bar{c}\hat{T}c, \tag{3.85}$$

of the amplitude matrices, where $\hat{T} = \mathrm{diag}(P_1, P_2, \dots, P_k, \dots)$. In quantum mechanics, *diagonal* elements of the density matrix,

$$w_n = \rho_{nn} = \sum P_k |c_{kn}|^2, \tag{3.86}$$

is density of probability to find observable in state $|n\rangle$, in the mixed ensemble. For example, in coordinate representation,

$$\rho(x, x', t) = \sum_k P_k \psi_k^*(x') \psi_k(x), \tag{3.87}$$

and

$$w(x, t) = \rho(x, x, t) = \sum_k P_k |\psi_k(x)|^2 \tag{3.88}$$

is density of probability for coordinate x, in the mixed state ensemble.

From the above sketch we see that pure states can be associated to the standard form of unit while mixed states can be associated to general form of unit (3.80), with the identification $P_k = 1/q_k$ and normalization condition Trace $\hat{T} = 1$. Then, ψ_k are seen as components of vector in space $\mathbb{R}^\infty_q$, and (3.88) is scalar product in $\mathbb{R}^\infty_q$ given by the dual metrics $g = \hat{T}$; see Eq.(3.32).

Also, we have many examples of using weight functions in functional analysis. For example, well known space $L_{2,\rho}[a,b]$ of complex functions is a Hilbert space if one define scalar product as [7]

$$\langle f, g\rangle = \int\limits_a^b dx \rho(x) f(x) \bar{g}(x), \tag{3.89}$$

where weight function $\rho(x)$ is real and positive, in the region $[a,b]$. Suppose that polynomials $p_n(x)$ constitute orthogonal system, i.e.

$$\delta_{mn} = \int\limits_a^b dx \rho(x) p_m(x) p_n(x). \tag{3.90}$$

Then, up to constant factors, for $\rho(x) = 1$, $a = -1$, $b = 1$ we obtain Legendre polynomials, for $\rho(x) = \exp\{-x^2\}$, $a = -\infty$, $b = +\infty$ we obtain Chebyshev-Hermite polynomials, and for $\rho(x) = \exp\{-x\}$, $a = 0$, $b = +\infty$ we obtain Chebyshev-Lagerre polynomials.

The above examples are given just to stress that some elements of infinite dimensional (discrete or continuous) case of $\mathrm{M}(n, \mathbb{C}, \hat{\times})$, where unit is not necessarily of standard form, are well established in quantum mechanics of mixed states and in functional analysis. In both the cases, their discrete finite-dimensional limit leads to consideration of $\mathrm{M}(n, \mathbb{C}, \hat{\times})$ equipped by unit of a general form (3.79).

In finite dimensional case, there is the following example where inhomogeneous dilation (3.33) is explicitly used. For system of N point particles with different masses, in three dimensional Euclidean space we have the following Lagrangian:

$$L = \frac{1}{2} \sum_{i=1}^{3N} m_i \dot{x}_i^2 - U(x_i), \tag{3.91}$$

where $\vec{x}_k = (x_k, x_{k+1}, x_{k+2})$ are coordinates of the particles, and $m_k = m_{k+1} = m_{k+2}$, $k = 1, \ldots, 3N-3$. Introducing $x'_i = \sqrt{m_i} x_i$, one can rewrite the above Lagrangian as

$$L = \frac{1}{2}\sum_{i=1}^{3N} \dot{x}'^2_i - U(x'_i). \tag{3.92}$$

The same transformation of coordinates can be used in the case of Schrödinger equation for system of N particles.

4 Examples

4.1 Algebra $so(3,\mathbb{R},\hat{\times})$ and group $\mathbf{SO}(3,\mathbb{R},\hat{\times})$

We start our consideration of examples of matrix Lie algebras and groups with the general form of unit by $so(3,\mathbb{R},\hat{\times})$.

Let us briefly recall usual $so(3,\mathbb{R})$ algebra. Basic elements of this algebra are 3×3 skew-symmetric matrices,

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.1}$$

They satisfy commutation relations, $[X_1, X_2] = X_3$, $[X_3, X_1] = X_2$, $[X_2, X_3] = X_1$.

One can construct *non-skew-symmetric* matrices

$$\hat{X}_i = X_i\hat{I}, \quad \text{or} \quad \hat{X}_i = \hat{I}X_i, \tag{4.2}$$

where the unit matrix is

$$\hat{I} = \mathrm{diag}(a_1, a_2, a_3), \tag{4.3}$$

identically satisfying the $\hat{\times}$-commutation relations,

$$[\hat{X}_1, \hat{X}_2]_{\hat{\times}} = \hat{X}_3, \quad [\hat{X}_3, \hat{X}_1]_{\hat{\times}} = \hat{X}_2, \quad [\hat{X}_2, \hat{X}_3]_{\hat{\times}} = \hat{X}_1. \tag{4.4}$$

Indeed, e.g., for the first $\hat{\times}$-commutator we have

$$\hat{X}_1\hat{T}\hat{X}_2 - \hat{X}_2\hat{T}\hat{X}_1 = X_1\hat{I}\hat{T}X_2\hat{I} - X_2\hat{I}\hat{T}X_1\hat{I} = (X_1X_2 - X_2X_1)\hat{I} = X_3\hat{I}. \tag{4.5}$$

However, these $\hat{X}_i$ matrices can not be used to construct the group $\mathrm{SO}(3, \mathbb{R}, \hat{\times})$, except for the trivialized case, $a_1 = a_2 = a_3$, which makes $\hat{X}_i$'s skew-symmetric. Indeed, only skew-symmetric matrices correspond to orthogonal group; see Eq.(3.69).

To construct appropriate $\hat{X}_i$'s with general values of the parameters a_i, we proceed as follows. First, we calculate for ordinary X_i's given by (4.1) the commutators, $X_i \hat{I} X_k - X_k \hat{I} X_i$. They are $X_1 \hat{I} X_2 - X_2 \hat{I} X_1 = a_3 X_3$, $X_3 \hat{I} X_1 - X_1 \hat{I} X_3 = a_2 X_2$, and $X_2 \hat{I} X_3 - X_3 \hat{I} X_2 = a_1 X_1$. Then, we use the duality property (2.21) of the $\hat{\times}$-commutator, and see that the matrices $\hat{X}_i$ of the form

$$\hat{X}_i = \hat{I} X_i \hat{I}, \tag{4.6}$$

are skew-symmetric by construction, namely,

$$\hat{X}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_2 a_3 \\ 0 & -a_2 a_3 & 0 \end{pmatrix}, \quad \hat{X}_2 = \begin{pmatrix} 0 & 0 & -a_1 a_3 \\ 0 & 0 & 0 \\ a_1 a_3 & 0 & 0 \end{pmatrix}, \tag{4.7}$$

$$\hat{X}_3 = \begin{pmatrix} 0 & a_1 a_2 & 0 \\ -a_1 a_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and satisfy the commutation relations

$$[\hat{X}_1, \hat{X}_2]_{\hat{\times}} = a_3 \hat{X}_3, \quad [\hat{X}_3, \hat{X}_1]_{\hat{\times}} = a_2 \hat{X}_2, \quad [\hat{X}_2, \hat{X}_3]_{\hat{\times}} = a_1 \hat{X}_1. \tag{4.8}$$

Corresponding matrix exponents, namely, $\hat{O}_i(t) = \hat{e}^{t\hat{X}_i}$, obtained by use of Eq.(3.56) are of the following form:

$$\hat{O}_1 = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 \cos\sqrt{a_2 a_3}t & \sqrt{a_2 a_3}\sin\sqrt{a_2 a_3}t \\ 0 & -\sqrt{a_2 a_3}\sin\sqrt{a_2 a_3}t & a_3 \cos\sqrt{a_2 a_3}t \end{pmatrix}, \tag{4.9}$$

$$\hat{O}_2 = \begin{pmatrix} a_1 \cos\sqrt{a_1 a_3}t & 0 & -\sqrt{a_1 a_3}\sin\sqrt{a_1 a_3}t \\ 0 & a_2 & 0 \\ \sqrt{a_1 a_3}\sin\sqrt{a_1 a_3}t & 0 & a_3 \cos\sqrt{a_1 a_3}t \end{pmatrix}, \tag{4.10}$$

$$\hat{O}_3 = \begin{pmatrix} a_1 \cos\sqrt{a_1 a_2}t & \sqrt{a_1 a_2}\sin\sqrt{a_1 a_2}t & 0 \\ -\sqrt{a_1 a_2}\sin\sqrt{a_1 a_2}t & a_2 \cos\sqrt{a_1 a_2}t & 0 \\ 0 & 0 & a_3 \end{pmatrix}. \tag{4.11}$$

Simple but tedious algebra shows that these matrices have determinants equal to Det $\hat{I} = a_1a_2a_3$, and satisfy orthogonality condition (3.7). We can conclude that linear combinations of these basic elements constitute group SO$(3, \mathbb{R}, \hat{\times})$.

