Confusion in cosmology and gravitation

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Abstract

In a series of papers, Santilli and collaborators released various strong statements against the general theory of relativity (GTR) and the standard ΛCDM model of cosmology. In this paper we show that such claims are due to fundamental misunderstandings of very basic concepts of gravitation and cosmology. In other words, we show that Santilli and collaborators demonstrated nothing. In particular, they demonstrated neither that the GTR is wrong, nor that the Universe is not expanding. We also show that the so-called iso-gravitation theory (IGT) of Santilli is in macroscopic

contrast with geodesic motion and, in turn, with the Equivalence Principle (EP) and must therefore be ultimately rejected. Finally, we show that, although the so called iso-redshift could represent an interesting alternative (similar to the tired light theory historically proposed by Zwicky) to the Universe expansion from a qualitative point of view, it must be rejected from a quantitative point of view because the effect of iso-redshift is 10^{-6} smaller than the effect requested to achieve the cosmological redshift.

Paper dedicated to the 80th birthday of Ruggero M. Santilli, hoping that this will permit Santilli to understand how GTR and ΛCDM cosmology really work and, in turn, will permit him to withdraw his very wrong claims in gravitation and cosmology.

Keywords: gravitation; cosmology; Equivalence Principle; expansion of the Universe.

1 Introduction

In a series of papers [1]-[6], Santilli claims to have found various flaws in Einstein's GTR [30] and that such flaws can be corrected through his so-called IGT [3]-[5]. In this paper we clarify that such supposed flaws are strong misconceptions by Santilli on the basic concepts of the GTR instead. In particular, Santilli does not know and/or does not understand the EP, on which, not only the GTR, but all the metric theories of gravity are founded [7]. We also show that the IGT is not viable, for at least two fundamental reasons. The first is theoretical, the latter experimental. From the theoretical point of view, Santilli [5] claims indeed that his theory is founded on the Freud Identity of Differential Geometry [8]. We show that Santilli's interpretation of the Freud Identity is wrong from both the mathematical and physical points of views. In fact, on one hand Santilli mathematically confuses true tensors with pseudo-tensors. This issue was previously clarified in [9], but here we reexamine it from a slightly different point of view, using the definitions of tensors and pseudo-tensors in the classical book [10]. This reexamination is necessary because Santilli and collaborators insist in a wrong mathematical interpretation of the Freud Identity in recent works [11, 12], generating further confusion. On the other hand, Santilli physically does not take into account the EP.

From the experimental point of view, the absence of space-time curvature in the IGT implies a macroscopic contrast with geodesic motion and, in turn, with the EP [7, 13] which is today tested with a precision of order 10^{-14} [14, 15]. We also stress that, starting from the historical experiments by Loránd Eötvös in 1890 [16], which tested the equivalence between inertial and gravitational mass with a precision of order 10^{-8}, the EP has been tested with always increasing precision; today the EP stands as a fundamental principle of nature [17]. This means that the IGT is in very strong contrast with tons of data collected in more than a century and therefore must be ultimately rejected.

In another series of papers and announcements [18, 22], Santilli and collaborators claim to have found various flaws in the standard ΛCDM model of

cosmology and that the expansion of the Universe can be dismissed through the so-called iso-redshift. In this paper we also clarify that such supposed flaws are strong misconceptions by Santilli and collaborators on the basic concepts of the standard ΛCDM model of cosmology. In particular, the claim that "Hubble's law establishes that the cosmological redshift is the same for all galaxies having the same distance from Earth in all directions in space. Consequently, the conjectures on the expansion of the universe, the acceleration of the expansion and the big bang necessarily imply a return to the Middle Ages with Earth at the center of the universe", or similar claims such as those in [18, 22], are well known to be completely wrong at the popularizing level of physics' folklore and at the high school level. At such levels the similarity between the Universe's expansion and the surface of an expanding balloon - which has no center - is explained [23]. From a technical point of view, the Universe is seen as a space-like hyper-surface having no center. Also, the so-called "conjectures" are not conjectures. Instead, they are applications of metric theories of gravity (which are the only viable applications, based on the extremely well-tested precision of the EP [7]), starting from the GTR) to the cosmological observations and to the Cosmological Principle which states that "that the distribution of matter in the Universe is homogeneous and isotropic when viewed on a large enough scale" [10, 24], i.e. exactly the opposite of Santilli's wrong claim that the Earth is the center of the Universe. Finally, we also show that, although the so-called iso-redshift [18, 22] could, in principle, represent an alternative (similar to the tired light theory historically proposed by Zwicky [25]) to the Universe expansion from a qualitative point of view, it must be rejected from a quantitative point of view because the effect of iso-redshift is 10^{-6} smaller than the effect requested to achieve the cosmological redshift.

