Chapter 3

LIE-ISOTOPIC BRANCH OF HADRONIC MECHANICS AND ITS ISODUAL

3.1 INTRODUCTION

3.1.1 Conceptual Foundations

As recalled in Chapter 1, the systems generally considered in the 20-th century are the conventional *exterior dynamical systems,* consisting of closed-isolated and reversible systems of constituents approximated as being point-like while moving in vacuum under sole action-at-a-distance potential interactions, as typically represented by planetary and atomic systems.

More technically, we can say that *exterior dynamical systems are characterized by the exact invariance of the Galilean symmetry for the nonrelativistic case and Poincaré symmetry for relativistic treatments,* with the consequential verification of the well known ten total conservation laws.

In this chapter we study the more general *interior dynamical systems of extended particles* and, separately, of *extended antiparticles,* consisting of systems that are also closed-isolated, thus verifying the same ten total conservation laws of the exterior systems, yet admit additional internal force of nonlocal-integral and nonpotential type due to actual contact and/or mutual penetration of particles, as it is the case for the structure of planets at the classical level (see Figure 3.1), and the structure of hadrons, nuclei, stars, and other systems at the operator level (see Figure 3.2).

To avoid excessive complexity, the systems considered in this chapter will be assumed to be *reversible,* that is, invariant under time reversal. The open-irreversible extension of the systems will be studied in the next chapter.

The most important methodological differences between exterior and interior systems are the following:

1) Exterior systems are completely represented with the knowledge of only *one* quantity, the Hamiltonian, while the representation of interior systems

Figure 3.1. A view of Jupiter, a most representative interior dynamical system, where one can see with a telescope the dramatic differences with exterior systems, such as internal exchanges of linear and angular momentum always in such a way to verify total conservation laws. As repeatedly stated in the literature on hadronic mechanics, the structure of Jupiter has been assumed as fundamental for the construction of new structure models of hadrons, nuclei and stars, and the development of their new clean energies and fuels.

requires the knowledge of the Hamiltonian for the potential forces, plus additional quantities for the representation of nonpotential forces, as done in the *true Lagrange and Hamilton equations,* those with external terms,

$$\frac{d}{dt}\frac{\partial L(t,r,v)}{\partial v_a^k} - \frac{\partial L(t,r,v)}{\partial r_a^k} = F_{ak}(t,r,v), \tag{3.1.1a}$$

$$\frac{dr_a^k}{dt} = \frac{\partial H(t,r,p)}{\partial p_{ak}}, \quad \frac{dp_{ak}}{dt} = -\frac{\partial H(t,r,p)}{\partial r_a^k} + F_{ak}(t,r,p), \tag{3.1.1b}$$

$$L = \Sigma_a \frac{1}{2} \times m_a \times v_{ak} \times v_a^k - V(t,r,v), \tag{3.1.1c}$$

$$H = \Sigma_a \frac{p_{ak} \times p_{ak}}{2 \times m_a} + V(t,r,p), \tag{3.1.1d}$$

$$V = \Sigma_a U(t,r)_{ak} \times v_a^k + U_o(t,r), \tag{3.1.1e}$$

$$F(t,r,v) = F(t,r,p/m), \tag{3.1.1f}$$

$$a = 1,2,3,\ldots,N; \quad k = 1,2,3.$$

Consequently, by their very conception, interior systems are structurally beyond the representational capability of classical and quantum Hamiltonian mechanics, in favor of covering disciplines.

2) Exterior systems are of *Keplerian type*, while interior systems are not, since they do not admit a Keplerian center (see, again, Figures 3.1 and 3.2). Consequently, also by their very conception, interior systems cannot be characterized by the Galilean and Poincaré symmetries in favor of covering symmetries.

3) Exterior systems are local-differential, that is, they describe a finite set of isolated points, thus being fully treatable with the mathematics of the 20-th century, beginning with conventional local-differential topologies. By contrast, interior systems are nonlocal-integral, that is, they admit internal interactions over finite surfaces or volumes that cannot be consistently reduced to a finite set of isolated points. Consequently, interior systems cannot be consistently treated via the mathematics of classical and quantum Hamiltonian mechanics in favor of a basically new mathematics.

4) The time evolution of the Hamiltonian treatment of exterior systems characterizes a *canonical transformation* at the classical level, and a *unitary transformation* at the operator level, that we shall write in the unified form

$$U \times U^\dagger = U^\dagger \times U = I, \tag{3.1.2}$$

where $\times$ represents the usual (associative) multiplication.[1] By contrast, the time evolution of interior systems, being non-Hamiltonian, characterizes *non-canonical transformations* at the classical level and *nonunitary transformations* at the operator level, that we shall jointly write

$$U \times U^\dagger \neq I. \tag{3.1.3}$$

In particular, the noncanonical-nonunitary character is necessary to exit from the class of equivalence of classical and quantum Hamiltonian theories.

5) The *invariance* (rather than "covariance") of exterior systems under the Galilean or Poincaré symmetry has the fundamental implication of preserving the basic units, predicting the same numerical values under the same conditions at different times, and admitting all conditions needed for consistent applications of the theory to experimental measurements. By comparison, the loss of the Galilean and Poincaré invariance, combined with the necessary noncanonical-nonunitary structure of interior systems activate the *theorems of catastrophic mathematical and physical inconsistencies* studied in Chapter 1 whenever treated with the mathematics of canonical-unitary theories.

In this chapter we report the rather long scientific journey that lead to a mathematically and physically consistent, classical and operator treatment of interior dynamical systems via the *isotopic branch of hadronic mechanics for matter,* and the *isodual isotopic branch for antimatter* including the resolution of all the above problems.

Besides a number of experimental verifications reviewed in this chapter, the achievement of a consistent treatment of interior systems offers basically new structure models of hadrons, nuclei, stars, Cooper pairs, molecules and other

[1]Since we shall use several types of multiplications, to avoid confusions, it is essential to identify the assumed multiplication in any mathematical treatment.

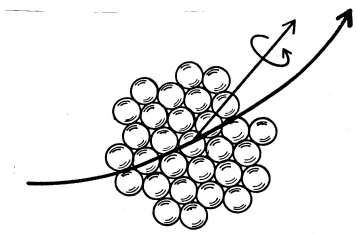

Figure 3.2. A schematic view of nuclei as they are in the physical reality, bound states of extended particles without a Keplerian center, under which conditions quantum mechanics cannot possibly be exact due to the breaking of the fundamental Galilean and Poincaré symmetries in favor of covering theories. As we shall see in this chapter, even though these breakings are small (because nucleons are in conditions of mutual penetration in nuclei of about 10^{-3} parts of their volumes), said breakings permit the prediction and industrial development of new clean energies and fuels that are prohibited by the exact validity of quantum mechanics.

interior structures. In turn, these new models permit quantitative studies of new clean energies and fuels already under industrial, let alone scientific development.

Stated in a nutshell, a primary aim of this chapter is to show that the assumption of a final character of quantum mechanics and special relativity beyond the conditions of their original conception (isolated point particles in vacuum) is the primary origin of the current alarming environmental problems.

The reader should be aware that, nowadays, the literature on hadronic mechanics is rather wast, having surpassed the mark of 15,000 pages of published research. As such, to avoid a prohibitive length, *the presentation in this chapter is restricted to the an outline of the origination* of each topic and of the *most important developments.* Scholars interested in a comprehensive list of literature are suggested to consult the quoted references as well as those of Chapter 1.

Also to avoid a prohibitive length, the presentation of this chapter is restricted to studies of direct relevance for hadronic mechanics, namely, research fundamentally dependent on a generalization of the basic unit. The quotation of related studies not fundamentally dependent on the generalization of the basic unit cannot be reviewed for brevity.

3.1.2 Closed Non-Hamiltonian Systems

The first step in the study of hadronic mechanics is the dispelling of the belief that nonpotential forces, being nonconservative, do not permit total

conservation laws, namely, that the external terms in the analytic equations (3.1.1) solely applies for open-nonconservative systems, such as an extended object moving within a resistive medium considered as external.

This belief was disproved, apparently for the first time, by Santilli in monographs [1,2]. Ref. [1] presented a comprehensive treatment of the integrability conditions for the existence of a potential or a Hamiltonian, *Helmholtz's conditions of variational selfadjointness,* according to which the total force is divided into the following two components

$$F(t, r, p, \dots) = F^{SA}(t, r, p) + F^{NSA}(t, r, p, \dots) \tag{3.1.4}$$

where the selfadjoint(SA) component F^{SA} admits a potential and the non-selfadjoint (NSA) component F^{NSA} does not.

We should also recall for clarity that, for be Newtonian as currently understood, a force should solely depend on time t, coordinates r and velocity $v = dr/dt$ or momenta $p = m \times v$, $F = F(t, r, v)$. Consequently, forces depending on derivatives of the coordinates of order bigger than the first, such as forces depending on the acceleration $F = F(t, r, v, a)$, $a = dv/dt$, are not generally considered Newtonian forces.

Ref. [2] then presented the broadest possible realization of the conditions of variational selfadjointness via analytic equations derivable from a variational principle, and included the first known identification of *closed non-Hamiltonian systems* (Ref. [2], pages 233–236), namely, systems that violate the integrability conditions for the existence of a Hamiltonian, yet verify all ten total conservation laws of conventional Hamiltonian systems.

Let us begin by recalling the following well known property:

THEOREM 3.1.1: Necessary and sufficient conditions for a system of N particles to be closed, that is, isolated from the rest of the universe, are that the following ten conservation laws are verified along an actual path

$$\frac{dX_i(t, r, p)}{dt} = \frac{\partial X_i}{\partial b^\mu} \times \frac{db^\mu}{dt} + \frac{\partial X_i}{\partial t} = 0, \tag{3.1.5a}$$

$$X_1 = E_{tot} = H = T + V, \tag{3.1.5b}$$

$$(X_2, X_3, X_4) = \mathbf{P}_{tot} = \Sigma_a \mathbf{p}_a, \tag{3.1.5c}$$

$$(X_5, X_6, X_7) = \mathbf{J}_{tot} = \Sigma_a \mathbf{r_a} \wedge \mathbf{p}_a, \tag{3.1.5d}$$

$$(X_8, X_9, X_{10}) = \mathbf{G}_{Tot} = \Sigma_a (m_a \times \mathbf{r}_a - t \times \mathbf{p}_a), \tag{3.1.5e}$$

$$i = 1, 2, 3, \dots, 10; \quad k = 1, 2, 3; \quad a = 1, 2, 3, \dots, N.$$

It is also well known that Galilean or Poincaré invariant systems do verify the above conservation laws since the X_i quantities are the generators of the

indicated symmetries. However, in this case all acting forces are derivable from a potential and the systems are Hamiltonian.