Action of the $\hat{O}_i$'s on vector $r = (x_1, x_2, x_3)$ is of the form $\hat{O}_i\hat{T}r$, so we present the matrices $\tilde{O}_i = \hat{O}_i\hat{T}$, which are of practical use, below.

$$\tilde{O}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\sqrt{a_2a_3}t & \sqrt{a_2/a_3}\sin\sqrt{a_2a_3}t \\ 0 & -\sqrt{a_3/a_2}\sin\sqrt{a_2a_3}t & \cos\sqrt{a_2a_3}t \end{pmatrix}, \tag{4.12}$$

$$\tilde{O}_2 = \begin{pmatrix} \cos\sqrt{a_1a_3}t & 0 & -\sqrt{a_1/a_3}\sin\sqrt{a_1a_3}t \\ 0 & 1 & 0 \\ \sqrt{a_3/a_1}\sin\sqrt{a_1a_3}t & 0 & \cos\sqrt{a_1a_3}t \end{pmatrix}, \tag{4.13}$$

$$\tilde{O}_3 = \begin{pmatrix} \cos\sqrt{a_1a_2}t & \sqrt{a_1/a_2}\sin\sqrt{a_1a_2}t & 0 \\ -\sqrt{a_2/a_1}\sin\sqrt{a_1a_2}t & \cos\sqrt{a_1a_2}t & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.14}$$

4.2 The groups SO$(2, \mathbb{R}, \hat{\times})$ and U$(1, \mathbb{C}, \hat{\times})$

According to the results of Sec. 4.1, elements of the group SO$(2, \mathbb{R}, \hat{\times})$ are of the form

$$\hat{O} = \begin{pmatrix} a_1\cos\sqrt{a_1a_2}t & \sqrt{a_1a_2}\sin\sqrt{a_1a_2}t \\ -\sqrt{a_1a_2}\sin\sqrt{a_1a_2}t & a_2\cos\sqrt{a_1a_2}t \end{pmatrix}. \tag{4.15}$$

This expression is instead of the matrix of usual rotation of Euclidean plane.

We are interested in the group U$(1, \mathbb{C}, \hat{\times})$, from which, by *making it real*, the group O$(2, \mathbb{R}, \hat{\times})$ can be obtained. We recall that in the usual case elements of U(1) are complex numbers of unit module, e^{it}. The representation (4.15) can be reproduced by the following *realization map*.

First, we note that for the 2×2 unit matrix $\hat{I} = \mathrm{diag}(a_1, a_2)$, we have Det $\hat{I} = a_1a_2$, and Det $\hat{O} = a_1a_2 =$ Det $\hat{I}$, as it should be for special orthogonal matrices. We introduce the complex number

$$\xi = \mathrm{Det}\ \hat{I}\exp\{i\sqrt{\mathrm{Det}\ \hat{I}}t\}, \tag{4.16}$$

and the matrix

$$\hat{J} = \sqrt{\mathrm{Det}\ \hat{I}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.17}$$

Observe that $|\xi| = \text{Det } \hat{I}$, and matrix $\hat{J}$ has the following properties:

$$\text{Det } \hat{J} = \text{Det } \hat{I}, \quad \hat{J^2} \equiv \hat{J}\hat{T}\hat{J} = -\hat{I}, \tag{4.18}$$

and *does not* commute with $\hat{I}$,

$$\hat{I}\hat{J} - \hat{J}\hat{I} = \sqrt{\text{Det } \hat{I}}(a_1 - a_2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.19}$$

However, it $\hat{\times}$-commute with $\hat{I}$, namely,

$$\hat{I}\hat{\times}\hat{J} - \hat{J}\hat{\times}\hat{I} = 0. \tag{4.20}$$

Also, note that particularly, $\hat{O}(0) = \hat{I}$ and $\hat{O}(\pi/2) = \hat{J}$.

Then, it can be verified that the realization map is given by

$$\hat{O} = r(\xi) \equiv \text{Re } \xi(\text{Det } \hat{T})\hat{I} + \text{Im } \xi(\text{Det } \hat{T})\hat{J}. \tag{4.21}$$

Indeed, $\text{Re } \xi = \text{Det } \hat{I} \cos\sqrt{a_1a_2}t$, $\text{Im } \xi = \text{Det } \hat{I} \sin\sqrt{a_1a_2}t$, and multiplying these by $(\text{Det } \hat{T})\hat{I}$ and $(\text{Det } \hat{T})\hat{J}$, respectively, we reproduce, after summing up, the representation (4.15). Note that determinant of the realization map matrix is

$$\text{Det } r(\xi) = |\xi|, \tag{4.22}$$

and matrix r $\hat{\times}$-commute with $\hat{J}$ due to Eq.(4.20) that means that $\hat{J}$ is indeed an operator of complex character.

Thus, elements of $\text{U}(1, \mathbb{C}, \hat{\times})$ are of the form (4.16), with product of the complex numbers given trivially by

$$\xi_3 = \xi_1\hat{\times}\xi_2 \equiv \xi_1(\text{Det } \hat{T})\xi_2, \quad \xi_{1,2,3} \in U(1, \mathbb{C}, \hat{\times}), \tag{4.23}$$

where $\text{Det } \hat{T} = 1/(a_1a_2)$ is a real number, and a_1 and a_2 are *fixed* positive real numbers.

Note that the matrix product in $\text{SO}(2, \mathbb{R}, \hat{\times})$ is $\hat{O}_1\hat{T}\hat{O}_2$ while in $\text{U}(1, \mathbb{C}, \hat{\times})$ the product is due to the above rule, where $\text{Det } \hat{T}$ is used instead of $\hat{T}$. Due to the realization map (4.21), the groups $\text{SO}(2, \mathbb{R}, \hat{\times})$ and $\text{U}(1, \mathbb{C}, \hat{\times})$ are isomorphic to each other.

An important remark here is that in the realization map (4.21) we used the fact that $-\hat{J}\hat{T}\hat{J} = \hat{I}$ due to Eq.(4.18). By this, we achieved isomorphism between $SO(2,\mathbb{R},\hat{\times})$ and $U(1,\mathbb{C},\hat{\times})$. Indeed, one can see from the form (4.16) of ξ that the group $U(1,\mathbb{C},\hat{\times})$ is characterized by *one* independent parameter, $a = \text{Det }\hat{I} = a_1a_2$, while $SO(2,\mathbb{R},\hat{\times})$ is characterized by *two* independent parameters, a_1 and a_2. So, if we were used the matrix $-\hat{J}^2 = \hat{J}\hat{J} = (\text{Det }\hat{I})I$ as a unit matrix we would obtain $SO(2,\mathbb{R},\hat{\times})$ characterized by the only parameter, a, instead of the two parameters, a_1 and a_2. Namely, the unit would be of trivialized form $a_1a_2\text{diag}(1,1)$, and elements of group $SO(2,\mathbb{R},\hat{\times})$ would be of form,

$$\hat{O} = \sqrt{a_1a_2}\begin{pmatrix} \cos\sqrt{a_1a_2}t & \sin\sqrt{a_1a_2}t \\ -\sin\sqrt{a_1a_2}t & \cos\sqrt{a_1a_2}t \end{pmatrix}. \tag{4.24}$$

Thus, the lesson is that we should use $-\hat{J}\hat{T}\hat{J}$ rather than $-\hat{J}\hat{J}$ to define the unit matrix in realization map. Certainly, we have some features stemming from real dimensionality two. See Sec. 4.5 for general consideration of the realization map, $n > 2$.