2 Confusion in gravitation

For the sake of clearness, we recall that Santilli calls "Einstein gravitation" the vacuum Einstein field equations while he calls "Einstein general relativity" the Einstein field equation in presence of sources [1]-[6], [11, 59]. After this clarifiyng, we can proceed with our analysis.

Some of Santilli's wrong claims are historical. For example, in [11], which is basically a review of [1]-[6], Santilli claims that being in the year 2015, we are "in connection with the centennial of the first geometric conception of gravitation". This is wrong. It is indeed well known that the GTR, of which this year is the centennial, was not the first geometric theory of gravitation. Historically, the relativistic scalar theory of gravitation introduced in 1912-13 by the Finnish physicist Gunnar Nordström in [26]-[28], has been the first geometric theory of gravity. In fact, it was derived three years before Einstein's GTR. This is well known in various papers throughout the literature, see for example the recent paper [29]. This is not the sole wrong historical mistake by Santilli and collaborators, as we will see in the following. In [11, 59] Santilli also claims that "Einstein general relativity is a scientic religion". This is completely wrong

and unacceptable. It is indeed well known that, although Einstein's GTR [30] achieved great success (see for example the opinion of Landau who says that GTR is, together with quantum field theory, the best scientific theory of all [10]) and withstood many experimental tests [7, 17], it also displayed many shortcomings and flaws which today make theoreticians question whether it is the definitive theory of gravity [29], [31]-[33]. Differently from other field theories like the electromagnetic theory, the GTR has not yet been quantized. This issue avoids treating gravitation like other quantum theories, precluding, in turn, the unification of gravitation with the other interactions. On the other hand, one defines Extended Theories of Gravity (ETG) as a group of semi-classical theories having the Lagrangian modified with respect to the standard Einstein-Hilbert gravitational Lagrangian. Such modifications include the addition of high-order terms in the curvature invariants (terms like R^2, $R^{\alpha\beta}R_{\alpha\beta}$, $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$, $R\ \ R$, $R\ \ ^{k}R$) or of terms with scalar fields being non-minimally coupled to the geometry (terms like $\phi^2 R$) [29], [31]-[33]. In fact, these kind of terms are usually considered in various approaches that attempt to perform the unification between gravitation and the other interactions. Another important issue is that, from the cosmological point of view, such extensions of GTR can generate inflationary approachs that result in very important attempts to solve various problems of the standard Universe model, starting from the historical work of Starobinsky [34]. We stress that, differently from Santilli [1]-[6] [11], we are not claiming that the GTR is wrong. It is well known that, even in the general framework of ETG, the GTR continues to serve as the most important part of the structure [29], [31]-[33]. The ETG approach only attempts to understand if (and how) weak modifications to the GTR structure can help to solve some theoretical and observational problems [29], [31]-[33]. Historically, the same Einstein claimed that the GTR structure could not be definitive [35]. It is indeed well known that, during his famous research on the unified field theory in the latest years of his life, Einstein attempted to realize a theory that he called the generalized theory of gravitation, but he claimed that mathematical difficulties prevented him from obtaining the final equations [35].