Assume now the most general possible dynamical systems, those according to the true Lagrange's and Hamilton equations (3.1.1) where the selfadjoint forces are represented with the Lagrangian or the Hamiltonian and the non-selfadjoint forces are external.

DEFINITION 3.1.1 [2]: Closed-isolated non-Hamiltonian systems of particles are systems of $N \geq 2$ particles with potential and nonpotential forces characterized by the following equations of motion

$$\frac{db_a^\mu}{dt} = \begin{pmatrix} dr_a^k/dt \\ dp_{ka}/dt \end{pmatrix} = \begin{pmatrix} p_{ak}/m_a \\ F_{ka}^{SA} + F_{ka}^{NSA} \end{pmatrix}, \tag{3.1.6}$$

verifying all conditions (3.1.5), where the term "non-Hamiltonian" denotes the fact that the systems cannot be entirely represented with the Hamiltonian, thus requiring additional quantities, such as the external terms.

The case $n = 2$ is exceptional, yet it admits solutions, and closed non-Hamiltonian systems with $N = 1$ evidently cannot exist (because a single free particle is always Hamiltonian).

Closed non-Hamiltonian systems can be classified into:

CLASS α: systems for which Eqs. (3.1.5) are first integrals;

CLASS β: systems for which Eqs. (3.1.5) are invariant relations;

CLASS γ: systems for which Eqs. (3.1.5) are subsidiary constraints.

The case of closed non-Hamiltonian systems of antiparticles are defined accordingly.

The study of closed non-Hamiltonian systems of Classes β and γ is rather complex. For the limited scope of this presentation it is sufficient to see that interior systems of Class α exist.

THEOREM 3.1.2 [2]: Necessary and sufficient conditions for the existence of a closed non-Hamiltonian systems of Class α are that the nonselfadjoint forces verify the following conditions:

$$\sum_a \mathbf{F}_a^{NSA} \equiv 0, \tag{3.1.7a}$$

$$\sum_a \mathbf{p}_a \otimes \mathbf{F}_a^{NSA} \equiv 0, \tag{3.1.7b}$$

$$\sum_a \mathbf{r}_a \wedge \mathbf{F}_a^{NSA} \equiv 0. \tag{3.1.7c}$$

Proof. Consider first the case $N > 2$ and assume first for simplicity that $\mathbf{F}_a^{SA} = 0$. Then, the first nine conservation laws are verified when

$$\frac{\partial X_i}{\partial p_{ka}} \times F_{ka}^{NSA} \equiv 0 \tag{3.1.8}$$

in which case the 10-th conservation law, Eq. (3.1.5e), is automatically verified, and this proves the *necessity* of conditions (3.1.7) for $N > 2$.

The sufficiency of the conditions is established by the fact that Eqs. (3.1.7) consist of seven conditions on 3N unknown functions F_{ka}^{NSA}. Therefore, a solution always exists for $N \geq 3$.

The case $N = 2$ is special inasmuch as motion occurs in a plane, in which case Eqs. (3.1.7) reduce to *five* conditions on *four* functions $\mathbf{F}_{ka}^{NSA}$, and the system appears to be overdetermined. Nevertheless, solutions always exist because the verification of the first four conditions (3.1.5) automatically implies the verification of the last one, Eqs. (4.1.5e). As shown in Ref. [2], Example 6.3, pages 272–273, a first solution is given by the *non-Newtonian force*

$$\mathbf{F}_1^{NSA} = -\mathbf{F}_2^{NSA} = K \times a = K \times \frac{dv}{dt}, \tag{3.1.9}$$

where K is a constant. Another solution is given by

$$\mathbf{F}_1^{NSA} = -\mathbf{F}_2^{NSA} =$$

$$= M \times \frac{dr}{dt} \times \phi(M \times \dot{r} + V), \quad M = \frac{m_1 \times m_2}{m_1 + m_2}. \tag{3.1.10}$$

Other solutions can be found by the interested reader. The addition of a non-null selfadjoint force leaves the above proof unchanged. **q.e.d.**

The search for other solutions is recommended to readers interested in acquiring a technical knowledge iof hadronic mechanics because such solutions are indeed useful for applications. A general solution of Eqs. (3.1.7), as well as of their operator counterpart and of their isodual images for antimatter will be identified later on in this chapter after the identification of the applicable mathematics.

It should be noted that the proof of Theorem 3.1.2 is not necessary because the existence of closed non-Hamiltonian systems is established by visual observations (Figure 3.1). At any rate, the representation of Jupiter's structure via one single function, the Lagrangian or the Hamiltonian, necessarily implies the belief in the perpetual motion within physical media, due to the necessary condition that constituents move inside Jupiter with conserved energy, linear momentum and angular momentum.

As recalled in Chapter 1, whenever exposed to departures from closed Hamiltonian systems, a widespread posture is the claim that the non-Hamiltonian character of the systems is "illusory" (*sic*) because, when the systems

are reduced to their elementary constituents, all nonpotential forces "disappear" (*sic*) and conventional Hamiltonian disciplines are recovered in full.

The political-nonscientific character of the above posture is established by the following property of easy proof by any graduate student in physics:

THEOREM 3.1.3 [3]: A classical non-Hamiltonian system cannot be consistently reduced to a finite number of quantum mechanical point-like particles and, vice-versa, a finite ensemble of quantum mechanical point-like particles cannot consistently characterize a classical non-Hamiltonian system.

The above property establishes that, rather than being "illusory," *nonpotential effect originate at the deepest and most elementary level of nature.* The property also establishes the need for the identification of methods suitable for the invariant treatment of classical and operator non-Hamiltonian systems in such a way to constitute a covering of conventional Hamiltonian treatments.

This chapter is devoted to the mathematical theoretical and experimental study of classical and operator interior system of particles and antiparticles, their experimental verifications and their novel applications.

3.1.3 Need for New Mathematics

By following the main guidelines of hadronic mechanics, we adapt the mathematics to nature, rather than adapting nature to preferred mathematics. For this purpose, we shall seek a mathematics capable of representing the following main features of interior dynamical systems:

1) Points have no dimension and, consequently can only have action-a-a-distance potential interactions. Therefore, the first need for the new mathematics is the representation of the *actual, extended, generally nonspherical shape of the wavepackets and/or of the charge distribution of the particles considered,* that we shall assume in this monograph for simplicity to have the shape of spheroidal ellipsoids with diagonal form

$$Shape_a = Diag.(n_{a1}^2, n_{a2}^2, n_{a3}^2), \quad a = 1, 2, 3, \ldots, N, \tag{3.1.11}$$

with more general non-diagonal expressions not considered for simplicity, where $n_{a1}^2, n_{a2}^2, n_{a3}^2$ represent the semiaxes of the spheroidal ellipsoids assumed as *deviation* from, or normalized with respect to the perfect spheridicity

$$n_{a1}^2 = n_{a2}^2 = n_{a3}^2 = 1. \tag{3.1.12}$$

The n's are called *characteristic quantities* of the particles considered. It should be stressed that, contrary to a rather popular belief, *the n-quantities are not parameters because they represent the actual shape as derived from experimental measurements.*

To clarify this important point, by definition a "parameter" can assume any value as derived form the fit of experimental data, while this is not the case for the characteristic quantities here considered. As an example, the use for the n's of value of the order of 10^{-16} cm to represent a proton would have no physical value because the proton charge distribution is a spheroidal ellipsoid of the order of 10^{-13} cm.

2) Once particles are assumed as being extended, there is the consequential need to represent their *density.* This task can be achieved via a fourth set of quantities

$$Density_a = n_{a4}^2, \tag{3.1.13}$$

representing the *deviation* of the density of the particle considered from the density of the vacuum here assumed to be one,

$$n_{Vacuum,4}^2 = 1. \tag{3.1.14}$$

Again, n_4 is not a free parameter because its numerical value is fixed by experimental data. As an example for the case of a hadron of mass m and radius r we have the density

$$n_4^2 = \frac{m \times c^2}{\frac{4}{3} \times \pi \times r^3}, \tag{3.1.15}$$

thus establishing that n_{a4} is not a free parameter capable of assuming.

Predictably, most nonrelativistic studies can be conducted with the sole use of the space components characterizing the shape. Relativistic treatments require the additional use of the density as the forth component, resulting in the general form

$$(Shape - Density)_a = Diag.(n_{a1}^2, n_{a2}^2, n_{a3}^2, n_{a4}^2), \quad a = 1, 2, 3, \dots, N. \tag{3.1.16}$$

3) Perfectly rigid bodies exist in academic abstractions, but not in the physical reality. Therefore, the next need is for a meaningful representation of the *deformation of shape* as well as *variation of density* that are possible under interior conditions. This is achieved via the appropriate functional dependence of the characteristic quantities on the energy E_a, linear momentum p_a, pressure P and other characteristics, and we shall write

$$n_{ak} = n_{ak}(E, p, P, \dots), \quad k = 1, 2, 3, 4. \tag{3.1.17}$$

The reader is suggested to meditate a moment on the fact that Lagrangian or Hamiltonian theories simply cannot represent the actual shape and density of particles. The impossibility of representing deformations of shapes and variations of density are well known, since the pillar of contemporary relativities, the rotational symmetry, is notoriously incompatible with the theory of elasticity.

4) Once particles are represented as they are in the physical reality (extended, nonspherical and deformable), there is the emergence of the following new class of interactions nonexistent for point-particles (for which reason these interactions have been generally ignored throughout the 20-th century), namely, interactions of:

I) *contact type,* that is, due to the actual physical contact of extended particle; consequently, of

II) *zero range type,* since all contacts are dimensionless; consequently of

III) *nonpotential type,* that is, not representable with any possible action-at-a-distance potential; consequently, of

IV) *non-Hamiltonian type,* that is, not representable with any Hamiltonian; consequently, of

V) *noncanonical type* at the classical level and *nonunitary type at the operator level;* as well as of

VI) *nonlinear type,* that is, represented via nonlinear differential equations, such as depending on power of the wavefunction greater than one; and, finally, of

VII) *nonlocal-integral type.* Interactions among point-particles are local-differential, that is, reducible to a finite set of isolated points, while contact interactions among extended particles and/or their wavepackets are, by conception, nonlocal-integral in the sense of being dependent on a finite surface or volume that, as such, cannot be reduced to a finite set of isolated points (see Figure 3.3).

5) Once the above new features of interior systems have been identified, there is the need not only of their mathematical representation, but above all of their *invariant representation* in order to avoid the theorem of catastrophic inconsistencies of Chapter 1.