For convenience and to have consistence with general definition (3.9), we take $\xi \in U(1,\mathbb{C},\hat{\times})$ in the form (4.16). Note that we can replace Det $\hat{I}$ by any real parameter but we are using Det $\hat{I}$ to keep explicit correspondence with $SO(2,\mathbb{R},\hat{\times})$.

In fact, it does not matter which non-zero value the *module* of ξ has because it can be absorbed by appropriate definition of the product (4.23) and associated realization map (4.21). For example, we can put $\xi = \sqrt{\text{Det }\hat{I}}\exp\{i\sqrt{\text{Det }\hat{I}}t\}$ provided that Det $\hat{T}$ in Eqs.(4.23) and (4.21) is replaced by $\sqrt{\text{Det }\hat{T}}$, obtaining the same result. Moreover, we can put simply $\xi = \exp\{i\sqrt{\text{Det }\hat{I}}t\}$, i.e. $|\xi| = 1$, and, accordingly, delete Det $\hat{T}$ in Eqs.(4.23) and (4.21).

This demonstrates the fact that group $U(1,\mathbb{C},\hat{\times})$ is isomorphic to usual U(1), *up to the factor* $\sqrt{\text{Det }\hat{I}}$ in the argument of complex number. This factor is of importance since $\sqrt{\text{Det }\hat{I}}$ is *fixed*, and

$$\exp\{i\sqrt{\text{Det }\hat{I}}t_1\}\exp\{i\sqrt{\text{Det }\hat{I}}t_2\} = \exp\{i\sqrt{\text{Det }\hat{I}}(t_1+t_2)\} \tag{4.25}$$

is again element of $U(1,\mathbb{C},\hat{\times})$ for *any* t_1 and t_2, while, for example,

$$\exp\{i\sqrt{\text{Det }\hat{I}}t_1\}\exp\{it_2\} \tag{4.26}$$

is not element of the group for *any* t_1 and t_2.

In other words, $\mathrm{U}(1,\mathbb{C},\hat{\times})$ consists of complex numbers ξ with modules $|\xi| = \mathrm{Det}\ \hat{I}$ and arguments $\mathrm{Arg}\ \xi$ dividable to $\sqrt{\mathrm{Det}\ \hat{I}}$, i.e. arguments modulo number $\sqrt{\mathrm{Det}\ \hat{I}}$.

4.3 Action of the group $\mathbf{U}(1,\mathbb{C},\hat{\times})$

Let us consider the action of group $\mathrm{U}(1,\mathbb{C},\hat{\times})$.

While $\mathrm{U}(1)$ linearly transforms $\mathbb{C}$ equipped by standard metrics $|z|^2 = x^2+y^2$, the group $\mathrm{U}(1,\mathbb{C},\hat{\times})$ must conserve, by definition (3.2), metrics $|z|^{\hat{2}} = a_1x^2+a_2y^2$, which is *not conformally equivalent* to standard metrics $|z|^2$. In fact, we see that group $\mathrm{U}(1) \subset \mathbb{C}$ and produces motion of $\mathbb{C} = \mathbb{R}^2$ while group $\mathrm{U}(1,\mathbb{C},\hat{\times}) \subset \mathbb{C}$ and produces, with the action defined by (3.19), motion of $\mathbb{C}_q = \mathbb{R}^2_q$, where $\mathbb{R}^2_q$ is Euclidean (flat) space equipped by the metrics $a_1x^2+a_2y^2$.

The fact that we can rescale module of $\xi \in U(1,\mathbb{C},\hat{\times})$ to 1 without loss of generality corresponds to conformal equivalence of metrics $a_1x^2+a_2y^2$ and $b(a_1x^2+a_2y^2)$, where b is a real constant.

We are interested to find out transformation of $\mathbb{C}_q$ corresponding to rotation (4.15) of the space $\mathbb{R}^2_q$.

Let us consider action of $\mathrm{SO}(2,\mathbb{R},\hat{\times})$ on $\mathbb{R}^2_q$. Action of (4.15) on vector $r=(x,y)$ reads $\hat{O}\hat{\times}r$, namely,

$$\hat{O}\hat{T}r = \begin{pmatrix} \cos\sqrt{a_1a_2}t & \sqrt{a_1/a_2}\sin\sqrt{a_1a_2}t \\ -\sqrt{a_2/a_1}\sin\sqrt{a_1a_2}t & \cos\sqrt{a_1a_2}t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{4.27}$$

The matrix $\tilde{O} = \hat{O}\hat{T}$ has the following particular values:

$$\tilde{O}(0) = I, \quad \tilde{O}(\pi/2) = \hat{J}\hat{T} = (\mathrm{Det}\ \hat{T})\hat{I}\hat{J}. \tag{4.28}$$

The transformation (4.27) is of linear form,

$$\begin{pmatrix} a & b \\ -c & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.29}$$

where a, b, and c are real parameters. However, linear transformation of complex space, $z \mapsto \lambda z$, where $\lambda = (a+ib)$, $z=(x+iy) \in \mathbb{C}$, results in

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{4.30}$$

Obviously, the use of modified transformation $z \mapsto \lambda \hat{\times} z = \lambda(\text{Det } \hat{T})z$ does not yield transformation of the form (4.29) because this causes just additional dilation by real factor Det $\hat{T}$.

So, we are led to consider transformation of the general form, $z \mapsto F(z, \bar{z})$. Let us consider standard $\mathbb{R}^2$ and make the inhomogeneous dilation (3.33) of its coordinates,

$$x' = x/\sqrt{a_1}, \quad y' = y/\sqrt{a_2}. \tag{4.31}$$

Then, $r^2 = x^2 + y^2$ becomes

$$r^2 = a_1 x'^2 + a_2 y'^2 = r' \hat{I} r' = r'^{\hat{2}}. \tag{4.32}$$

Since $a_{1,2} > 0$ the transformation (4.31) is invertible and well defined. Associated Jacobi matrix is the same as the transformation matrix of (4.31), namely,

$$\begin{pmatrix} 1/\sqrt{a_1} & 0 \\ 0 & 1/\sqrt{a_2} \end{pmatrix} = +\sqrt{\hat{T}} \tag{4.33}$$

and Jacobian is Det $\hat{T}$. This transformation obviously provides the map $\mathbb{R}^2 \mapsto \mathbb{R}^2_q$ due to Eq.(4.32); $r = (x, y) \in \mathbb{R}^2$, $r' = (x', y') \in \mathbb{R}^2_q$.

In terms of complex coordinates, using $x = (z + \bar{z})/2$, $y = (z - \bar{z})/2i$ we obtain from transformation (4.31)

$$z' = \frac{z + \bar{z}}{2\sqrt{a_1}} + \frac{z - \bar{z}}{2\sqrt{a_2}}, \quad \bar{z}' = \frac{z + \bar{z}}{2\sqrt{a_1}} - \frac{z - \bar{z}}{2\sqrt{a_2}}, \tag{4.34}$$

or

$$z' = (\frac{1}{2\sqrt{a_1}} + \frac{1}{2\sqrt{a_2}})z + (\frac{1}{2\sqrt{a_1}} - \frac{1}{2\sqrt{a_2}})\bar{z} \equiv f(z, \bar{z}), \tag{4.35}$$

$$\bar{z}' = (\frac{1}{2\sqrt{a_1}} + \frac{1}{2\sqrt{a_2}})z - (\frac{1}{2\sqrt{a_1}} - \frac{1}{2\sqrt{a_2}})\bar{z} = f(z, -\bar{z}). \tag{4.36}$$

Function f in transformation (4.35) depends on $\bar{z}$, and thus it is *not* complex analytic function, $\partial f/\partial \bar{z} \not\equiv 0$, which thus makes *non complex analytic transformation* of complex plane $\mathbb{C}$. We write for this case $\mathbb{C} \mapsto \mathbb{C}_q$, only for the purpose to be not confused by the usual convention that *transformation* of complex plane, $\mathbb{C} \mapsto \mathbb{C}$, means *complex analytic* transformation. Accordingly, we write $z' \in \mathbb{C}_q$.