Considering the general context of cosmological observations, one also finds other considerations and approaches that suggest extensions to the GTR [29], [31]-[33]. In fact, today the Universe appears appears to be undergoing a period of accelerated expansion. The cosmological dynamics seem to be dominated by the so-called Dark Energy, which gives a large negative pressure [36]-[40]. This new ingredient in the standard picture is considered as a source of the right side of the Friedmann–Lemaître–Robertson–Walker (FLRW) field equations. In the standard Universe model the global dynamics are driven by this un-clustered non-zero vacuum energy together with the clustered Dark Matter [36]-[40]. The global framework is called the "concordance model" (ΛCDM) and gives, together with the CMBR, LSS and SNeIa data, the most general accepted tapestry of the Universe as it is observed today. On the other hand, ΛCDM cosmology shows various shortcomings as the well known "coincidence" and "cosmological constant" problems [40]. The alternative approach of ETG changes the left side of the field equations instead, attempting to achieve the observed cosmic

dynamics through the extension of the GTR [29], [31]-[33]. In the ETG context we do not need to search candidates for Dark Energy and Dark Matter, that, currently, have not yet been found. Only the "observed" ingredients, being curvature and baryon matter, have to be taken into account. Thus, one can think that gravity could be different at different scales [41] and there is room for alternative theories. As a result, various Dark Energy and Dark Matter models can be achieved considering f (R) theories of gravity, where R is the Ricci curvature scalar, and/or scalar-tensor gravity (STG) [29], [31]-[33], where STG is a generalization of the historical Brans-Dicke theory [42]. It has been shown in [31] that if advanced projects on the detection of gravitational waves (GWs) will improve their sensitivity - allowing to perform a GWs astronomy through accurate angular and frequency dependent response functions of interferometers for GWs arising from various theories of gravity, i.e. GTR and ETG - then this will be the ultimate test for the GTR.

We stress that we have been forced to insert this digression on the ETG to falsify Santilli's claim that "Einstein general relativity is a scientic religion" [11].

Another claim by Santilli is that there is a "lack of clear compatibility of general relativity with special relativity" [1]-[6], [11]. This is another elementary mistake. There is no lack of clear compatibility of the GTR with the special relativity theory (SRT) if one possesses a thorough understanding of the EP, which is any of the various connected concepts involving, on one hand, the equivalence of inertial and gravitational mass, and on the other hand, the observation by Einstein that the gravitational "force" that an observer experiences locally when standing in the gravitational field generated by a massive body (for example the Earth) is equivalent to the pseudo-force experienced in an accelerated (non-inertial) reference frame [24]. One of the various interpretations of the EP implies that a globally curved space-time is locally flat [24]. In other words, gravitational effects are always locally negligible and, in a local Lorentz frame, where the space-time can be considered flat in an excellent approximation, the SRT works very well [24].

Santilli defines the "First historical insufficiency of general relativity" as "Ignoring the refraction of star light passing through the Sun chromosphere, with consequential lack of evidence that space is curved" [11, 59]. That Santilli calls "historical" this and the following supposed insufficiences of GTR is a mystery and completely new to us. In our knowledge Santilli is indeed the sole person who claims that GTR has these supposed insufficiences. In any case, here are the various misconceptions.

1. It is wrong that the GTR claims that "the 0.87 arc-seconds deviation is caused by Newton gravitation" and "the remaining 0.87 arc-seconds deviation have been known for a century to be due to the curvature of space" [11, 59]. Instead, the GTR demonstrates that the whole 1.75 arc-second "bending" of star light passing near the Sun is due to the space-time curvature, not to the curvature of space [24]. In the GTR the global space-time is curved, not only the spatial surfaces [24]. In fact, the GTR is not based

on spatial curvature but on space-time curvature [24]. There are indeed solutions - for example the cosmological ones with $k = 0$ - where, although the spatial section is Euclidean at constant time, the global space-time structure is curved, i.e. it is not Lorentzian, but only conformally flat [24].