As an illustration, Coulomb interactions have reached their towering position in the physics of the 20-th century because *the Coulomb potential is invariant under the basic symmetries of physics,* thus predicting the same numerical values under the same conditions at different times with consequentially consistent physical applications. The same occurs for other interactions derivable from a potential)except gravitation represented with curvature as shown in Section 1.4).

Along the same lines, any representation of the extended, nonspherical and deformable character of particles, their densities and their novel nonlinear, nonlocal and nonpotential interactions cannot possibly have physical value unless it is also *invariant,* and not "covariant," again, because the latter would activate the theorems of catastrophic inconsistencies of Chapter 1.

It should be indicated that an extensive search conducted by the author in in 1978–1983 in the advanced libraries of Cambridge, Massachusetts, identified

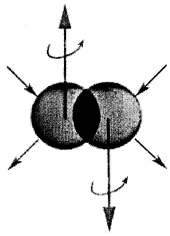

Figure 3.3. A schematic view of the fundamental interactions studied in this monograph, those originating from deep wave-overlappings of the wavepackets of particles also for the case with point-like charge as occurring in electron valence bonds, Cooper pairs in superconductivity, Pauli's exclusion principle, and other basic structures. These interactions have been ignored throughout the 20-th century, resulting in the problematic aspects or sheer inconsistencies identified in Chapter 1. As we shall see in this chapter, the representation of the new interactions here depicted with generalized units of type (3.1.19) permits the achievement of the first known, exact and invariant representation of molecular data and other data that have escaped an exact and invariant representation via quantum mechanics for about one century.

numerous integral geometries and other nonlocal mathematics. However, none of them verifies all the following conditions necessary for physical consistency:

CONDITION 1: *The new nonlocal-integral mathematics must admit the conventional local-differential mathematics as a particular case under a well identified limit procedure,* because new physical advances must be a covering of preceding results. This condition alone is not verified by any integral mathematics the author could identify.

CONDITION 2: *The new nonlocal-integral mathematics must permit the clear separation of the contributions of the new nonlocal-integral interactions from those of local-differential interactions.* This second condition too was not met by any of the integral mathematics the author could identify.

CONDITION 3: *The new nonlocal-integral mathematics must permit the invariant formulation of the new interactions.* This latter condition was also violated by all integral mathematics the author could identify, thus ruling them out in a final form for consistent physical applications.

After clarifying that the mathematics needed for the correct treatment of interior systems was absent, the author was left with no other choice than that of constructing the needed mathematics. After extensive search, Santilli [4,5] suggested as the *only* possible or otherwise known solution, the invariant representation of nonlinear, nonlocal and nonpotential interactions via a generalization of the trivial unit of conventional theories. The selection was

based on the fact that, whether conventional or generalized, the unit is the basic invariant of any theories. We reach in this way the following:

FUNDAMENTAL ASSUMPTION OF HADRONIC MECHANICS [4-10]: The actual, extended, nonspherical and deformable shape of particles, their variable densities and their nonlinear, nonlocal and nonpotential interactions can be invariantly represented with a generalization of the basic spacetime unit of conventional Hamiltonian theories

$$I = Diag.(1, 1, 1, 1), \tag{3.1.18}$$

into nowhere singular, sufficiently smooth, most general possible integro-differential forms, today called "Santilli isounit", of the type here expressed for simplicity for the case of two particles:

$$\begin{aligned} \hat{I} = \hat{I}^\dagger = \hat{I}_{1-2} = Diag.(n_{11}^2, n_{12}^2, n_{13}^2, n_{14}^2)\times \\ \times Diag.(n_{21}^2, n_{22}^2, n_{23}^2, n_{24}^2)\times \\ \times e^{\Gamma(t,r,\psi,\psi^\dagger,\dots)\times\int dr^3\times\psi^\dagger(r)\times\psi(r)} = 1/\hat{T} > 0, \end{aligned} \tag{3.1.19}$$

with trivial generalizations to multiparticle and nondiagonal forms, where the n_{ak}^2 represents the semiaxes of the spheroidal shape of particle a, n_{a4}^2 represents its density, the expression $\Gamma(t, r, \psi, \psi, \dots)$ represents the nonlinearity of the interaction and $\int dr^3 \times \psi^\dagger(r) \times \psi(r)$ provides a simple representation of its nonlocality. The corresponding features of antiparticles are represented by Santilli's isodual isounit

$$\hat{I}^d = -\hat{I}^\dagger = -\hat{I} < 0, \tag{3.1.20}$$

and mixed states of particles and antiparticles are represented by the tensorial product of the corresponding units and their isoduals.

Explicit examples of classical (operator) systems with nonpotential forces represented via generalized units will be given in Section 2.3 (Section 2.4).

As we shall see, the entire structure of hadronic mechanics follows uniquely and unambiguously from the assumption of the above basic unit. As a matter of fact, some the main features of hadronic mechanics can already be derived from the above basic assumption.

First, the maps, called in the literature *Santilli liftings*

$$I \;\rightarrow\; \hat{I}, \;\; I^d \;\rightarrow\; \hat{I}^d; \tag{3.1.21}$$

(where $I^d = -I$ is the isodual unit of Chapter 2 [8]) require two corresponding generalizations of the totality of the mathematical and physical formulations

of conventional classical and quantum Hamiltonian theories without any exception known to this author (to avoid catastrophic inconsistencies).

As we shall see in this chapter, even basic notions such as trigonometric functions, Fourier transforms, differentials, etc. have to be lifted into two form admitting the new quantity $\hat{I}$ and $\hat{I}^d$ as the correct left and right units.

In view of the assumed Hermiticity and positive-definiteness of $\hat{I}$, the resulting new mathematics is called in the literature *Santilli's isotopic mathematics* or *isomathematics* for short, with the corresponding *isodual isomathematics* for antimatter in interior conditions. The resulting new physical formulations are known as *Santilli isotopic mechanics* or *isomechanics* for short for the case of particles, with the *isodual isomechanics* for antiparticles.

Again in view of the fact that $\hat{I}$ is Hermitean and positive-definite, at the abstract, realization-free level there is no topological difference between I and $\hat{I}$ and, for this reason $\hat{I}$ is called *Santilli isotopic unit* or *isounit* for short.

Consequently, the new mathematical and physical formulations are expected to be *new realizations of the same axioms of conventional Hamiltonian mechanics,* and they should not be intended as characterizing "new theories" since they do not admit new abstract axioms. This illustrates the name of *isotopic mathematics* from the Greek meaning of preserving the topology.[2]

Finally, Santilli isounit $\hat{I}$ identifies in full the *covering* nature of isomechanics over conventional mechanics, as well as the type of resulting covering. This covering character is illustrated by the fact that at sufficiently large mutual distances of particles the integral in the exponent of Eq. (3.1.19) is null

$$\lim_{r>>1Fm} \int dr^3 \times \psi^\dagger(r) \times \psi(r) = 0, \tag{3.1.22}$$

in which case the actual shape of particles has no impact in the interactions and the generalized unit recovers the conventional unit[3]

$$\lim_{r>>Fm} \hat{I} = I = Diag.(1, 1, 1, 1). \tag{3.1.23}$$

under which limit hadronic mechanics recovers conventional quantum mechanics identically and uniquely.

The above limits also identify the important feature according to which *hadronic mechanics coincides with quantum mechanics for all mutual distances of particles sufficiently bigger than their wavepackets, while at mutual distances*

[2]When $\hat{I}$ is no longer Hermitean, we have the more general *genotopic mathematics* studied in Chapter 4.

[3]When the exponent of Eq. (3.1.19) is null, that is, when the mutual distances of particles are large, the characteristic quantities are constant and, consequently, terms such as $Diag.(n_{11}^{-2}, n_{12}^{-2}, n_{13}^{-2}, n_{14}^{-2})$ factor out of all equations, resulting in reduction (3.1.23).

below that value hadronic mechanics provides a generally small corrections to quantum mechanics (see Figure 3.3).

In this chapter we review the long and laborious scientific journey by mathematicians, theoreticians and experimentalists (see the bibliography of Chapter 1) for the achievement of maturity of formulation of the isotopic branch of hadronic mechanics, its experimental verification, its novel industrial applications, and its isodual for antimatter.

We shall begin with a review of recent developments in the construction of isomathematics that have occurred following the publication of the second edition of Vol. I of this series in 1995 [6] since these developments have important implications. We shall then identify the recent developments in physical theories occurred since the second edition of Vol. II of this series [7]. We shall then review the novel industrial applications developed since the appearance of Volumes I and II.

It should be noted that in this chapter we shall merely present recent developments. As a consequence, Volumes I and II of this series [6,7] remain useful for all detailed aspects that will not be repeated in this final volume.

A primary motivation of this volume is to present *industrial applications.* Consequently, we have selected the simplest possible mathematical treatment accessible to any experimentalists. Readers interested in utmost mathematical rigor are suggested to consult the specialized mathematical literature in the field.

Finally, the literature on the mathematics, physics and chemistry of classical and quantum Hamiltonian theories is so vast to discourage discriminatory quotations. For this reason, unless there is a contrary need, we shall abstain from quotations of works on pre-existing methods since their knowledge is a pre-requisite for the understanding of this monograph in any case.

3.2 ELEMENTS OF SANTILLI'S ISOMATHEMATICS AND ITS ISODUAL

3.2.1 Isounits, Isoproducts and Their Isoduals

As indicated earlier, *Santilli isotopic mathematics*, [4–10] or *isomathematics* for short, is characterized by the map, called *lifting,* of the trivial unit $I = +1$ into a generalized unit $\hat{I}$

N-dimensional unit

$$I = +1 \rightarrow \hat{I}(t, r, p, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \dots), \tag{3.2.1}$$

or, more generally, by the lifting of N-dimensional units

$$I = (I^i_j) = Diag.(1, 1, 1, \dots),\ i, j = 1, 2, \dots, N$$

of conventional Hamiltonian theories[4] into a nowhere singular, Hermitean and positive-definite, matrix $\hat{I}$ of the same dimension N whose elements $\hat{I}^i_j$ have an arbitrary, nonlinear and integral dependence on time t, space coordinates r, momenta p, wavefunctions ψ, their derivatives $\partial\psi$, and any other needed quantity [*loc. cit.*]

$$I \;=\; (I^i_j) = Diag.(1,1,\dots) > 0 \rightarrow$$

$$\rightarrow \hat{I} \;=\; (\hat{I}^j_i) \;=\; \hat{I}(t,r,p,\psi,\psi^\dagger,\partial\psi,\partial\psi^\dagger,\dots) \;=\; 1/\hat{T} \;>\; 0. \tag{3.2.2}$$

Isomathematics can then be defined as the lifting of all possible branches of mathematics with left and right unit I into forms admitting $\hat{I}$ has the new left and right unit.