Function $f(z, \bar{z})$ is a sum of holomorphic and antiholomorphic functions, $f(z, \bar{z}) = f_1(z) + f_2(\bar{z})$, each of which is a linear function of its argument.

Below, we make various linear transformations of complex plane $\mathbb{C}$, namely, $\mathbb{C} \mapsto \mathbb{C}$, and analyze what kind of transformations they induce in complex plane $\mathbb{C}_q$, namely, $\mathbb{C}_q \mapsto \mathbb{C}_q$.

Let us make linear complex analytic transformation of $\mathbb{C}$, namely, $z \mapsto \lambda z$, and see its image in $\mathbb{C}_q$. We observe from (4.35) that $f_1(z)$ becomes $\lambda f_1(z)$ while $f_2(\bar{z})$ remains intact, and thus we have *no* linear transformation of $\mathbb{C}_q$ which is of the form $z' \mapsto \lambda z'$. By construction, this describes the action of group U(1), for $|\lambda| = 1$. So, the image of U(1)-action on $\mathbb{C}$ is some *non-linear* transformation in $\mathbb{C}_q$.

By making linear *non* complex analytic transformation of $\mathbb{C}$, namely, $z \mapsto \lambda(z + \bar{z})$, we readily obtain that the image of this transformation in $\mathbb{C}_q$ is *linear complex analytic* transformation, $z' \mapsto \lambda z'$, of $\mathbb{C}_q$. However, this still does not describe the action of group $\mathrm{U}(1, \mathbb{C}, \hat{\times})$.

By making linear *non* complex analytic transformation of $\mathbb{C}$, namely, $z \mapsto \lambda(z + \bar{z}) + \mu(z - \bar{z})$, we obtain the image of this transformation in $\mathbb{C}_q$ which is *linear non complex analytic* transformation of $\mathbb{C}_q$ that corresponds to the action of group $\mathrm{U}(1, \mathbb{C}, \hat{\times})$.

Indeed, by choosing complex numbers in the form

$$\lambda = \frac{1}{2}(\sqrt{a_1}a + i\sqrt{a_2}b), \quad \mu = \frac{i}{2}(\sqrt{a_1}b - i\sqrt{a_2}a), \tag{4.37}$$

where a and b are arbitrary real numbers, and making non complex analytic transformation of $\mathbb{C}_q$ as above,

$$z' \mapsto \lambda(z' + \bar{z}') + \mu(z' - \bar{z}'), \tag{4.38}$$

we obtain directly the following associated transformation of $\mathbb{R}^2_q$:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} \mapsto \begin{pmatrix} a & \sqrt{a_2/a_1}b \\ -\sqrt{a_1/a_2}b & a \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}, \tag{4.39}$$

which is of the desired form (4.29).

4.4 Group $\mathbf{SO}(1, 1, \mathbb{R}, \hat{\times})$

In the usual setting of group $\mathrm{SO}(1, 1, \mathbb{R})$, we have the generator of the form

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.40}$$

and the usual matrix exponent gives us elements of the proper group in the form

$$O = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}. \tag{4.41}$$

This matrix is pseudo-orthogonal, $OGO^t = G$, where $G = \mathrm{diag}(1, -1)$, and Det $O = 1$, so that the scalar product $xGx = (x^0)^2 - (x^1)^2$ is conserved; x^0 and x^1 are local coordinates of two-dimensional pseudo-Euclidean (Minkowski) space M^2. Also, there is a smooth path from O to $I = \mathrm{diag}(1, 1)$.

In the group SO$(1, 1, \mathbb{R}, \hat{\times})$, the unit is $\hat{I} = \mathrm{diag}(a_0, a_1)$; $\hat{T} = (\hat{I})^{-1}$. The generator can be chosen here as $\hat{X} = \hat{I}X\hat{I}$ (see Sec. 4.1),

$$\hat{X} = a_0 a_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.42}$$

which supplies us, by the help of the matrix exponent $\hat{e}^{\phi X} = e^{\phi X\hat{T}}\hat{I}$, with elements of the group having the form (cf. Eq.(4.15))

$$\hat{O} = \begin{pmatrix} a_0\cosh\sqrt{a_0a_1}\phi & \sqrt{a_0a_1}\sinh\sqrt{a_0a_1}\phi \\ \sqrt{a_0a_1}\sinh\sqrt{a_0a_1}\phi & a_1\cosh\sqrt{a_0a_1}\phi \end{pmatrix}. \tag{4.43}$$

This matrix conserves the scalar product

$$x\hat{G}x = a_0(x^0)^2 - a_1(x^1)^2, \tag{4.44}$$

where metrics is

$$\hat{G} = \hat{I}G = \begin{pmatrix} a_0 & 0 \\ 0 & -a_1 \end{pmatrix}, \tag{4.45}$$

in the sense that

$$\hat{O}\hat{\times}\hat{G}\hat{\times}\hat{O}^t \equiv \hat{O}\hat{T}\hat{G}\hat{T}\hat{O}^t = \hat{G}. \tag{4.46}$$

Also, $\hat{O}$ can be continuously connected to the identity transformation $\hat{I}$ by $\phi \to 0$. It is instructive to check the above pseudo-orthogonality (4.46), where $\hat{T} = \mathrm{diag}(1/a_0, 1/a_1)$ is used for the product, since $\hat{O}(\psi)\hat{\times}\hat{I}\hat{\times}\hat{O}^t(\psi) \neq \hat{I}$, as it might seem; in fact, this is equal to $\hat{O}$ with double angle (!), $\hat{O}(\psi)\hat{\times}\hat{I}\hat{\times}\hat{O}^t(\psi) = \hat{O}(2\psi)$. Moreover, similar (simple but tedious) calculations show that in addition to Eq.(4.46), we have

$$\hat{O}\hat{W}\hat{G}\hat{W}\hat{O}^t = \hat{G}, \tag{4.47}$$

where we have denoted

$$\hat{W} = \hat{G}^{-1} = \begin{pmatrix} 1/a_0 & 0 \\ 0 & -1/a_1 \end{pmatrix}. \tag{4.48}$$

Our comment here is that we can use the pair $(\hat{G}, \hat{W})$ instead of $(\hat{I}, \hat{T})$.

Action of the matrix $\hat{O}$ on $r = (x^0, x^1)$ is $\hat{O}\hat{T}r$, namely,

$$x'^0 = x^0 \cosh\,\psi + x^1 \sqrt{\frac{a_0}{a_1}} \sinh\,\psi, \quad x'^1 = \sqrt{\frac{a_1}{a_0}} x^0 \sinh\,\psi + x^1 \cosh\,\psi, \tag{4.49}$$

where we have denoted $\psi = \sqrt{a_0 a_1}\phi$, for brevity. At $x^1 = 0$, we have from Eq.(4.49)

$$\frac{x'^1}{x'^0} = \sqrt{\frac{a_1}{a_0}} \tanh\,\psi. \tag{4.50}$$

In the context of *special relativity in two dimensions*, $x^0 = ct$, the l.h.s. of Eq.(4.50) is a relative speed, v/c, of the frame of reference (ct, x^1) in respect to the frame of reference (ct', x'^1). Thus,

$$\tanh\,\psi = \sqrt{\frac{a_0}{a_1}} \frac{v}{c}. \tag{4.51}$$

Inserting this to Eq.(4.49), we obtain by use of trigonometric relations $\sinh\,\psi = \tanh\,\psi/\sqrt{1 - (\tanh\,\psi)^2}$ and $\cosh\,\psi = 1/\sqrt{1 - (\tanh\,\psi)^2}$,

$$t' = t \frac{1}{\sqrt{1 - \hat{\beta}^2}} + x^1 \frac{a_0}{a_1} \frac{v/c^2}{\sqrt{1 - \hat{\beta}^2}}, \quad x'^1 = t \frac{v}{\sqrt{1 - \hat{\beta}^2}} + x^1 \frac{1}{\sqrt{1 - \hat{\beta}^2}}, \tag{4.52}$$

where we have denoted

$$\hat{\beta} = \sqrt{\frac{a_0}{a_1}} \frac{v}{c}. \tag{4.53}$$

Since only the ratio a_0/a_1 is present in Eq.(4.52), we denote

$$a = \sqrt{\frac{a_1}{a_0}} \tag{4.54}$$

and rewrite it in the form

$$t' = (t + \frac{v}{a^2 c^2} x^1)\hat{\gamma}, \quad x'^1 = (x^1 + vt)\hat{\gamma}, \tag{4.55}$$

where $\hat{\gamma}$-factor

$$\hat{\gamma} = \frac{1}{\sqrt{1 - \hat{\beta}^2}}. \tag{4.56}$$

and $\hat{\beta} = v/(ac)$.