2. Claiming that "0.87 arc-seconds deviation is caused by Newton gravitation" and "The remaining 0.87 arc-seconds deviation is due to to the refraction of star light when passing through the Sun chromosphere" [11, 59] is very wrong for various reasons. First of all Newton gravitation admits instantaneous propagation, and this is in contrast with the SRT, as this theory requests interactions to have finite velocities of propagation. Second, today the bending of the light is carefully tested for many stars which are very far from the Sun chromosphere. The entire sky is indeed slightly distorted due to the gravitational deflection of light caused by the Sun (except the anti-Sun direction). This effect has been observed by the European Space Agency astrometric satellite Hipparcos [43]. It measured the positions of about 10^5 stars [43]. The results agree with the prevision of the GTR at the level of 0.3 percent [43]. Clearly, as the Newtonian value is exactly half of the Einsteinian one, a precision of the level of 0.3 percent rules out in an ultimate way the possibility to consider the gravitational bending of the light in a purely Newtonian context. Third, the Sun chromosphere was very different in the past and it will be very different in the future. Thus, if the author should be correct the consequence should be that, as the contribution of the Sun chromosphere is **exactly** the same as that of the contribution of the Newtonian theory at the present time, we are currently living in a very special period in the history of the solar system, because that contribution was very different in the past and will be very different in the future. Clearly, this issue cannot be taken seriously into account.

3. It is well known that also the Newtonian theory of gravity can be written in the language of curved space-time [24]. In fact, the EP is not unique to the GTR description of the concepts of gravity [24]. What is unique to Einstein's vision is the combination of the EP with the local Lorentz geometry [24]. Let us return to the Newtonian "universal time" [24]. For the trajectories of test particles, the Newtonian theory of gravity gives [24]

$$\frac{d^2x^j}{dt^2} = \frac{\partial V}{\partial x^j} \tag{1}$$

where V is the Newtonian potential. The most famous interpretation of the Newtonian gravitational theory is that eq. (1) describes the "curved paths" $x^j(t)$ along which test particles move in the flat Euclidean space (**not** space-time) [24]. On the other hand, there exists an alternative description, which is due to Cartan [24], which interprets the trajectories of eq. (1) as geodesics $t(\lambda)\ x^j(\lambda)$ in curved space-time [24]. Details of

this important issue, which still remark that the claims by Santilli against curvature are wrong, can be found in [24]. Here we limit ourselves to recall that the source equation for the Newtonian potential

$$\nabla^2 V \equiv \sum_j V_{jj} = -4\pi\rho \tag{2}$$

can be rewritten in the geometric form [24]

$$R_{00} = 4\pi\rho \tag{3}$$

where $R_{\alpha\beta}$ is the Ricci tensor.

On the other hand, and contrary to the Santilli's claims in [1]-[6], [11, 59], the bending from the light by the Sun is not considered to be the most important experimental proof that space-time is curved. Instead, the most important experimental proof is the EP. Before clarifying this point, we need to stress another important issue. Santilli claims in [11] that the inertial mass is bigger than the gravitational mass and explicity wrote down this point in eq. (7) of [11]. This is in strong contrast with tons and tons of experimental and observational data. Starting from the famous, historical, experiments by Eötvös in 1908, who found that the difference is less than 1 part in 10^8 [16], we have the experiments by Roll, Krotkov and Dicke, who found that the difference is less than 1 part in 10^{11} [44], and the experiments by Braginsky and Panov [45], who found that difference is less than 1 part in 10^{12} etc. To date, the most precise experimental results are those obtained by Baessler et al. [14, 15], who found that, if a difference should exist, it must be less than 1 part in 10^{14} In our knowledge, the equivalence between inertial and gravitational mass is one of the most precise instances of experimental evidence in the whole history of science, and it is today considered to be a fundamental principle of nature [17]. Thus, even admitting that eq. (7) of [11] is correct, the difference between the inertial mass and the gravitational mass must be less than 1 part in 10^{14} in order to not contradict the experimental results (we stress that Santilli and collaborators always stressed the Galileian statement that a scientist must never assume positions in contrast with experimental results [46, 47] in accordance to the scientific method). On the other hand, such a difference is completely negligible in the framework of classical gravitational theories.