Recall that I is the right and left unit under the conventional *associative product* $A \times B = AB$, where A, B are generic quantities (e.g., numbers, vector-fields, operators, *etc.*) for which $I \times A = A \times I = A$ for all element A of the considered set.

It is easy to see that $\hat{I}$ cannot be a unit under the same product because $\hat{I} \times A \neq A$. Therefore, for consistency, the conventional associative product $A \times B$ must be lifted into the new form first proposed by Santilli in Ref. [5] of 1978,

$$A \times B \rightarrow A \hat{\times} B = A \times \hat{T} \times B = A \times (1/\hat{I}) \times B, \tag{3.2.3}$$

where $\hat{T}$ is fixed for the set considered, under which product $\hat{I}$ is indeed the correct left and right new unit,

$$I \times A = A \times I = A \rightarrow \hat{I} \hat{\times} A = A \hat{\times} \hat{I} = A, \tag{3.2.4}$$

for all elements A of the considered set. In this case (only) $\hat{I}$ is called *Santilli's isotopic unit*, or *isounit* for short, and $\hat{T}$ is called *Santilli's isotopic element,* or *isoelement* for short.

Isomathematics was first submitted by Santilli in memoirs [*loc. cit.*] of 1978 and then worked out in various additional contributions by the same author, as well as by numerous mathematicians and theoreticians (see the references of Chapter 1 as well as of this section).

The most salient feature of Santilli's liftings (3.2.2) and (3.2.3) is that they are *axiom preserving,* from which feature they derived their name "isotopic" [*loc. cit.*], recently contracted to the prefix "iso."

In fact, $\hat{I}$ preserves the basic topological characteristics of I. Therefore, isomathematics is expected to provide *new realizations* of the abstract axioms

[4]For instance, Hamiltonian theories in 3-dimensional Euclidean space are based on the unit $I = Diag.(1,1,1)$ of the rotational and Euclidean symmetries, while Hamiltonian theories in Minkowski space are based on the unit $I = Diag.(1,1,1,1)$ that is at the foundation of Lie's theory of the Lorentz and Poincaré symmetries.

of the mathematics admitting I as left and right unit. In particular, the preservation of the original abstract axioms is an important guiding principle in the consistent construction of isomodels and their applications.

At this introductory stage the axiom-preserving character of generalized product (3.2.3) is easily verified by the fact that it preserves all basic axioms of the original product. In fact, the isoproduct verifies the *right and left isoscalar laws*

$$n\hat{\times}(A\hat{\times}B) = (n\hat{\times}A)\hat{\times}B, \tag{3.2.5a}$$

$$(A\hat{\times}B)\hat{\times}n = A\hat{\times}(B\hat{\times}n), \tag{3.2.5b}$$

the *right and left isodistributive laws*[5]

$$A\hat{\times}(B+C) = A\hat{\times}B + A\hat{\times}C, \tag{3.2.6a}$$

$$(A+B)\hat{\times}C = A\hat{\times}C + B\hat{\times}C, \tag{3.2.6b}$$

and the *isoassociative law*

$$A\hat{\times}(B\hat{\times}C) = (A\hat{\times}B)\hat{\times}C. \tag{3.2.7}$$

A verification of the preservation of the axioms of all subsequent constructions is crucial for a serious study and application of hadronic mechanics.

The simplest method for the construction of isomathematics as needed for various applications is given by the use of a positive-definite N-dimensional *noncanonical transform* at the classical level or a *nonunitary transform* at the operator level, here written in the unified form

$$U \times U^{\dagger} \neq I, \tag{3.2.8}$$

and its identification with the basic isounit of the theory

$$\hat{I} = U \times U^{\dagger} = 1/\hat{T} > 0, \tag{3.2.9}$$

realization first introduced by Santilli in Ref. [6,7] of 1993.

In this case, the Hermiticity of $\hat{I}$ is guaranteed because of the property,

$$(U \times U^{\dagger})^{\dagger} = U \times U^{\dagger}. \tag{3.2.10}$$

Therefore, realization (3.2.9) of the isounit only requires that $U \times U^{\dagger}$ be a positive-definite N-dimensional matrix other than the unit matrix, from which the nowhere singularity follows, e.g., via condition

$$\mathrm{Det}(\mathrm{U} \times \mathrm{U}^{\dagger}) > 0, \neq \mathrm{I}. \tag{3.2.11}$$

[5]The reader should keep in mind that the verification of the right and left scalar and distributive laws are necessary for any product to characterize an *algebra* as commonly understood in contemporary mathematics.

Once the fundamental realization (3.2.9) is assumed, the construction of of isomathematics follows in a simple, unique and unambiguous way. In fact, *isomathematics can be constructed by submitting conventional mathematics with left and right unit I to said noncanonical-nonunitary transform,* with very few exception, such as the isodifferential calculus that escapes construction via noncanonical-nonunitary transforms.

To begin, the isounit itself is simply given by said noncanonical-nonunitary transform of the conventional unit,

$$I \rightarrow U \times I \times U^{\dagger} = \hat{I}, \tag{3.2.12}$$

the isoproduct too is simply given by said noncanonical-nonunitary transform of the conventional product

$$A \times B \rightarrow U \times (A \times B) \times U^{\dagger} =$$

$$= (U \times A \times U^{\dagger}) \times (U \times U^{\dagger})^{-1} \times (U \times B \times U^{\dagger}) =$$

$$= \hat{A} \times \hat{T} \times \hat{B} = \hat{A} \hat{\times} \hat{B}, \tag{3.2.13}$$

and the same simple transform holds for the construction of other aspects of isomathematics, as illustrated in this section.

As a matter of fact, the use of the above transform provides a method for the construction of isomathematics that is more rigorous than empirical liftings. For instance, by comparing Eqs. (3.2.3) and (3.2.13), we see that the lifting of the unit $I \rightarrow \hat{I} = U \times I \times U^{\dagger}$ implies not only the lifting of the associative product $\times \rightarrow \hat{\times} = \times (U \times U^{\dagger})^{-1} \times$, but also the lifting of all elements of the set considered, $A \rightarrow \hat{A} = U \times A \times U^{\dagger}$.

In view of the above, the claim often expressed in the nontechnical physics literature that "the mathematics of hadronic mechanics is too difficult to comprehend" is just a case of venturing judgment without any serious knowledge of the topic.

The reader should be aware that other generalizations of the associative product, such as

$$A \bigotimes B = \hat{T} \times A \times B, \tag{3.2.14a}$$

$$A \bigodot B = A \times B \times \hat{T}, \tag{3.2.14b}$$

are unacceptable because they violate either the right or the left distributive and scalar laws, thus being unable to characterize an algebra. As such, liftings (3.2.14) are not isotopic in Santilli's sense [*loc. cit.*].

Examples of isounits have been given in Section 3.1.3. Additional examples will be provided in Sections 3.3 and 3.4. Note that, since they are Hermitean by assumption, isounits can always be diagonalized into the form of type (3.1.19).

Santilli isodual isomathematics [6-10] is the image of isomathematics under the anti-isomorphic *isodual map* of an arbitrary quantity

$$A(t, r, p, \psi, \psi^\dagger, \dots) \to A^d(t^d, r^d, p^d, \psi^d, \psi^{\dagger d}, \dots) =$$
$$\to -A^\dagger(-t, -r^t, -p^t, -\psi^\dagger, -\psi^\dagger, \dots), \tag{3.2.15}$$

(where t denotes transposed) first submitted by Santilli in Ref. [8] of 1985 (see also Chapter 2).

The basic quantity of isodual isomathematics is then the *isodual isounit*

$$\hat{I}^d = -\hat{I}^\dagger(-t, -r^\dagger, -p^\dagger, -\psi^\dagger, -\partial\psi^\dagger, \dots) = 1/\hat{T}^d. \tag{3.2.16}$$

Similarly, we have the *isodual isoproduct*

$$B^\dagger \times \hat{T}^d \times A^\dagger = B^\dagger \hat{\times}^d A^\dagger, \tag{3.2.17}$$

under which $\hat{I}^d$ is indeed the right and left unit,

$$\hat{I}^d \hat{\times}^d A = A \hat{\times}^d \hat{I}^d = A, \tag{3.2.18}$$

for all A of the considered set.

Note that, *isodual map (3.2.15) must be applied for consistency to the totality of quantities of isomathematics as well as of their operations.* As an illustration, the application of the isodual map only to the quantities A, B of a product $A \times B$ and not to the product itself $\times$, leads to a host of inconsistencies.

For this and other reasons the conventional associative product is written in this monograph with the explicit notation $A \times B$ rather than the conventional notation AB. In fact, the latter would lead to gross misunderstandings and inconsistencies under the various liftings of hadronic mechanics.

Also, the construction of isomathematics is indeed recommended for physicists to be done via a noncanonical-nonunitary transform (3.2.9), while the construction of isodual isomathematics is recommended via the isodual map (3.2.15) and not via the use of an anti-isomorphic transform.

In fact, the use of anti-isomorphic transforms causes ambiguities in the very central issue, the achievement of equivalence of the isodual operator theory with charge conjugation due to ambiguities and other technical aspects. In turn, this occurrence illustrates the significance and uniqueness of Santilli isodual map (3.2.15).

Note also that isodual isomathematics preserves the axioms, not of conventional mathematics, but of the isodual mathematics of Chapter 2, that with the simplest possible isounit unit $I^d = -I$.

Needless to say, mathematicians do not need the above elementary construction of isomathematics and its isodual since they can be formulated on abstract realization-free grounds from basic axioms.

3.2.2 Isonumbers, Isofields and Their Isoduals

The first necessary isotopic lifting following that of the basic unit and product, is that of ordinary numbers. The resulting new numbers were first presented by Santilli at the 1980 meeting in Clausthal, Germany, on *Differential Geometric Methods in Mathematical Physics* and then published in a variety of papers, such as Ref. [8] of 1985, Vols. [15,16] of 1991, memoir [9] of 1993 and other works. A comprehensive presentation is available in Vol. I [6] of 1995 that also presents industrial applications of the new numbers for cryptograms and other fields. As a result of these contributions the new numbers are today known as *Santilli's isonumbers.*

The new numbers have also been studied by various authors. An important contribution has been made by E. Trell [11] in 1998 consisting in a proof of Fermat's celebrated theorem that is the simplest on record and, therefore, credibly conceivable by Fermat (as compared to other proof requiring mathematics basically unknown during Fermat's time). Unfortunately, Fermat left no record of the proof of his celebrated theorem and, therefore, there is no evidence that Fermat first studied numbers with arbitrary units. Nevertheless, Trell's proofs of Fermat's theorems remains the most plausible known to this author for being conceived during Fermat's time.