The following remarks are in order.

(a) We took without proof that the trigonometric relations used above in the case of metrics $\mathrm{diag}(a_1, -a_2)$ are the same as they are in the case of standard pseudo-Euclidean metrics $\mathrm{diag}(1, -1)$, and note only that the spaces are flat in both the cases.

(b) Despite the fact that $\hat{I}$ depends on two parameters, a_0 and a_1, and the generator $\hat{X}$ depends on the product $a_1 a_2$, only their *ratio* (4.54) appeared in the transformations (4.55).

(c) We see from Eq.(4.55) that the only distinction from the conventional Lorentz transformations is that the constant c is replaced by ac. This can be understood as follows. Making inhomogeneous dilation (rescalements) of the coordinates, $x^0 \mapsto x^0/\sqrt{a_0}$ and $x^1 \mapsto x^1/\sqrt{a_1}$ to obtain metrics G from $\hat{G}$, we change, by this, slope of the isotropic line, $x^1 = ct$ to $x^1 = act$.

(d) One can suppose that such properties extend to consideration of action of the higher dimensional pseudo-orthogonal group, $\mathrm{SO}(3, 1, \mathbb{R}, \hat{\times})$, on the corresponding four-dimensional Minkowski space-time, with three different coefficients appearing at c in three main space axises, Ox^1, Ox^2, and Ox^3, of a chosen coordinate system (space anisotropic behavior). Certainly, this should be checked by a direct consideration.

4.5 Realization map

The realization map constructed in Sec. 4.2 can be extended to higher dimensions in the following way.

First, we note that $\mathrm{GL}(n, \mathbb{C}, \hat{\times})$ and $\mathrm{GL}(m, \mathbb{R}, \hat{\times})$, $m = 2n$, in general have the units parameterized by n and $2n$ parameters, respectively. In the two-dimensional case it appeared fortunately that $-\hat{J}^{\hat{2}} = \hat{I}$ exactly for special choice of the parameters. However, this is not the case in higher dimensions, $n > 2$, and some reparameterization is needed to match between the parameters.

Indeed, the $2n \times 2n$ matrix $\hat{J}$ has the form,

$$\hat{J} = \begin{pmatrix} 0 & \hat{I} \\ -\hat{I} & 0 \end{pmatrix}. \tag{4.57}$$

where $n \times n$ matrix $\hat{I} = \mathrm{diag}(q_1, q_2, \dots, q_n)$ is unit in $\mathrm{GL}(n, \mathbb{C}, \hat{\times})$. Note that $\mathrm{Det}\ \hat{J} = (\mathrm{Det}\ \hat{I})^2$. The corresponding $2n \times 2n$ unit $\hat{I}_{2n}$ in $\mathrm{GL}(2n, \mathbb{R}, \hat{\times})$ can be found by squaring the matrix $\hat{J}$ with the help of unit matrix $\hat{I}_{gen}$,

$$\hat{I}_{gen} = \mathrm{diag}(a_1, a_2, \dots, a_{2n}), \tag{4.58}$$

of $\mathrm{GL}(m, \mathbb{R}, \hat{\times})$, $m = 2n$. Namely,

$$\hat{I}_{2n} = -\hat{J}\hat{\times}\hat{J} = -\hat{J}\hat{T}_{gen}\hat{J} = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}, \tag{4.59}$$

where $\hat{T}_{gen} = \hat{I}_{gen}^{-1}$ and

$$K_1 = \mathrm{diag}(q_1^2/a_{n+1}, q_2^2/a_{n+2}, \dots, q_n^2/a_{2n}), \tag{4.60}$$

$$K_2 = \mathrm{diag}(q_1^2/a_1, q_2^2/a_2, \dots, q_n^2/a_n). \tag{4.61}$$

Then, the unit matrix $\hat{I}_{2n}$ depends on $2n$ independent parameters, with extra n independent parameters coming from $\hat{I}_{gen}$. The realization map for matrix $M = A + iB$, $M \in GL(n, \mathbb{C}, \hat{\times})$, is given by

$$r(M) = \hat{I}_{2n}A + \hat{J}B, \quad r(M) \in GL(2n, \mathbb{C}, \hat{\times}). \tag{4.62}$$

However, $\hat{I}_{2n}$ is not equal to $\hat{I}_{gen}$ even if we identify $q_i = a_i$, $i = 1, \dots, n$. Indeed, in this case we have $K_1 = \mathrm{diag}(a_1^2/a_{n+1}, a_2^2/a_{n+2}, \dots, a_n^2/a_{2n})$ and $K_2 = \mathrm{diag}(a_1, a_2, \dots, a_n)$. Only after the reparameterizing,

$$a_i^2/a_{n+i} \mapsto a_{n+i}, \quad i = 1, \dots, n, \tag{4.63}$$

we obtain $K_1 = \mathrm{diag}(a_{n+1}, a_{n+2}, \dots, a_{2n})$, and thus achieve the identification $\hat{I}_{2n} = \hat{I}_{gen}$.

Let us consider the case $n = 4$ for an illustrative purpose. Let $\mathrm{GL}(2, \mathbb{C}, \hat{\times})$ and $\mathrm{GL}(4, \mathbb{R}, \hat{\times})$ have the units

$$\hat{I} = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}, \quad \hat{I}_{gen} = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}, \tag{4.64}$$

respectively, and $\hat{T}_{gen} = \hat{I}_{gen}^{-1}$. Then, $\hat{J}$ is

$$\hat{J} = \begin{pmatrix} 0 & 0 & q_1 & 0 \\ 0 & 0 & 0 & q_2 \\ -q_1 & 0 & 0 & 0 \\ 0 & -q_2 & 0 & 0 \end{pmatrix}, \tag{4.65}$$

and unit $\hat{I}_4 = \hat{J}\hat{T}_{gen}\hat{J}$ has the form

$$\hat{I}_4 = \begin{pmatrix} q_1^2/a_3 & 0 & 0 & 0 \\ 0 & q_2^2/a_4 & 0 & 0 \\ 0 & 0 & q_1^2/a_1 & 0 \\ 0 & 0 & 0 & q_2^2/a_2 \end{pmatrix}. \tag{4.66}$$

Putting $q_1 = a_1$, $q_2 = a_2$, and reparameterizing, $a_1^2/a_3 \mapsto a_3$ and $a_2^2/a_4 \mapsto a_4$, we reproduce $\hat{I}_4 = \hat{I}_{gen}$.

4.6 Matrix algebra $\mathrm{M}(2, \mathbb{C}, \hat{\times})$

Let us consider $\mathrm{M}(2, \mathbb{C})$ matrix algebra consisting of all 2×2 matrices over the field of complex numbers $\mathbb{C}$.

Additive basis of the matrix algebra $\mathrm{M}(2, \mathbb{C})$ consists of unit 2×2 matrix I and matrices σ_i,

$$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{4.67}$$

This means that the algebra with the basis

$$\{I,\ \sigma_1,\ \sigma_2,\ \sigma_3\}, \tag{4.68}$$

and relations

$$\sigma_i\sigma_j + \sigma_j\sigma_i = -2I\delta_{ij}, \tag{4.69}$$

over $\mathbb{C}$, i.e. the universal enveloping algebra of $su(2, \mathbb{C})$, is isomorphic to $\mathrm{M}(2, \mathbb{C})$. Note that the above σ-matrices are traceless and skew-Hermitean in $\mathrm{M}(2, \mathbb{C})$, and are related to usual Pauli matrices by factor i, with labels 1 and 3 interchanged.

