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Necessary and Sufficient Conditions for the Existence of a Lagrangian in Field Theory. I. Variational Approach to Self-adjointness for Tensorial Field Equations*

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In this paper we identify some of the most significant references on the inverse problem of the calculus of variations for single integrals and initiate the study of the generalization of the underlying methodology to classical field theories. We first classify Lorentz-covariant tensorial field equations into nonlinear, quasi-linear, and semilinear forms, and then introduce their systems of equations of variation and adjoint systems. The necessary and sufficient conditions for the self-adjointness of class $\mathscr{C}^2$, regular, tensorial, nonlinear, quasi-linear and semilinear forms are worked out. We study the Lagrange equations, their system of equations of variations (Jacobi equations) and their adjoint system by proving that, for class $\mathscr{C}^4$ and regular Lagrangian densities, they are always self-adjoint. We then introduce a concept of analytic representation which occurs when the Lagrange equations coincide with the field equations up to equivalence transformations and refine the definition by particularizing it as direct or indirect and ordered or nonordered. Some of the conventional cases of tensorial fields are considered and we prove, in particular, that the conventional representation of the complex scalar field in interaction with the electromagnetic field is of the ordered indirect type. For the objective of identifying our program we recall the two classes of equivalence transformations of the Lagrangian densities which are primarily used nowadays, namely, the Lorentz (coordinate) transformations and the gauge transformations (transformations of fields within a fixed coordinate system), and postulate the existence of a third class, which we term isotopic transformations of the Lagrangian density and which consist of equivalence transformations within a fixed coordinate system and gauge. We finally outline the objectives of our program, which essentially consist of the identification of the necessary and sufficient conditions for the existence of a Lagrangian in field theories and their first application to the transformation theory within the framework of our variational approach to self-adjointness.

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Necessary and Sufficient Conditions for the Existence of a Lagrangian in Field Theory
II. Direct Analytic Representations of Tensorial Field Equations*

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By using a characterization of the concept of analytic representation and a variational approach to self-adjointness introduced in a preceding paper, we prove a theorem, according to which a necessary and sufficient condition for a class $\mathscr{C}^2$, regular, tensorial, quasi-linear system of field equations to admit an ordered direct analytic representation in terms of the Lagrange equations in a region R of its variables is that the system is self-adjoint in R. We point out as a first corollary that if the ordering requirement is removed from the definition of analytic representation, then the condition of self-adjointness of the field equations is only sufficient for the existence of a Lagrangian density. We then provide as a second corollary a methodology for the computation of the Lagrangian density for the representation of self-adjoint quasi-linear tensorial field equations. This methodology is also particularized for ordinary semilinear systems of tensorial field equations through a third corollary. The above results are interpreted from the viewpoint of interactions. We first recover, through a fourth corollary, the conventional structure of the total Lagrangian density $\mathscr{L}_{\text{Tot}} = \sum_{1}^{n}{}_{a} \mathscr{L}_{\text{Free}}^{(a)} + \mathscr{L}_{\text{Int}}$ for the semilinear form of the field equations, and then introduce through a fifth corollary a generalized structure of the type $\mathscr{L}_{\text{Tot}} = \sum_{1}^{n}{}_{a} \mathscr{L}_{\text{Int,I}}^{(a)} \mathscr{L}_{\text{Free}}^{(a)} + \mathscr{L}_{\text{Int,II}}$ for the representations of the field equations in the quasi-linear form. Therefore, our analysis seems to indicate that a general form of representing interacting fields is characterized by $(n+1)$-interaction terms in the Lagrangian: n multiplicative terms and one additive term to the Lagrangian for the free fields.

1. INTRODUCTION

In a preceding paper [1], we have studied class $\mathscr{C}^2$, regular, Lorentz-covariant, tensorial field equations in (a) the *nonlinear form*:

$$F_{a_1}(x_\alpha, \phi^a, \phi^{a:}{}_\alpha, \phi^{a:}{}_{\alpha\beta}) = 0,$$
$$a_1, a = 1, 2, \ldots, n, \qquad \alpha, \beta = 0, 1, 2, 3,$$
$$\phi^{a:}{}_\alpha \equiv \frac{\partial \phi^a}{\partial x^\alpha}, \qquad \phi^{a:}{}_{\alpha\beta} \equiv \frac{\partial^2 \phi^a}{\partial x^\alpha\, \partial x^\beta}; \tag{1.1}$$

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Necessary and Sufficient Conditions for the Existence of a Lagrangian in Field Theory. III. Generalized Analytic Representations of Tensorial Field Equations*

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In this paper we first study the equivalence transformations of class $\mathscr{C}^2$, regular, tensorial, quasi-linear systems of field equations which (a) preserve the continuity, regularity, and quasi-linear structure of the systems; and (b) occur within a fixed system of Minkowski coordinates and field components. We identify, among the transformations of this class, those which either induce or preserve a self-adjoint structure of the field equations and we term them genotopic and isotopic transformations, respectively. We then give the necessary and sufficient conditions for an equivalence transformation of the above type to be either genotopic or isotopic. By using this methodology, we then extend the theorem on the necessary and sufficient condition for the existence of ordered direct analytic representations introduced in the preceding paper to the case of ordered indirect analytic representations in terms of the conventional Lagrange equations; we introduce a method for the construction of a Lagrangian, when it exists, in this broader context; and we explore some implications of the underlying methodology for the problem of the structure of the Lagrangian capable of representing interactions within the framework of the indirect analytic representations. Some of the several aspects which demand an inspection prior to the use of this analytic approach in actual models are pointed out. In particular, we indicate a possible deep impact in the symmetries and conservation laws of the system generated by the use of the concept of indirect analytic representation. As a preparatory step prior to the analysis of these problems, we study some methodological aspects which underlie the generalized Lagrange equations postulated in the first paper of this series for the case when they are regular, namely, when they are simple equivalence transformations of the conventional Lagrange equations. We first introduce a generalization of the action principle capable of inducing the generalized as well as the conventional equations. In this way we establish that the former equations are "bona fide" analytic equations. Finally, as our most general analytic framework for the case of unconstrained field equations, we work out the necessary and sufficient condition for the existence of ordered direct analytic representations of quasi-linear systems in terms of the generalized analytic equations and study their relationship to the conventional representations.

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