Numerous additional studies on isonumbers have been conducted by other authors. For a complete bibliography we refer interested readers to the monograph on *Santilli isonumber theory* by C.-X. Jiang [12] of 2002. Additional studies on isonumbers have occurred for their use as basis of other isostructures. Related references will be quoted in the appropriate subsequent sections.

Santilli's isonumbers have also been subjected to a generalization called *pseudo-isonumbers* identified in Ref. [9] and studies by various authors, including N. Kamiya [13] and others. However, the latter generalization violates the axioms of a field and, as such, it cannot be used for hadronic mechanics.

The reader should be aware that in this section we merely present the minimal possible properties of isonumbers sufficient for industrial applications.

Let us consider: the field $R(n, +, \times)$ of *real numbers* n with ordinary sum $+$ and product $\times$; the field $C(c, +, \times)$ of *complex numbers* $c = n_1 + i \times n_2$ where i is the imaginary unit and $n_1, n_2 \in R$; and the field $Q(q, +, \times)$ of *quaternions* $q = i_o + i_1 \times n_1 + i_2 \times n_2 + i_3 \times n_3$, where i_o is the 2-dimensional unit matrix, $i_k, k = 1, 2, 3$ are Pauli's matrices and $n_1, n_2, n_3 \in R$. These fields are hereon represented with the unified notation[6]

$$F(a, +, \times) : a = n,\ c,\ q, \tag{3.2.19}$$

[6]Octonions are not considered "numbers" because they violate the associativity property of the axioms of a field.

In this section we present first the simplest possible method for the lifting of numbers via the use of a positive-definite (thus invertible) noncanonical-nonunitary transform identified with Santilli's isounit

$$I \rightarrow \hat{I} = U \times I \times U^{\dagger} = 1/\hat{T} > 0, \quad U \times U^{\dagger} \neq I. \tag{3.2.20}$$

We shall then pass to a mathematical presentation.

The isotopic lifting of ordinary numbers is easily achieved via the above map resulting in *Santilli isonumbers* for the characterization of *matter*

$$a \rightarrow \hat{a} = U \times a \times U^{\dagger} = a \times (U \times U^{\dagger}) = a \times \hat{I}, \tag{3.2.21}$$

and related *isoproduct*

$$a \times b \rightarrow U \times (a \times b) \times U^{\dagger} = \hat{a} \times \hat{T} \times \hat{b} = \hat{a} \hat{\times} \hat{b}, \tag{3.2.22}$$

under which $\hat{I}$ is the correct right and left isounit, Eq. (3.2.4), with the element *isozero* coinciding with the ordinary zero

$$0 \rightarrow \hat{0} = U \times 0 \times U^{\dagger} \equiv 0, \tag{3.2.23}$$

and, consequently, the *isosum* coinciding with the ordinary sum,

$$a + b \rightarrow U \times (a + b) \times U^{\dagger} = \hat{a} \hat{+} \hat{b} \equiv \hat{a} + \hat{b}. \tag{3.2.24}$$

The above liftings result in: *Santilli isofield* $\hat{R}(\hat{n}, \hat{+}, \hat{\times})$ of *isoreal isonumbers*; the isofield $\hat{C}(\hat{c}, \hat{+}, \hat{\times})$ of *isocomplex isonumbers*; and the isofield $\hat{Q}(\hat{q}, \hat{+}, \hat{\times})$ of *isoquaternionic isonumbers*; hereon represented with the unified notation

$$\hat{F}(\hat{a}, \hat{+}, \hat{\times}), \quad \hat{a} = \hat{n}, \hat{c}, \hat{q}. \tag{3.2.25}$$

Needless to say, the liftings of the unit and of the product require a corresponding lifting of all conventional operations of numbers depending on the multiplication. By using the above noncanonical-nonunitary map, one can easily prove the *isopowers*

$$\hat{a}^{\hat{n}} = \hat{a} \hat{\times} \hat{a} \hat{\times} \ \ldots \ \hat{\times} \hat{a} \ \ (n \ \text{times}) = a^n \times \hat{I}. \tag{3.2.26}$$

An important particular case is the property that *isopowers of the isounits reproduce the isounit identically,*

$$\hat{I}^{\hat{n}} = \hat{I} \hat{\times} \hat{I} \hat{\times} \ldots \hat{\times} \hat{I} \equiv \hat{I}. \tag{3.2.27}$$

Similarly we have the *isosquare isoroot*

$$\hat{a}^{\hat{1/2}} = a^{1/2} \times \hat{I}^{1/2}; \tag{3.2.28}$$

the *isoquotient*

$$\hat{a}\hat{/}\hat{b} = (\hat{a}/\hat{b}) \times \hat{I} = (a/b) \times \hat{I}; \tag{3.2.29}$$

and the *isonorm*

$$\hat{|}\hat{a}\hat{|} = |a| \times \hat{I}, \tag{3.2.30}$$

where $|a|$ is the conventional norm. All these properties were first introduced by Santilli in Refs. [6–9]. The reader can now easily construct the desired isotopic image of any other operation on numbers.

Despite their simplicity, isonumbers are nontrivial. As an illustration, the assumption of the isounit $\hat{I} = 3$ implies that "2 multiplied by 3" = 18, while 4 becomes a prime number.

The best way to illustrate the nontriviality of the new numbers is to indicate the **industrial applications of Santilli's isonumbers,** that are a primary objective of this monograph as indicated earlier.

To begin, *all* applications of hadronic mechanics are based on isonumbers, and they will be presented later on in this chapter. In addition to that, *Santilli's isonumbers have already found a direct industrial application consisting of the isotopic lifting of cryptograms used by the industry to protect secrecy, including banks, credit cards. etc.* This industrial application was first presented by Santilli in Appendix 2.C of the second edition of Vol. I [6] of 1995, and will be reviewed later on in this chapter.

At this moment we merely mention that all cryptograms based on the multiplication depend on only one value of the unit, the quantity +1 dating back to biblical times. A mathematical theorem establishes that a solution of any cryptogram can be identified in a finite period of time. As a result of this occurrence, banks and other industries are forced to change continuously their cryptograms to properly protect their secrecy.

By comparison, *Santilli's isocryptograms* are based on the isoproduct and, as such, they admit an *infinite number of possible isounits,* such as, for instance, the values

$$\hat{I} = 7.2; \;\; 0.98364; \;\; 236; \;\; 1,293' \;\; 576; \;\; etc. \tag{3.2.31}$$

Consequently, it remains to be seen whether Santilli isocryptograms can be broken in a finite period of time under the availability of an infinite number of possible isounits.

Independently from that, with the use of isocryptograms banks and other industries do not have to change the entire cryptogram for security, but can merely change the value of the isounit to keep ahead of possible hackers, and even that process can be computerized for frequent automatic changes of the isounit, with clearly added safety.

Finally, another application of Santilli isocryptograms permitted by their simplicity is their use to protect the access to personal computers.

It is hoped this illustrates the industrial significance of Santilli isonumbers *per se,* that is, independently from their basic character for hadronic mechanics.

We now pass to a mathematical presentation of the new numbers.

DEFINITION 3.2.1 [9]: Let $F = F(a, +, \times)$ be a field of characteristic zero as per Definition 2.2.1. Santilli's isofields are rings $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times})$ with: elements

$$\hat{a} = a \times \hat{I}, \tag{3.2.32}$$

where $a \in F$, $\hat{I} = 1/\hat{T}$ is a positive-definite quantity generally outside F and $\times$ is the ordinary product of F; the isosum $\hat{+}$ coincides with the ordinary sum $+$,

$$\hat{a}\hat{+}\hat{b} \equiv \hat{a} + \hat{b}, \ \ \forall\, \hat{a}, \hat{b} \in \hat{F}, \tag{3.2.33}$$

consequently, the element $\hat{0} \in \hat{F}$ coincides with the ordinary $0 \in F$; and the isoproduct $\hat{\times}$ is such that $\hat{I}$ is the right and left isounit of $\hat{F}$,

$$\hat{I}\hat{\times}\hat{a} = \hat{a}\hat{\times}\hat{I} \equiv \hat{a}, \ \ \forall\, \hat{a} \in \hat{F}. \tag{3.2.34}$$

Santilli's isofields verify the following properties:

1) For each element $\hat{a} \in \hat{F}$ there is an element $\hat{a}^{-\hat{1}}$, called isoinverse, for which

$$\hat{a}\hat{\times}\hat{a}^{-\hat{1}} = \hat{I}, \forall \hat{a} \in \hat{F}; \tag{3.2.35}$$

2) The isosum is isocommutative

$$\hat{a}\hat{+}\hat{b} = \hat{b}\hat{+}\hat{a}, \tag{3.2.36}$$

and isoassociative

$$(\hat{a}\hat{+}\hat{b}) + \hat{c} = \hat{a}\hat{+}(\hat{b}\hat{+}\hat{c}), \ \ \forall \hat{a}, \hat{b}, \hat{c} \in \hat{F}; \tag{3.2.37}$$

3) The isoproduct is not necessarily isocommutative

$$\hat{a}\hat{\times}b \neq \hat{b}\hat{\times}\hat{a}, \tag{3.2.38}$$

but isoassociative

$$\hat{a}\hat{\times}(\hat{b}\hat{\times}\hat{c}) = (\hat{a}\hat{\times}\hat{b})\hat{\times}\hat{c}, \quad \forall \hat{a}, \hat{b}, \hat{c} \in \hat{F}; \tag{3.2.39}$$

4) The set $\hat{F}$ is closed under the isosum,

$$\hat{a}\hat{+}\hat{b} = \hat{c} \in \hat{F}, \tag{3.2.40}$$

the isoproduct,

$$\hat{a}\hat{\times}\hat{b} = \hat{c} \in \hat{F}, \tag{3.2.41}$$

and right and left isodistributive compositions,

$$\hat{a}\hat{\times}(\hat{b}\hat{+}\hat{c}) = \hat{d} \in \hat{F}, \tag{3.2.42a}$$

$$(\hat{a}\hat{+}\hat{b})\hat{\times}c = \hat{d} \in \hat{F}, \quad \forall \hat{a}, \hat{b}, \hat{c}, \hat{d} \in \hat{F}; \tag{3.2.42b}$$

5) The set $\hat{F}$ verifies the right and left isodistributive law

$$\hat{a}\hat{\times}(\hat{b}\hat{+}\hat{c}) = (\hat{a}\hat{+}\hat{b})\hat{\times}\hat{c} = \hat{d}, \ \ \forall \hat{a}, \hat{b}, \hat{c}, \hat{d} \in \hat{F}. \tag{3.2.43}$$

Santlli's Isofields are called of the first (second) kind when $\hat{I}$ is (is not) an element of F.