After having clarified this fundamental issue, now we explain why today the EP is considered to be the most important experimental proof that space-time is curved. The key point is that the EP implies that test masses must follow geodesic lines. This point is very intuitive, but has been also rigorously demonstrated by Weinberg [13]. Before writing the derivation of this fundamental issue we stress its important consequence: in the absence of space-time curvature geodesic motion is given by straight lines! But instead, of course, all astrophysical observations show that the gravitational motion is not given by straight lines. Hence, the only possibility is that space-time is curved. In other words, Santilli's assumption of the absence of space-time curvature should therefore indicate a macroscopic violation of the equivalence between the inertial mass and

the gravitational mass, which, instead, is tested with the enormous precision of 1 part in 10^{14} [14, 15]. Clearly, considering also the experiments [16, 44, 45] etc., it is obvious that Santilli's claim of the absence of space-time curvature is in very strong contrast with tons of data collected in more than a century. Now, let us show that the EP implies that test masses must follow geodesic lines. This will be also an excellent example of how dynamical equations and causal structures arise from observations. We stress that in the following derivation we closely follow [13]. Let us start supposing that no particles are accelerating in the neighborhood of a point-event with respect to a freely falling coordinate system (X^{μ}) [13]. Putting $T = X^{0}$ one writes down the following equation that is locally applicable in free fall [13]

$$\frac{d^2X^{\mu}}{dT^2} = 0 \tag{4}$$

Using the chain rule one gets [13]

$$\frac{dX^{\mu}}{dT} = \frac{dx^{\nu}}{dT}\frac{\partial X^{\mu}}{\partial x^{\nu}} \tag{5}$$

Differentiating eq. (5) with respect to T one gets [13]

$$\frac{d^2X^{\mu}}{dT^2} = \frac{d^2x^{\nu}}{dT^2}\frac{\partial X^{\mu}}{\partial x^{\nu}} + \frac{dx^{\nu}}{dT}\frac{dx^{\alpha}}{dT}\frac{\partial^2 X^{\mu}}{\partial x^{\nu}\partial x^{\alpha}} \tag{6}$$

Combining eqs. (4) and (6) one immediately gets [13]

$$\frac{d^2x^{\nu}}{dT^2}\frac{\partial X^{\mu}}{\partial x^{\nu}} = -\frac{dx^{\nu}}{dT}\frac{dx^{\alpha}}{dT}\frac{\partial^2 X^{\mu}}{\partial x^{\nu}\partial x^{\alpha}} \tag{7}$$

Multiplying both sides of eq. (7) by $\frac{\partial x^{\lambda}}{\partial X^{\mu}}$ one gets [13]

$$\frac{d^2x^{\lambda}}{dT^2} = -\frac{dx^{\nu}}{dT}\frac{dx^{\alpha}}{dT}\frac{\partial^2 X^{\mu}}{\partial x^{\nu}\partial x^{\alpha}}\frac{\partial x^{\lambda}}{\partial X^{\mu}} \tag{8}$$

Setting $t = x^{0}$ and using again the chain rule, T can be eliminated in favor of the coordinate time t [13]

$$\frac{d^2x^{\lambda}}{dt^2} = -\frac{dx^{\nu}}{dt}\frac{dx^{\alpha}}{dt}\frac{\partial^2 X^{\mu}}{\partial x^{\nu}\partial x^{\alpha}}\frac{\partial x^{\lambda}}{\partial X^{\mu}} + \frac{dx^{\nu}}{dt}\frac{dx^{\alpha}}{dt}\frac{dx^{\lambda}}{dt}\frac{\partial^2 X^{\mu}}{\partial x^{\nu}\partial x^{\alpha}}\frac{\partial x^{0}}{\partial X^{\mu}} \tag{9}$$