We define new unit matrix $\hat{I}$,

$$\hat{I} = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}, \quad q_{1,2} > 0, \quad \hat{I} \in M(2, \mathbb{C}), \tag{4.70}$$

and the associated $\hat{\times}$-product between the matrices,

$$M \hat{\times} N = M \hat{T} N, \quad M, N \in M(2, \mathbb{C}), \tag{4.71}$$

where

$$\hat{T} = \hat{I}^{-1} = \begin{pmatrix} 1/q_1 & 0 \\ 0 & 1/q_2 \end{pmatrix}, \quad \hat{T} \in M(2, \mathbb{C}). \tag{4.72}$$

Explicitly,

$$M \hat{\times} N = M \hat{T} N = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} q_1^{-1} & 0 \\ 0 & q_2^{-1} \end{pmatrix} \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \tag{4.73}$$

$$= \begin{pmatrix} \frac{m_{11} n_{11}}{q_1} + \frac{m_{12} n_{21}}{q_2} & \frac{m_{11} n_{12}}{q_1} + \frac{m_{12} n_{22}}{q_2} \\ \frac{m_{21} n_{11}}{q_1} + \frac{m_{22} n_{21}}{q_2} & \frac{m_{21} n_{12}}{q_1} + \frac{m_{22} n_{22}}{q_2} \end{pmatrix}. \tag{4.74}$$

We would like to construct an additive basis,

$$\{\hat{I},\ i\hat{\sigma}_1,\ i\hat{\sigma}_2,\ i\hat{\sigma}_3\}, \tag{4.75}$$

in terms of which elements of the algebra $\mathrm{M}(2, \mathbb{C}, \hat{\times})$ are presented as *linear combinations*. Namely,

$$M = x_0 \hat{I} + x_1 i\hat{\sigma}_1 + x_2 i\hat{\sigma}_2 + x_3 i\hat{\sigma}_3, \quad M \in M(2, \mathbb{C}, \hat{\times}), \tag{4.76}$$

where x_i are parameters. Note that we are not using $\hat{\times}$-product to multiply matrices by parameters (numbers) in Eq.(4.76) because parameters are not elements of the matrix algebra, and we are not considering action of matrices on a vector.

The criterium to determine $\hat{\sigma}$-matrices is that they, together with unit $\hat{I}$, must form additive basis in algebra $\mathrm{M}(2, \mathbb{C}, \hat{\times})$.

The possible way to have such a basis is that $\hat{\sigma}$-matrices must satisfy the following anticommutation relations, instead of standard (4.69),

$$\hat{\sigma}_i \hat{T} \hat{\sigma}_j + \hat{\sigma}_j \hat{T} \hat{\sigma}_i = 2 \hat{I} \delta_{ij}, \tag{4.77}$$

or, defining $\hat{\times}$-anticommutator,

$$\{M,N\}_{\hat{\times}} = M\hat{\times}N + N\hat{\times}M = M\hat{T}N + N\hat{T}M, \tag{4.78}$$

we rewrite the above as

$$\{\hat{\sigma}_i, \hat{\sigma}_j\}_{\hat{\times}} = 2\hat{I}\delta_{ij}. \tag{4.79}$$

We have two formal algebraic solutions for these equations,

$$\hat{\sigma}_i = \sigma_i \hat{I}, \tag{4.80}$$

i.e.

$$\hat{\sigma}_1 = \begin{pmatrix} iq_1 & 0 \\ 0 & -iq_2 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & q_2 \\ -q_1 & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 0 & iq_2 \\ iq_1 & 0 \end{pmatrix}, \tag{4.81}$$

and

$$\hat{\sigma}_i = \hat{I}\sigma_i, \tag{4.82}$$

i.e.

$$\hat{\sigma}_1 = \begin{pmatrix} iq_1 & 0 \\ 0 & -iq_2 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & q_1 \\ -q_2 & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 0 & iq_1 \\ iq_2 & 0 \end{pmatrix}. \tag{4.83}$$

Indeed, we have identically for (4.82)

$$\hat{\sigma}_i\hat{T}\hat{\sigma}_j + \hat{\sigma}_j\hat{T}\hat{\sigma}_i = \hat{I}\sigma_i\hat{T}\hat{I}\sigma_j + \hat{I}\sigma_j\hat{T}\hat{I}\sigma_i = \hat{I}(\sigma_i\sigma_j + \sigma_j\sigma_i) = -\hat{I}2\delta_{ij}, \tag{4.84}$$

and similarly for (4.80).

Note that the relations (4.84) hold for *any* invertible 2×2 matrix $\hat{I}$, not only for those having the above mentioned diagonal form (4.70). This means that the algebraic solutions (4.80) and (4.82) are formal.

We stress that the spaces $\mathrm{M}(2,\mathbb{C})$ and $\mathrm{M}(2,\mathbb{C},\hat{\times})$ are isomorphic to each other. The difference is that they have different bases, $B = \{I,\ \sigma_1,\sigma_2,\sigma_3\}$ and $\hat{B} = \{\hat{I},\ \hat{\sigma}_1,\hat{\sigma}_2,\hat{\sigma}_3\}$, respectively, and different definitions of matrix product. In view of the solutions (4.80) and (4.82), these bases are related to each other simply by

$$\hat{B} = B\hat{I}, \tag{4.85}$$

and

$$\hat{B} = \hat{I}B, \tag{4.86}$$

respectively. From this point of view, if one change unit I by some transformation matrix, the same matrix should be used to change remaining elements of the basis. This justifies partially the choice of algebraic solutions in the forms (4.80) and (4.82).

An important note is that in general matrices $\hat{\sigma}_i$ are not skew-Hermitean and not traceless. However, due to Eq.(3.58), we must have Trace $\hat{\sigma}_i \hat{T} = 0$ for matrices (4.80) to meet the condition $|\text{Det } U| = \text{Det } \hat{I}$ for the associated Lie group. This is indeed trivially the case. The problem of the lack of skew-Hermiticity concerns algebra $su(2, \mathbb{C}, \hat{\times})$, and is considered in the next Section.

4.7 Algebra $su(2, \mathbb{C}, \hat{\times})$ and group SU$(2, \mathbb{C}, \hat{\times})$

Let us find norm of vector,

$$X = x_1 i\hat{\sigma}_1 + x_2 i\hat{\sigma}_2 + x_3 i\hat{\sigma}_3. \tag{4.87}$$

in space $su(2, \mathbb{C}, \hat{\times})$. Using Eq.(4.81) and noting from Eq.(4.77) that $\hat{\sigma}_1^{\hat{2}} = -\hat{I}$, $\hat{\sigma}_2^{\hat{2}} = -\hat{I}$, and $\hat{\sigma}_3^{\hat{2}} = -\hat{I}$, we have (the Killing metrics)

$$|X|^2 = \text{Det } X = q_1 q_2 (x_1^2 + x_2^2 + x_3^2) = (\text{Det } \hat{I})(x_1^2 + x_2^2 + x_3^2). \tag{4.88}$$

Transformation

$$X \mapsto u \hat{\times} X \hat{\times} u^{-1}, \quad u \in su(2, \mathbb{C}, \hat{\times}), \tag{4.89}$$

is orthogonal in the sense of the scalar product (4.88), namely,

$$\text{Det } X = \text{Det } (u \hat{\times} X \hat{\times} u^{-1}). \tag{4.90}$$

Thus, any matrix $Z \in su(2, \mathbb{C}, \hat{\times})$ makes linear transformation ad $Z = [Z, X]_{\hat{\times}}$ of three-dimensional space $su(2, \mathbb{C}, \hat{\times})$.

Metric tensor of space $su(2, \mathbb{C}, \hat{\times})$ due to Eq.(4.88) is, evidently,

$$\hat{\delta}_{ij} = \delta_{ij} \text{Det } \hat{I}, \tag{4.91}$$

which is conformally equivalent to the usual Euclidean metrics δ_{ij} of three-dimensional Euclidean space $\mathbb{R}^3$.