The basic axiom-preserving character of the isotopies of numbers is illustrated by the following:

LEMMA 3.2.1 [9]: Isofields of first and second kind are fields (namely, they verify all axioms of a field).

Note that the isotopic lifting does indeed change the *operation* of the multiplication but not that of the sum because the isotopies here considered do change the multiplicative unit I, but not the additive unit 0, Eq. (3.2.23). This is a crucial property of hadronic mechanics best illustrated by the following property:

LEMMA 3.2. [9]: Nontrivial liftings of the additive unit 0 *and related sum violates the axioms of a field (for which reason, they are called "pseudoisofields")*

In fact, suppose that one wants to change the value of the element 0, e.g.,

$$0 \rightarrow \hat{0} = K \neq 0, \ \ K \in F. \tag{3.2.44}$$

Then, for $\hat{0}$ to remain the new additive unit, one must alter the sum into a new form admitting $\hat{0}$ as left and right additive unit, e.g.,

$$a\hat{+}b = a + (-\hat{0}) + b, \tag{3.2.45}$$

under which

$$a\hat{+}\hat{0} = \hat{0}\hat{+}a \equiv a, \ \ \forall a \in F. \tag{3.2.46}$$

However, there is no single lifting of the product such that

$$\hat{0}\hat{\times}a \neq \hat{0}, \ \ \forall a \in F, \tag{3.2.47}$$

under which there is the loss of the distributive axiom of a field, i.e.,

$$(a\hat{+}b) \times c \neq a \times c \ \hat{+} \ b \times c. \tag{3.2.48}$$

In turn, the loss of the distributive law causes very serious physical inconsistencies, such as preventing experimental applications of the theory. Therefore, *being axiom-preserving, hadronic mechanics is solely based on the isotopic lifting of the multiplicative unit and related product, but not on any lifting of the additive unit and related sum.*

Santilli's isodual isonumbers for the characterization of *antimatter* can be uniquely and unambiguously characterized via the isodual map (3.2.15). They are characterized by the *additive and multiplicative isodual isounit*

$$\hat{0} \rightarrow \hat{0}^d \equiv 0, \tag{3.2.49a}$$

$$\hat{I}^d = -\hat{I} < 0, \tag{3.2.49b}$$

where one should recall that $\hat{I}$ is real valued and positive-definite, thus Hermitean. Isodual isonumbers are then explicitly given by

$$\hat{a}^d = -\hat{a}^\dagger = -\hat{I} \times \hat{a}^\dagger. \tag{3.2.50}$$

The isodual isonumbers were first introduced by Santilli in Ref. [8] of 1985, treated mathematically in Ref. [9] of 1993 and studied extensively in Vol. I of this series [6].

The use of the same isodual map then identifies the *isodual isosum*

$$\hat{a}^d \hat{+}^d \hat{b}^d = \hat{a}^d + \hat{b}^d, \tag{3.2.51}$$

the *isodual isoproduct*

$$(\hat{a}\hat{\times}\hat{b})^d = \hat{b}^d \times^d \hat{T}^d \times^d \hat{A}^d = -\hat{b}^d \hat{\times} \hat{a}^d = -\hat{b}^\dagger \hat{\times} \hat{a}^\dagger, \tag{3.2.52}$$

and the *isodual isonorm*

$$\hat{|}\hat{a}\hat{|}^d = -\hat{|}\hat{a}\hat{|} = -|a| \times \hat{I}. \tag{3.2.53}$$

that is always *negative-definite.*

The above liftings result in: *Santilli's isodual isofield* $\hat{R}^d(\hat{n}^d, \hat{+}^d, \hat{\times}^d)$ of *isodual isoreal isonumbers*; the isodual isofield $\hat{C}^d(\hat{c}^d, \hat{+}^d, \hat{\times}^d)$ of *isodual isocomplex isonumbers*; and the isodual isofield $\hat{Q}^d(\hat{q}^d, \hat{+}^d, \hat{\times}^d)$ of *isodual isoquaternionic isonumbers*; hereon represented with the unified notation

$$\hat{F}^d(\hat{a}^d, \hat{+}^d, \hat{\times}^d), \hat{a}^d = \hat{n}^d, \hat{c}^d, \hat{q}^d. \tag{3.2.54}$$

DEFINITION 3.2.3 [9]: Let $\hat{F}(\hat{a}, \hat{+}, \hat{\times})$ be an isofield as per Definition 3.2.1. Then Santilli isodual isofields $\hat{F}^d(\hat{a}^d, \hat{+}^d, \hat{\times}^d)$ are the image of $\hat{F}$ under the isodual map (3.2.15).

LEMMA 3.2.3 [9]: Isodual isofields are fields (that is, they verify all axioms of a field).

LEMMA 3.2.4 [9]: Isodual isofields are anti-isomorphic to isofields.

As we shall see in this chapter, the latter property, jointly with the anti-isomorphic character of the isodual map, will result to be crucial for a consistent treatment of antimatter composed of extended particles with potential and nonpotential internal forces.

The above properties establish the fact (first identified in Ref. [8] that, by no means, the axioms of a field require that the multiplicative unit to be the trivial unit $+1$, because the basic unit can be a negative-definite quantity -1 as it occurs for the isodual mathematics of Chapter 2, an arbitrary positive-definite quantity $\hat{I} > 0$ as occurring in isomathematics, or an arbitrary negative-definite quantity $\hat{I}^d < 0$ as it occurs for the isodual isomathematics.

The reader should be aware that an in depth knowledge of Santilli's isonumbers and their isoduals requires an in depth study of memoir [9] or of Chapter 2 of Vol. I of this series, Ref. [6], and that an in depth knowledge of Santilli's isonumbers theory requires a study of Jiang's monograph [12].

Finally, the reader should meditate a moment on the viewpoint expressed several times his his writing to the effect that *there cannot be really new physical theories without new mathematics, and there cannot be really new mathematics without new numbers.* The basic novelty of hadronic mechanics can, therefore, be reduced to the novelty of Santilli's isonumbers.

By remembering that all "numbers" have been fully identified centuries ago, the novelty of hadronic mechanics can be reduced to the discovery that the axioms of conventional fields admit new realizations with nonsingular, but otherwise arbitrary multiplicative units.

3.2.3 Isospaces and Their Isoduals

Following the lifting of units, products and fields, the next necessary lifting is that of N-dimensional *metric or pseudo-metric spaces* with local coordinates r and Hermitean, thus diagonalized metric m over a field F, here written in the unified notation

$$S(r, m, F): \quad r = (r^k), \quad m = [m_{ij}(r, \dots)] = Diag.(m_{11}, m_{22}, \dots, m_{NN}), \tag{3.2.55}$$
$$i, j, k = 1, 2, \dots, N,$$

basic invariant

$$r^2 = (r^i \times m_{ij} \times r^j) \times I = (r^t \times m \times r) \times I \quad \in \quad F(a, +, \times), \tag{3.2.56}$$

(where t stands for transposed) and fundamental N-dimensional unit[7]

$$I = Diag.(1, 1, \dots, 1). \tag{3.2.57}$$

As now familiar, isotopies are based on the lifting of the above N-dimensional unit via a positive-definite noncanonical-nonunitary transform in the same dimension with an otherwise unrestricted functional dependence

$$I = Diag.(1, 1, \dots, 1) \rightarrow \hat{I}(t, r, p, \psi, \psi^\dagger, \dots) =$$

$$= U \times I \times U^\dagger = 1/\hat{T} > 0, \tag{3.2.58}$$

The above liftings requires that of spaces $S(r, m, R)$ into *isotopic spaces,* or *isospaces* for short, for the treatment of *matter,* hereon denoted $\hat{S}(\hat{r}, \hat{M}, \hat{F})$, where $\hat{r}$ denotes the *isocoordinates,* and $\hat{M}$ denotes the *isometric* defined on the isofields $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times})$ of Section 3.2.2.

Isospaces were first proposal by Santilli in Ref. [14] of 1983 for the axiom-preserving isotopies of the Minkowskian spacetime and special relativity that are at the foundations of hadronic mechanics. Isospaces were then used by Santilli for the liftings of the various spacetime and internal symmetries (such as $SU(2)$, $SO(3)$, $SO(3.1)$, $SL(2.C)$, $G(3.1)$, $P(3.1)$, $SU(3)$, etc.) as studied later on in this chapter.

A comprehensive presentation of isospaces first appeared in monographs [15,16] of 1991 and in the first edition of Volumes I and II of this series, Ref. [6,7] of 1993 (see the second edition of 1995 for various upgradings). A mathematical study of isospaces by Santilli was presented in memoir [10] of 1996. In view of all these c contributions, the new spaces are today known as *Santilli's isospaces.*

Following the appearances of these contributions, isospaces have been also studied by a number of authors for both mathematical and physical applications to be studied in subsequent sections, including the definition of isocontinuity, isotopology, isomanifolds, etc. The related literature will be presented in the appropriate subsequent sections.

[7]The basic character of the unit should be recalled here. For the case of the three-dimensional Euclidean space, $I = Diag.(1, 1, 1)$ is not only the basic geometric unit, but also the unit of the entire Lie theory of the rotational and Euclidean symmetries. Similarly, for the case of the Minkowski spacetime, the unit $I = Diag.(1, 1, 1, 1)$ is at the foundations of the entire Lie theory for the Lorentz and Poincaré symmetries. We begin to see in this way the far reaching implications of isotopic generalization of the basic unit.

In this section we identify the basic notions of Santilli isospaces. Specific types of isospaces needed for applications will be studied in subsequent sections.

The coordinates r of ordinary spaces $S(r, m, F)$ are defined on the base field $F = F(a, +, \times)$, thus being real numbers for $F = R$, complex numbers for $F = C$ and quaternionic numbers for $F = Q$.

Consequently, the *isocoordinates* $\hat{r}$ on isospaces $\hat{S}(\hat{r}, \hat{m}, \hat{F})$ must be defined on the isofields $\hat{F} = \hat{F}(\hat{a}, \hat{+}, \hat{\times})$, namely, must be *isonumbers* and, more particularly, be isoreal isonumbers for $\hat{F} = \hat{R}$, isocomplex isonumbers for $\hat{F} = \hat{C}$, and isoquaternionic isonumbers for $\hat{F} = \hat{Q}$.