Recalling that the bracketed terms involving the relationship between local coordinates X and general coordinates x are functions of the general coordinates, eq. (9) gives immediately the geodesic equation of motion using the coordinate time t as parameter) [13]

$$\frac{d^2x^{\lambda}}{dt^2} = -\Gamma^{\lambda}_{\nu\alpha}\frac{dx^{\nu}}{dt}\frac{dx^{\alpha}}{dt} + \Gamma^{0}_{\nu\alpha}\frac{dx^{\nu}}{dt}\frac{dx^{\alpha}}{dt}\frac{dx^{\lambda}}{dt} \tag{10}$$

which is equivalent to the standard geodesic equation written in terms of the scalar parameter s [13]

$$\frac{d^2x^{\lambda}}{ds^2} = -\Gamma^{\lambda}_{\nu\alpha}\frac{dx^{\nu}}{ds}\frac{dx^{\alpha}}{ds} \tag{11}$$

Clearly, based on the extreme precision on which the EP is today tested and verified, the demonstration that we have reviewed here - i.e. that geodesic motions arise from the EP - ultimately rules out Santilli's IGT, which is instead founded on the absence of curvature. Notice that, based on our criticisms [58], Santilli attempted to take into due account the EP in [59] claiming that "This raises the question as to whether Einstein's Equivalence Principle also holds for exterior isogravitation with a source. Einstein supporters quickly voice their opinion that this is not the case for the intent of invalidating isogravitation" and that "In particular, it is easy to see that Einstein's Equivalence Principle is maintained in its integrity in multiple ways. First of all, the projection of isogravitation on the conventional Riemannian space over a conventional field coincides with Einstein gravitation with consequential trivial validity of Einstein's Equivalence Principle. Additionally, the Equivalence principle independently holds also on the Minkowski-Santilli isospace over isofield by very conception of isotopies". Again, Santilli misunderstands the key point. The problem is not the potential contrast between the EP and the "exterior isogravitation with a source". As we have shown above, instead the real problem is that the absence of curvature is in macroscopic contrast with the EP. In addition, some of Santilli's claims contradict each other. In fact on one hand Santilli claims that the vacuum Einstein field equations are wrong, see [1]-[6], [11, 59] and the below discussions. On the other hand, in order to attempt preserving the EP, he is forced to reduce the IGT to the vacuum Einstein field equations [59].

Other misconceptions by Santilli result from the very wrong claims that "Irrespective of the above, the conjecture of curvature of space has been unable to represent without ambiguities truly basic gravitational events, such as the free fall of masses that has to be necessarily along a "straight" radial line, the weight of bodies in a gravitational field, and other basic events that are clearly represented by Newtonian gravitation" [11]. Of course, these are very elementary mistakes. First of all, we stress that a fundamental constraint used not only by Einstein in the derivation of the field equations [10, 24, 30], but also by Schwarzschild [48] in the derivation of his famous and fundamental solution to the Einstein field equations in vacuum was that, in the weak field approximation, i.e. at large distances, the general relativistic gravitational field must reduce to the Newtonian gravitational field. Clearly, the weak field approximation works very well near the Earth's surface. Thus, neglecting the higher-order terms in the general relativistic gravitational field and considering the Newtonian approximation permits to recover both of the free fall of masses, the weight of bodies in a gravitational field, and all of the other basic events of Newtonian gravitation. Second, again Santilli does not understand how the EP works. As the motion obeys the geodesic of eq. (11), locally a geodesic becomes a "straight" radial line and the free fall of masses is completely found also in the full GTR. In other words, for short distances where the gravitational field can be considered constant, the geodesics of a curved space-time are extremely well approximated by "straight" radial lines of a flat space-time in the same way that a curve line is locally well approximated by a straight line. This is another consequence of Einstein Equivalence Principle: in an inertial frame of reference bodies (and light)

obey Newton's first law, moving at constant velocity in straight lines. Analogously, in a curved spacetime the world line of an inertial particle or pulse of light is as straight as possible (in space and time) [88]. On the other hand, the issue that the weight of bodies in a gravitational field can be achieved also by the full GTR has been well explained in [49].