Formally, $\hat{\sigma}$-matrices form representation of algebra $su(2, \mathbb{C}, \hat{\times})$. To prove this we must verify the following commutation relations

$$[\hat{\sigma}_1, \hat{\sigma}_2]_{\hat{\times}} = 2\hat{\sigma}_3, \quad [\hat{\sigma}_3, \hat{\sigma}_1]_{\hat{\times}} = 2\hat{\sigma}_2, \quad [\hat{\sigma}_2, \hat{\sigma}_3]_{\hat{\times}} = 2\hat{\sigma}_1. \tag{4.92}$$

One can easily verify by using (4.80), or (4.82), that these relations trivially hold.

Note, however, that in general the matrices $\hat{\sigma}_i$ are *not skew-Hermitean*. This give us no possibility to construct associated *unitary* group with their help.

They become skew-symmetric in the particular case, $q_1 = q_2 = q$, that leads, however, to reduction of $\hat{I}$ to scalar matrix

$$\hat{I} = \mathrm{diag}(q, q) = qI, \tag{4.93}$$

and therefore somewhat trivializes the attempt.

Application of the duality method developed for $so(3, \mathbb{R}, \hat{\times})$ in Sec. 4.1 to the case of $su(2, \mathbb{C}, \hat{\times})$ does not seem provide us with an appropriate algebraic solution. Some obstacle is made by σ_1 matrix, which has a *diagonal* form whereas none of X_i's has a diagonal form. Explicit calculations show that

$$\sigma_1\hat{I}\sigma_2 - \sigma_2\hat{I}\sigma_1 = (q_1 + q_2)\sigma_3, \quad \sigma_1\hat{I}\sigma_3 - \sigma_3\hat{I}\sigma_1 = -(q_1 + q_2)\sigma_2, \quad \sigma_2\hat{I}\sigma_3 - \sigma_3\hat{I}\sigma_2 = q_1q_2\hat{T}\sigma_1, \tag{4.94}$$

implying that the matrices

$$\hat{\sigma}_i = \hat{I}\sigma_i\hat{I} \tag{4.95}$$

having explicit *skew-Hermitean* form,

$$\hat{\sigma}_1 = \begin{pmatrix} iq_1^2 & 0 \\ 0 & -iq_2^2 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & q_1q_2 \\ -q_1q_2 & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 0 & iq_1q_2 \\ iq_1q_2 & 0 \end{pmatrix}, \tag{4.96}$$

satisfy

$$[\hat{\sigma}_1, \hat{\sigma}_2]_{\hat{\times}} = (q_1 + q_2)\hat{\sigma}_3, \quad [\hat{\sigma}_1, \hat{\sigma}_3]_{\hat{\times}} = -(q_1 + q_2)\hat{\sigma}_2, \quad [\hat{\sigma}_2, \hat{\sigma}_3]_{\hat{\times}} = q_1q_2\hat{T}\hat{\sigma}_1. \tag{4.97}$$

Here, the last equation includes matrix $\hat{T}$, and thus is unusual so that these commutation relations are seemed to be not Lie-algebraic. Also, direct calculations show that $\hat{\times}$-anticommutators between these $\hat{\sigma}$-matrices are of the form

$$\{\hat{\sigma}_1, \hat{\sigma}_1\}_{\hat{\times}} = -2\hat{I}^3, \quad \{\hat{\sigma}_2, \hat{\sigma}_2\}_{\hat{\times}} = -2q_1q_2\hat{I}, \quad \{\hat{\sigma}_3, \hat{\sigma}_3\}_{\hat{\times}} = -2q_1q_2\hat{I}, \tag{4.98}$$

$$\{\hat{\sigma}_1, \hat{\sigma}_2\}_{\hat{\times}} = iq_1q_2(q_1 - q_2)\sigma_2, \quad \{\hat{\sigma}_1, \hat{\sigma}_3\}_{\hat{\times}} = iq_1q_2(q_1 - q_2)\sigma_3, \quad \{\hat{\sigma}_2, \hat{\sigma}_3\}_{\hat{\times}} = 0. \tag{4.99}$$

We note that some non-zero values appear in Eq.(4.99).

Thus, the problem to construct general solution for $\hat{\sigma}$-matrices which obey appropriate $\hat{\times}$-anticommutation and/or $\hat{\times}$-commutation relations, and *are skew-Hermitean*, is opened. Note that we have explicit construction for $so(3, \mathbb{R}, \hat{\times})$ with general values of the parameters a_i, $i = 1, 2, 3$; see Sec. 4.1. And this is a candidate to the algebra isomorphic to $su(2, \mathbb{C}, \hat{\times})$, with unrestricted parameters q_i, $i = 1, 2$. However, we should to note that number of the parameters a_i and q_i is different.

Elements of the group $\mathrm{SU}(2, \mathbb{C}, \hat{\times})$ can be represented by using the matrix exponent (see Sec. 3.3),

$$M = \hat{e}^{\frac{1}{2}t^i\hat{\sigma}_i}, \quad M \in SU(2, \mathbb{C}, \hat{\times}), \tag{4.100}$$

where t^i are real parameters.

Direct calculations show that for the representations (4.80) and (4.82) we obtain matrix exponents, which indeed exhibit the property $\mathrm{Det}\ M = \mathrm{Det}\ \hat{I}$, but they are *not unitary*. Evidently the latter is a consequence of the fact that these representations are not skew-Hermitean matrices. In the case of unit of the form of scalar matrix (4.93), the group $\mathrm{SU}(2, \mathbb{C}, \hat{\times})$ is simply isomorphic to the ordinary group $\mathrm{SU}(2, \mathbb{C})$ since q can be absorbed by the parameters t^i.

In the case of the representation (4.96), we have $\mathrm{Trace}\ \hat{\sigma}_1\hat{T} = (q_1 - q_2)$, $\mathrm{Trace}\ \hat{\sigma}_2\hat{T} = 0$, $\mathrm{Trace}\ \hat{\sigma}_1\hat{T} = 0$, that means that $\mathrm{Det}\ M \neq \mathrm{Det}\ \hat{I}$ for $M = \hat{e}^{\frac{1}{2}t^1\hat{\sigma}_1}$, and thus we can not construct group $\mathrm{SU}(2, \mathbb{C}, \hat{\times})$ despite the fact that these $\hat{\sigma}$-matrices are skew-Hermitean. Instead, we could construct $\mathrm{U}(2, \mathbb{C}, \hat{\times})$ but only if the $\hat{\times}$-commutation relations (4.97) are acceptable.

A Appendix A

Distributivity implies that abstract product, $f(M, N)$, is a *linear* function in both the matrices, $f(M_1 + M_2, N) = f(M_1, N) + f(M_2, N)$, $f(M, N_1 + N_2) = f(M, N_1) + f(M, N_2)$, restricting possible functions $f(M, N)$ by a polynomial in M and N. Let us define the product in the form

$$f(M, N) = \tau_1 M \tau_2 + \tau_3 N \tau_4 + \tau_5 M \tau_6 N \tau_7 + \tau_8 N \tau_9 M \tau_{10}, \tag{A.1}$$

where τ_i are fixed matrices, $M, N, \tau_i \in \mathcal{M}(n, \mathbb{C})$. Axiom of left and right unit gives us two equations, $f(\hat{I}, N) = \hat{I}$, $f(N, \hat{I}) = N$, which should be *solvable* equations for any N to have the algebra with unit. Namely, for (A.1) we have

$$\tau_1\hat{I}\tau_2 + \tau_3 N\tau_4 + \tau_5\hat{I}\tau_6 N\tau_7 + \tau_8 N\tau_9\hat{I}\tau_{10} = N, \tag{A.2}$$

$$\tau_1 N\tau_2 + \tau_3\hat{I}\tau_4 + \tau_5 N\tau_6\hat{I}\tau_7 + \tau_8\hat{I}\tau_9 N\tau_{10} = N, \tag{A.3}$$

from which we see that to satisfy identically the equations each term in the l.h.s. of them must be considered separately. Namely,