Since isocoordinates are isonumbers, they can be easily constructed via the same lifting used for isonumbers, resulting in the simple definition

$$r \to \hat{r} = U \times r \times U^{\dagger} = r \times (U \times U^{\dagger}) = r \times \hat{I}. \tag{3.2.59}$$

Similarly, the metric m on $S(r, m, F)$ is an ordinary matrix in N-dimension whose elements m_{ij} are functions defined on the base field F, thus being real, complex or quaternionic functions depending on the corresponding character of F.

As we shall see shortly, a necessary condition for $\hat{S}(\hat{r}, \hat{M}, \hat{F})$ to preserve the geometric axioms of $S(r, m, F)$ (that is, for $\hat{S}$ to be an isotope of S), is that, when the unit is lifted in the amount $I \to \hat{I} = 1/\hat{T}$, the metric is lifted by the *inverse* amount $m \to \hat{m} = \hat{T} \times m$, thus yielding the transform (where the diagonal character of m is taken into account)

$$m \to U^{\dagger -1} \times m \times U^{-1} = (U \times U^{\dagger})^{-1} \times m =$$

$$= \hat{T} \times m = (\hat{m}_{ij}) = (\hat{T}_i^k \times m_{kj}), \tag{3.2.60}$$

However, in this case the elements $\hat{m}_{ij}$ are not properly defined on $\hat{S}$ because they are not isonumbers on $\hat{F}$. For this purpose, the correct definition of the *isometric* is given by

$$\hat{M} = \hat{m} \times \hat{I} = (\hat{m}_{ij} \times \hat{I}) = (\hat{m}_{ij}) \times \hat{I}. \tag{3.2.61}$$

As we shall see in the next section, the above definition is independently confirmed by the isotopies of matrices. We, therefore, have the following

DEFINITION 3.2.3 [14]: Let $S(r, m, F)$ be an N-dimensional metric or pseudo-metric space with contravariant coordinates $r = (r^k)$, metric $m = (m_{ij})$ and invariant $r^2 = (r_k \times r^k) \times I = (r^i \times m_{ij} \times r^j) \times I$ over a field F with trivial unit I. Then, Santilli's isospaces are the N-dimensional isovector spaces

$$\hat{S}(\hat{r}, \hat{M}, \hat{F}) : \quad \hat{r} = (\hat{r}^k) = (r^k) \times \hat{I} \ \in \hat{F}, \tag{3.2.62a}$$

$$\hat{M} = (\hat{T} \times m) \times \hat{I} = (T_i^k \times m_{ki}) \times \hat{I} \ \in \ \hat{F}, \ \ \hat{M}^{ij} = [(\hat{M}_{pq})^{-1}]^{ij} \ \in \hat{F}, \quad (3.2.62b)$$

$$\hat{r}^k = \hat{M}^{ki} \hat{\times} \hat{r}_i = \hat{m}^{ki} \times r_i \times \hat{I}, \quad \hat{r}_k = \hat{M}_{ki} \hat{\times} \hat{r}^i = \hat{m}_{ki} \times r^i \times \hat{I}, \quad (3.2.62c)$$

$$\hat{r}^{\hat{2}} = \hat{r}^k \hat{\times} \hat{r}_k = \hat{r}^i \hat{\times} \hat{M}_{ij} \hat{\times} \hat{r}^j = (r^i \times \hat{m}_{ij} \times r^j) \times \hat{I} \ \in \ \hat{F}, \quad (3.2.62d)$$

$$i, j, k, p, q = 1, 2, \ldots, N,$$

and its projection on the original space $S(r, m, F)$, is characterized by

$$\hat{S}(r, \hat{m}, F): \ \ r = (r^k) = (r^k) \times I \ \in \ F; \quad (3.2.63a)$$

$$\hat{m} = \hat{T} \times m = (\hat{T}_i^k \times m_{kj}) \ \in F, \ \ \hat{m}^{ij} = [(\hat{m}_{ps})^{-1}]^{ij} \ \in \ F, \quad (3.2.63b)$$

$$r^k = \hat{m}^{ki} \times r_i \ \in \ R, \ \ r_k = \hat{m}_{ki} \times r^1 \ \in \ F, \quad (3.2.63c)$$

$$r^2 = r^i \times \hat{m}_{ij} \times r^j) \times I = r^i \times (\hat{T}_i^k \times m_{kj}) \times r^j) \times I \ \in \ F. \quad (3.2.63d)$$

As one can see, expression (3.2.62) is the proper formulation of the isoinvariant on isospaces over the base isofield, and we shall write $\hat{S}(\hat{r}, \hat{M}, \hat{F})$, while expression (3.2.63) is the "projection" of the preceding space in the original space S, and we shall write $\hat{S}(r, \hat{m}, F)$, because the latter space is defined with conventional coordinates, units and products over the conventional field F by construction.

It should be stressed that *isospaces are mathematical spaces and, therefore, all physical calculations and applications will be done in the projection of isospaces over conventional spaces.* In fact, experimental measurements and events can only occur in our space time. Therefore, all physical applications of isospaces can only occur in their projection in our spacetime.

A simple visual inspection of invariants (3.2.56) and (3.2.62) establish the following

THEOREM 3.2.1 [10]: All line elements of metrics or pseudo-metric spaces with metric m and unit I, and all their isotopes possess the following invariance property

$$I \to \hat{I} = n^2 \times I, \ \ m \to \hat{m} = n^{-2} \times m, \quad (3.2.64)$$

where n is a non-null parameter.

This property too will soon acquire fundamental character, since it permits the identification, for the first time, of the property that *the Galilean and Poincaré symmetries are "eleven" dimensional,* and not ten-dimensional as believed throughout the 20-th century.

In particular, the 11-th invariance is "hidden" in conventional line elements and will permit the first and perhaps only known, axiomatically consistent

grand unification of electroweak and gravitational interactions, as studied later on in this chapter.

The nontriviality of isospaces is then expressed by the following

THEOREM 3.2.2 [14]: Even though preserving all topological properties of m (from the positive-definiteness of $\hat{I}$), the projection $\hat{m}$ of the isometric $\hat{M}$ on $\hat{S}$ over $\hat{F}$ into the original space S over F acquires an unrestricted functional dependence on any needed local variables or quantities,

$$\hat{M} \rightarrow \hat{m} = \hat{m}(t, r, p, \psi, \psi^{\dagger}, \dots). \tag{3.2.65}$$

As we shall see, the above property has truly fundamental implications, since it will permit the first and only known *geometric unification of the Minkowskian and Riemannian geometries with the consequential unification of special and general relativities,* and other applications of manifestly fundamental nature.

By recalling that the basic invariant r^2 represents the square of the "distance" in S, from Eqs. (3.2.56) and (3.2.62) we derive the following additional property

THEOREM 3.2.3 [6,7,10]: The basic invariant of a metric or pseudometric space has the structure:

$$Invariant \;\; = \;\; [Length]^2 \times [Unit]^2 \tag{3.2.66}$$

The above property will soon have deep geometric implications, such as permitting different shapes, sizes and dimension for the same object under inspection by different observers, all in a way compatible with our sensory perception.

Note that invariant structure (3.2.66) is indeed new because identified for the first time by the isotopies, since the multiplication of the invariant by the unit is trivial for conventional studies and, as such, it was ignored.

It is now important to indicate the *differences between Santilli isospaces $\hat{S}(\hat{r}, \hat{M}, \hat{F})$ or $\hat{S}(r, \hat{m}, F)$ and deformed spaces* that, as well known, are given by the sole deformations of the metric, for which we use the notation $S(r, \hat{m}, F)$.

It is easy to see that *deformed spaces $S(r, \hat{m}, F)$ have a conventional noncanonical or nonunitary structure,* thus activating the theorems of catastrophic inconsistencies of Section 3.4. By comparison, Santilli isospaces have been constructed precisely to resolve these catastrophic inconsistencies via the reconstruction of canonicity or unitarity on isospaces over isofields.

Moreover, *deformed metric spaces $S(r, \hat{m}, F)$ necessarily break the symmetries of the original spaces $S(r, m, F)$, while, as we shall soon see, isospaces $\hat{S}(\hat{r}, \hat{M}, \hat{F})$ reconstruct the exact symmetries of $S(r, g, F)$.*

The implications of the latter property alone are far reaching because *all symmetries believed to be broken in the 20-th century can be proved to remain exact on suitable isospaces over isofields.* In different terms, the "breakings of spacetime and internal symmetries" studies through the 20-th century are a direct manifestation of the adaptation of new physical events to a rather limited, pre-existent mathematics because, the underlying mathematics is suitably lifted, all believed breakings cease to exist, as already proved in Vol. II of this series [7] and updated in this volume.

Santilli's isodual isospaces for the treatment of *antimatter* are the anti-isomorphic image of isospaces under the isodual map (3.2.15) and can be written

$$\hat{S}^d(\hat{r}^d, \hat{M}^d, \hat{F}^d): \quad \hat{r}^d = -\hat{r}^\dagger, \quad \hat{M}^d = -\hat{M}, \tag{3.2.67a}$$

$$\hat{r}^{\hat{2}d} = \hat{r}^d \hat{\times}^d \hat{M}^d \hat{\times}^d \hat{r}^{t,d}. \tag{3.2.67b}$$

Isodual isospaces were introduced in Vol. I of this series [6] and then treated in various other works (see, e.g., [10,17,18]). As we shall see, they play a crucial role for the treatment of antimatter in interior conditions. The tensorial product of isospaces and their isoduals appears to be significant for basic advances in biology, e.g., to achieve a quantitative mathematical representation of bifurcations and other biological behavior.

As we shall see, all **industrial applications** of hadronic mechanics are based on isospaces to such an extend that the new isogeometries have acquired evident relevance for new patents assuredly without prior art, evidently in view of their novelty.

3.2.4 Isofunctional Analysis and its Isodual

The lifting of fields evidently requires a corresponding lifting of functional analysis into a form known as *Kadeisvili isofunctional analysis* since it was first studied by J. V. Kadeisvili [19,20] in 1992. Additional studies were done by A. K. Aringazin *et al.* [21] in 1995 and other authors.

A detailed study of isofunctional analysis was also provided in monographs [6,7] of 1995. A knowledge of these studies is necessary for any application of hadronic mechanics because all conventional functions and transforms have to be properly lifted for consistent applications, while the use of conventional (or improperly lifted) functions and transforms leads to catastrophic inconsistencies.

In essence, the consistent formulation of isofunctional analysis requires not only the preservation of the original axioms, but also the preservation of the original numerical values when formulated on isospaces over isofields, under which conditions the broadening of conventional formulations emerge in the projection of the isotopic treatment in the original space.