Another wrong claim by Santilli is that "Despite one century of studies, the "actual" orbits of planets in our Solar system have not been represented in an accurate, unique and time invariant way via Einstein gravitation, while they are exactly and unambiguously represented by Newton's gravitation and Kepler's laws. In fact, calculations based on the Riemannian geometry of the actual orbits of planets, besides not being unique due to the non-linearity of the theory, are different than physical orbits, and are not the same over time". To falsify this misconception let us write down the Schwarzschild line element in geometrized units as [24]

$$ds^2 = (1 - \frac{2M_S}{r})dt^2 - r^2(\sin^2\theta d\phi^2 + d\theta^2) - \frac{dr^2}{1 - \frac{2M_S}{r}} \tag{12}$$

where M_S is the solar mass. Setting $\theta = \frac{\pi}{2}$ in order to consider test bodies moving in the "equatorial plane" one computes to order $\frac{M_S}{r}$ the shape $r(\phi)$ of the nearly Keplerian, nearly elliptical geodesic orbit as [24]

$$r = \frac{1 - e^2\ a}{1 + e\cos\ \left(1 - \frac{\delta\phi_0}{2\pi}\right)\ \phi} \tag{13}$$

where e and a are constants of integration and [24]

$$\delta\phi_0 = \frac{6\pi M_S}{(1 - e^2)\, a} \tag{14}$$

Contrary to Santilli's claims and based on Birkhoff's theorem [24], it is well known that the line element of eq. (12) is unique despite the non-linearity of the theory, see also [50] for further details. Eq. (13) is more precise than the corresponding Newtonian counterpart which is

$$r = \frac{1 - e^2\ a}{1 + e\cos\phi} \tag{15}$$

corresponding to $\delta\phi_0 = 0$ In fact, eq. (13) takes into due account the presence of the precession (14). On the other hand, eq. (15), which is the Newtonian limit of eq. (13), is a perfect Keplerian ellipse having the semi-major axis a and eccentricity e [24].

Santilli also claims that the "second historical insufficiency of general relativity is ignoring the electromagnetic origin of the mass, with consequential invalidation of Einstein's reduction of gravitation to pure curvature without sources" [11]. Here Santilli generates massive confusion and attempts to propagate it. He indeed claims "to have identified the electromagnetic origin of the mass via the

full use of quantum electrodynamics, including advanced and retarded treatments and showed that such an origin requires the necessary presence in the r.h.s. of the field equations of source tensor of first order in magnitude, irrespective of whether the body is charged or neutral" [11]. In other words, Santilli claims that the vacuum Einstein field equations are

$$G_{ik} = R_{ik} - \frac{R}{2} g_{ik} = 0 \quad \text{with} \quad i\ j = 1\ 2\ 3\ 4 \tag{16}$$

where G_{ik}, R_{ik}, R and g_{ik} are the Einstein tensor, the Ricci tensor, the Ricci scalar and the metric tensor, respectively (see [10, 13, 24] for details), and are in contrast with his equations [11]

$$G_{ij} = kT_{ij\ elm} \tag{17}$$

where Santilli claims that k is a unit-dependent constant and $T_{ij\ elm}$ should be the "source tensor of first order in magnitude" arising from quantum electrodynamics [11]. The terms "first order in magnitude" should be "referred to the condition of entirely representing the gravitational mass of the body considered" [11]. In particular Santilli claims that "the mass of the electron is of entirely electromagnetic origin" and, as a consequence, the vacuum Einstein field equations of eq. (16) should be insufficient to represent the gravitational field of the electron in favor of his eq. (17) [11].

First of all, we stress that the GTR is a classical theory, which by definition does not take into account quantum effects. On the other hand, we show that, even considering the tensors found by Santilli in a classical approach, Santilli is wrong. In fact, within a classical framework we stress that vacuum is vacuum, i.e. we cannot force a source tensor to be always present in vacuum. To clarify this issue, we proceed as follows. Let us consider the electron in the framework of classical theories in terms of a sphere having the classical Compton radius $r_e \cong 2\ 8 * 10^{-15}$ meters We use this approach because the Compton radius is much greater than the Planck length, i.e. $r_e \gg l_p \cong 1\ 6 * 10^{-35}$ meters and we know that the GTR breaks down at the Planck scale [24]. Thus, assuming spherical symmetry and the correctness of Santilli's stress-tensor $T_{ij\ elm}$, we have the following framework for the field equations of the electron's gravitational field:

$$\begin{cases} \text{unknown} \quad \text{for } 0 \le r \le l_p \ (\text{we need a theory of quantum gravity}) \\ G_{ik} = kT_{ij\ elm} \quad \text{for } l_p \le r \le r_e \\ G_{ik} = 0 \quad \text{for } r \ge r_e \end{cases} \tag{18}$$