$$\tau_1\hat{I}\tau_2 = 0, \tag{A.4}$$

$$\tau_3 = I, \quad \tau_4 = I, \tag{A.5}$$

$$\tau_5\hat{I}\tau_6 = I, \quad \tau_7 = I, \tag{A.6}$$

$$\tau_8 = I, \quad \tau_9\hat{I}\tau_{10} = I, \tag{A.7}$$

$$\tau_3\hat{I}\tau_4 = 0, \tag{A.8}$$

$$\tau_1 = I, \quad \tau_2 = I, \tag{A.9}$$

$$\tau_5 = I, \quad \tau_6\hat{I}\tau_7 = I, \tag{A.10}$$

$$\tau_8\hat{I}\tau_9 = I, \quad \tau_{10} = I. \tag{A.11}$$

Therefore, since we assume $\hat{I} \neq 0$ we have $\tau_1 = 0$ or $\tau_2 = 0$ and $\tau_3 = 0$ or $\tau_4 = 0$, that rules out first two terms in (A.1). Note that the same result can be obtained by using the distributivity condition. Further, we obtain

$$\hat{I} = \tau_6^{-1}, \quad \text{or} \quad \hat{I} = \tau_9^{-1}, \tag{A.12}$$

where we have assumed that τ_6 and τ_9 are invertible matrices. This means that we are leaved with the following two forms of product,

$$f(M,N) = M\tau_6 N \quad \text{or} \quad f(M,N) = N\tau_9 M, \tag{A.13}$$

which are in essence equivalent to each other.

Putting of the constant terms $\tau_1\hat{I}\tau_2$ and $\tau_3\hat{I}\tau_4$ to zero is an obvious requirement, while putting of the remaining terms of Eqs.(A.2) and (A.3) *separately* equal to N needs some comments. To see more closely on the above made separation of the terms let us check the associativity condition, $f(f(M,N),P) = f(M,f(N,P))$,

$$\begin{aligned} f(f(M,N),P) = {} & \tau_5(\tau_5 M\tau_6 N\tau_7 + \tau_8 N\tau_9 M\tau_{10})\tau_6 P\tau_7 \\ & +\tau_8 P\tau_9(\tau_5 M\tau_6 N\tau_7 + \tau_8 N\tau_9 M\tau_{10})\tau_{10}, \end{aligned} \tag{A.14}$$

$$f(M,f(N,P)) = \tau_5 M\tau_6(\tau_5 N\tau_6 P\tau_7 + \tau_8 P\tau_9 N\tau_{10})\tau_7 + \tau_8(\tau_5 N\tau_6 P\tau_7 + \tau_8 P\tau_9 N\tau_{10})\tau_9 M\tau_{10}. \tag{A.15}$$

Obviously, the associativity condition is not satisfied for this general form of the product. The term $\tau_5\tau_8 N\tau_9 M\tau_{10}\tau_6 P\tau_7$ is present in Eq.(A.14) while such a term is absent in Eq.(A.15) so to meet the associativity condition we must put one of fixed matrices in this term equal to zero. This leads to discarding either third ($\tau_5 = 0$ or $\tau_6 = 0$ or $\tau_7 = 0$) or fourth ($\tau_8 = 0$ or $\tau_9 = 0$ or $\tau_{10} = 0$) term in the definition (A.1), thus yielding its separate consideration. For $\tau_8 = 0$ or $\tau_9 = 0$ or $\tau_{10} = 0$, the associativity condition reads

$$\tau_5\tau_5 M\tau_6 N\tau_7\tau_6 P\tau_7 = \tau_5 M\tau_6\tau_5 N\tau_6 P\tau_7\tau_7, \tag{A.16}$$

from which we find again $\tau_5 = \tau_7 = I$. For $\tau_5 = 0$ or $\tau_6 = 0$ or $\tau_7 = 0$, we find similarly $\tau_8 = \tau_{10} = I$. As the conclusion, we obtain the product in the form (2.3).

For completeness, let us consider higher degrees (> 2) in M or N in definition of the product. Using axiom of left and right unit, $f(\hat{I},N) = f(N,\hat{I}) = N$, one can see that there is no possibility to have these equations identically satisfied for fixed τ_i and any N. For example, the definition

$$f(M,N) = \tau_1 M\tau_2 N\tau_3 N + N\tau_4 M\tau_5 N\tau_6 + N\tau_7 N\tau_8 M\tau_9, \tag{A.17}$$

where τ_i are fixed matrices, $M, N, \tau_i \in \mathcal{M}(n,\mathbb{C})$, implies

$$\tau_1\hat{I}\tau_2 N\tau_3 N + N\tau_4\hat{I}\tau_5 N\tau_6 + N\tau_7 N\tau_8\hat{I}\tau_9 = N, \tag{A.18}$$

$$\tau_1 N\tau_2\hat{I}\tau_3\hat{I} + \hat{I}\tau_4 N\tau_5\hat{I}\tau_6 + \hat{I}\tau_7\hat{I}\tau_8 N\tau_9 = N \tag{A.19}$$

These two equations can not be identically satisfied for fixed τ_i and arbitrary N. Indeed, in the first equation, matrix N appears two times in each term of the l.h.s. so that some of τ_i must be of the form N^{-1} to satisfy this equation. However, we assume that τ_i's are *fixed* matrices so that they can not be of the form N^{-1}, where N is an *arbitrary* matrix. The same reason rules out any higher degree in M or N. Thus, the form (2.3) is the most general form for associative and distributive product in matrix algebra with unit.

B Appendix B

Below, we present the proof of the statement that matrices I and $\hat{I}$ are not similar to each other. It is based on the construction of the associated invariant polynomials [2].

Let us consider the matrices of the form $(I\lambda - A)$, where A is 2×2 matrix and λ is a real number.

Two matrices A and B are *similar* to each other iff the matrices $X = (I\lambda - A)$ and $Y = (I\lambda - B)$ have the same invariant polynomials.

For the case under study, $A = I$ and $B = \hat{I}$, we have

$$X = (\lambda - 1)I = \mathrm{diag}(\lambda - 1, \lambda - 1).) \tag{B.1}$$

and

$$Y = (I\lambda - \hat{I}) = \mathrm{diag}(\lambda - q_1, \lambda - q_2). \tag{B.2}$$

Let us find invariant polynomials of X. Main minors of X are

$$\begin{array}{rc} \text{2nd order minors :} & (\lambda-1)^2, \\ \text{1st order minors :} & (\lambda-1), \quad (\lambda-1). \end{array} \tag{B.3}$$

Largest common quotients of the minors are

$$d_2(\lambda) = (\lambda - 1)^2, \quad d_1(\lambda) = (\lambda - 1). \tag{B.4}$$

Then, the invariant polynomials are

$$i_2(\lambda) \equiv d_2/d_1 = (\lambda - 1), \quad i_1(\lambda) \equiv d_1 = (\lambda - 1). \tag{B.5}$$

Let us find invariant polynomials of Y. Main minors of Y are

$$\begin{array}{rc} \text{2nd order minors :} & (\lambda-q_1)(\lambda-q_2), \\ \text{1st order minors :} & (\lambda-q_1), \quad (\lambda-q_2). \end{array} \tag{B.6}$$

Then, largest common quotients of the minors of are

$$d_2(\lambda) = (\lambda - q_1)(\lambda - q_2), \quad d_1(\lambda) = 1. \tag{B.7}$$

So, invariant polynomials are

$$i_2(\lambda) = d_2/d_1 = (\lambda - q_1)(\lambda - q_2), \quad i_1(\lambda) = d_1 = 1. \tag{B.8}$$

We see that the invariant polynomials of I given by (B.5) and of $\hat{I}$ given by (B.8) are different. So, the matrices I and $\hat{I}$ are not similar to each other in the sense of (2.60) and (2.61).

Perhaps, the above presented exercise is not necessary to see that I and $\hat{I}$ are not similar to each other. However, we have seen that I and $\hat{I}$ are homotopically equivalent in the space of matrices, for $q_{1,2} > 0$, that could be thought of as they are related to each other by similarity condition $I = V^{-1}\hat{I}V$ for some matrix V. We have seen that this is not the case.

One can easily prove that this property holds for general n-dimensional case, $\mathrm{M}(n, \mathbb{C})$, by noting that $n = 2$ case forms subspace of the higher dimensional cases.

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