In other words, assuming that Santilli's field equations of eq. (17) are correct, they are not in contrast with Einstein's field equations of eq. (16). One must merely use eq. (17) in the electron's interior and eq. (16) for the external geometry. We stress that the vacuum Einstein field equations sometimes generate

some confusion because one can ask: how is it possible that eq. (16) works? In other words, how is it possible that a gravitational field can exist without a source? This key point is clarified in a very enlightening way by t' Hooft in [51], verbatim "Einstein's equations are non-linear, and this is why gravitational fields can be the source of an additional amount of gravity, so that a gravitational field can support itself." In other words, it is the non-linear part of eq. (16) which acts as a self-source of the gravitational field.

Santilli attempts to endorse his strong misunderstanding of this above discussed issue by claiming that the Freud identity of differential geometry [8] should establish "the need on purely mathematical grounds of a source tensor of first order in magnitude in the r.h.s of the field equations" according to his eq. (17) [11]. This is another misconception that we clarify immediately, but before we stress that such a misconception has been clarified from a purely mathematical point of view in [9]. As previously emphasized above, we now reanalyze this issue from a slightly different point of view with respect to [9], by using the definitions of tensors and pseudo-tensors in the classical book [10]. As Santilli and collaborators insist in wrong mathematical and physical interpretations of the Freud Identity in the recent works [11, 12, 59] (which generate further confusion), this reanalyzing is necessary. In addition we also discuss the physical counter part of the mathematical examination, which is missed in [9], which again concerns the EP.

Santilli [1, 6, 11] claims that eq. (16) violates the Freud identity of differential geometry [8] which is [8, 9]

$$2U_\kappa^\iota = \delta_\kappa^\iota \sqrt{-g}\, R + g^{\mu\nu}\left(\Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu} - \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma}\right) - 2 - \sqrt{-g}R^\iota_\kappa + \Gamma^\iota_{\mu\nu}\partial_\chi\left(\sqrt{-g}g^{\mu\nu}\right) - \Gamma^\nu_{\mu\nu}\partial_\chi\left(\sqrt{-g}g^{\mu\iota}\right) \tag{19}$$

with [8, 9]

$$U^\nu_\mu \equiv \frac{\partial}{\partial x_\rho} U^{\nu\rho}_\mu \tag{20}$$

being [8, 9]

$$U^{\nu\rho}_\mu \equiv \sqrt{-g} S^{\nu\rho}_\mu \tag{21}$$

the Freud superpotentials, with [8, 9]

$$S^{\lambda\rho}_\mu \equiv \frac{1}{2}\det \begin{pmatrix} \delta^\lambda_\mu & \delta^\sigma_\mu & \delta^\iota_\mu \\ g^{\lambda\kappa} & g^{\sigma\kappa} & g^{\iota\kappa} \\ \Gamma^\lambda_{\kappa\iota} & \Gamma^\sigma_{\kappa\iota} & \Gamma^\iota_{\iota\kappa} \end{pmatrix} \tag{22}$$

see [8, 9] for details.

Rearranging eq. (19), Santilli writes down the Freud identity as (eq. (3.10) in [6])

$$R^\alpha_\beta - \frac{R}{2}\delta^\alpha_\beta - \frac{1}{2}\Theta\delta^\alpha_\beta = U^\alpha_\beta + \frac{\partial V^{\alpha\rho}_\beta}{\partial x^\rho} = kT^\alpha_\beta \tag{23}$$