The latter mathematical requirement has deep physical implications, such as the preservation of the speed of light *in vacuum* as the universal invariant on *isospaces over isofield,* with consequential preservation under isotopies of all axioms of special relativity, while locally varying speeds of light within physical media emerge in the *projection* of the isospace in our spacetime, as we shall see in subsequent sections.

The scope of this section is essentially that of providing the guidelines for the updating of Refs. [19,20,16,6,7] along the above requirements so as so achieve compatibility with the main lines of this presentation.

DEFINITION 3.2.4 [19,20,21, 6,7] Let $f(x)$ be an ordinary (sufficiently smooth) function on a vector space S with local variable x (such as a coordinate) over the reals R. The isotopic image of $f(x)$, called isofunctions, can be constructed via the use of a noncanonical-nonunitary transform

$$U \times f(x) \times U^\dagger = f(x) \times \hat{I} \ \in \ \hat{F}, \tag{3.2.68}$$

reformulated on isospace $\hat{S}(\hat{x}, \hat{F})$ over the isofield $\hat{F}$

$$f(x) \times \hat{I} = f(\hat{T} \times \hat{x}) \times \hat{I} = \ \hat{f}(\hat{x}) \in \hat{F}, \tag{3.2.69}$$

with projection in the original space $S(x, F)$

$$f(\hat{T} \times x) \in F. \tag{3.2.70}$$

As one can see, expression (3.2.68) coincides with the definition of isofunction in the quoted references. A feature identified since that time is the re-interpretation in such a way that the function $f(x)$ preserves its numerical value when formulated as $\hat{f}(\hat{x})$ on the isospace $\hat{S}$ over the isofield $\hat{F}$ because the variable $\hat{x}$ is multiplied by $\hat{T}$ while the unit to which such a variable is referred to is multiplied by the *inverse* amount $\hat{I} = 1/\hat{T}$. All numerical differences emerge in the *projection* of $\hat{f}(\hat{x})$ in the original space.

This is essentially the definition of isofunctions that will allow us to preserve the basic axioms of special relativity on isospaces over isofields and actually expand their applicability from motion in empty space to motion within physical media.

For the case of the simple function $f(x) = x$ we have the lifting

$$\hat{x} = U \times x \times U^\dagger = x \times (U \times U^\dagger) = x \times \hat{I} = \hat{T} \times \hat{x} \times \hat{I} \ \in \ \hat{F}, \tag{3.2.71}$$

with the projection in the original space S being simply given in this case by $\hat{T} \times x$.

More instructive is the lifting of the exponentiation into the *isoexponentiation* given by

$$
\begin{aligned}
e^x \to U \times e^x \times U^\dagger = \\
= U \times (I + x/1! + x \times x/2! + \dots) \times U^\dagger = \\
= \hat{I} + \hat{x}\hat{/}\hat{1}! + \hat{x}\hat{\times}\hat{x}\hat{/}\hat{2}! + \dots) = \\
= \hat{e}^{\hat{x}} = (e^{\hat{x}\times\hat{T}}) \times \hat{I} = \hat{I} \times (e^{\hat{T}\times\hat{x}}) \ \in \ \hat{F},
\end{aligned}
\tag{3.2.72}
$$

with projection in the original space S given by

$$
\hat{e}^x = (e^x \times \hat{T}) \times I = I \times (e^{\hat{T}\times x}) \ \in F, \tag{3.2.73}
$$

where one should note that the function in isospace is computed over $\hat{F}$ while its projection in the original space is computed in the original; field F.

The above lifting is nontrivial because of the appearance of the nonlinear integro-differential quantity $\hat{T}(t, x, \psi, \partial\psi, \dots)$ in the exponent. As we shall see shortly, this feature permits the first known extension of the linear and local Lie theory to nonlinear and nonlocal formulations.

Let $M(x) = (M_{ij}(x))$ be an N-dimensional matrix with elements $M_{ij}(x)$ on a conventional space $S(x, F)$ with local coordinates x over a conventional field F with unit I. Then, the isotopic image of $M(x)$ or it isomatrix, is defined by

$$
\hat{M}(\hat{x}) = (\hat{M}_{ij}(\hat{x})) = M(\hat{T} \times \hat{x}) \times \hat{I}, \ \ \hat{M}_{ij} \in \hat{F}, \tag{3.2.74}
$$

Similarly, the *isodeterminant* of $\hat{M}$ is defined by

$$
\hat{\mathrm{Det}}\hat{M} = [\mathrm{Det}(\hat{T} \times M)] \times \hat{I} \tag{3.2.75}
$$

where Det represents the conventional determinant, with the preservation of the conventional axioms, e.g.,

$$
\hat{\mathrm{Det}}(\hat{M}_1 \hat{\times} \hat{M}_2) = \hat{\mathrm{Det}}(\hat{M}_1) \hat{\times} \hat{\mathrm{Det}}(\hat{M}_2); \tag{3.2.76a}
$$

$$
\hat{\mathrm{Det}}(\hat{M}^{-\hat{I}}) = (\hat{\mathrm{Det}}\hat{M})^{-\hat{I}}, \tag{3.2.76b}
$$

Note that, by construction, isomatrices and isodeterminant preserve the original values on isospaces over isofields, although show deviations when the same quantities are observed from the original space, that is, referred to the original unit.

Similarly, the *isotrace* of $\hat{M}$ is defined by[8]

$$
\hat{Tr}\hat{M} = [Tr(\hat{T} \times M)] \times \hat{I}, \tag{3.2.77}
$$

[8]The isodeterminant introduced in Ref. [6], Eq. (6.3.19) is the correct form as in Eq. (3.2.77) above. However, the isotrace introduced in Eq. (6.3.20a) of Ref. [6] preserves the axioms of a trace, but not its value, as a consequence of which it is not fully invariant, the correct definition of isotrace being given by Eq. (3.2.77) above.

where Tr is the conventional trace, and it also verifies the conventional axioms, such as

$$\hat{Tr}(\hat{M}_1 \hat{\times} \hat{M}_2) = \hat{Tr}\hat{M}_1 \hat{\times} \hat{Tr}\hat{M}_2, \tag{3.2.78a}$$

$$\hat{Tr}(\hat{M}^{-\hat{I}}) = (\hat{Tr}\hat{M})^{-\hat{I}}. \tag{3.2.78b}$$

The *isologarithm* is hereon defined by[9]

$$\hat{\log}_{\hat{e}} \hat{a} = \log_e a \times \hat{I}, \tag{3.2.79}$$

and admit the unique solution

$$\hat{\log}_{\hat{e}} \hat{a} = \log_e (\hat{T} \times a) \times \hat{I}, \tag{3.2.80}$$

under which the conventional axioms are preserved,

$$\hat{e}^{\hat{\log}_{\hat{e}} \hat{a}} = \hat{a}, \tag{3.2.81a}$$

$$\hat{\log}_{\hat{e}} \hat{e} = \hat{I}, \;\; \hat{\log}_{\hat{e}} \hat{I} = 0, \tag{3.2.81b}$$

$$\hat{\log}_{\hat{e}} (\hat{a} \hat{\times} \hat{b}) = \hat{\log}_{\hat{e}} \hat{a} + \hat{\log}_{\hat{e}} \hat{b}, \tag{3.2.81c}$$

$$\hat{\log}_{\hat{e}} (\hat{a} \hat{/} \hat{b}) = \hat{\log}_{\hat{e}} \hat{a} - \hat{\log}_{\hat{e}} \hat{b}, \tag{3.2.81d}$$

$$\hat{\log}_{\hat{e}} (\hat{a}^{-\hat{I}}) = -\hat{\log}_{\hat{e}} \hat{a}, \tag{3.2.81e}$$

$$\hat{b} \hat{\times} \hat{\log}_{\hat{e}} \hat{a} = \hat{\log}_{\hat{e}} (\hat{a}^{\hat{b}}). \tag{3.2.81f}$$

The lifting of trigonometric functions is intriguing and instructive (see Chapter 6 of Ref. [6] and Chapter 5 of Ref. [7] whose results in this case require no upgrading). Let $E(r, \delta, R)$ be a conventional two-dimensional Euclidean space with coordinates $r = (x, y)$ on the reals R and polar representation $x = r \times \cos\theta$ and $y = r \times \sin\theta$, $x^2 + y^2 = r^2 \times (\cos^2\theta + \sin^2\theta) = r^2$. Consider now the *isoeuclidean space* in two dimension

$$\hat{E}(\hat{r}, \hat{\delta}, \hat{R}) : \;\; \hat{\delta} = Diag.(n_1^{-2}, n_2^{-2}), \;\; \hat{I} = Diag.(n_1^2, n_2^2), \tag{3.2.82a}$$

$$\hat{r}^{\hat{2}} = (x^2/n_1^2 + y^2/n_2^2) \times \hat{I} \;\; \in \;\; \hat{R}. \tag{3.2.82b}$$

Then, the *isopolar coordinates* and related *isotrigonometric functions* on $\hat{E}$ are defined by

$$\hat{x} = \hat{r} \hat{\times} \hat{\cos\phi}, \tag{3.2.83a}$$

$$\hat{\cos\phi} = n_1 \times \cos(\phi / n_1 \times n_2), \tag{3.2.83b}$$

$$\hat{y} = \hat{r} \hat{\times} \hat{\sin\phi}, \tag{3.2.83c}$$

$$\hat{\sin\phi} = n_2 \times \sin(\phi / n_1 \times n_2), \tag{3.2.83d}$$

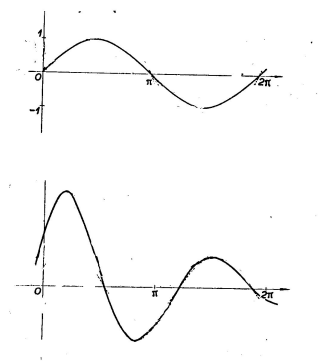

Figure 3.4. A schematic view of the conventional sinus function in Euclidean and isoeuclidean spaces (top view) and of the projection of a possible example of the isosinus function in the conventional space.

and they preserve the axioms of conventional trigonometric functions, such as,

$$\hat{r}^{\hat{2}} = (x^2/n_1^2 + y^2/n_2^2) \times \hat{I} \quad = r^2 \times \hat{I} \in \quad \hat{R}. \tag{3.2.84}$$

The isotopy of spherical coordinates are treated in detail in Section 5.5 of Ref. [7]. For self-sufficiency of this volume we recall that their definition requires a three-dimensional iso-Euclidean space

$$\hat{E}(\hat{r}, \hat{\delta}, \hat{R}): \quad \hat{\delta} = Diag.(n_1^{-2}, n_2^{-2}, n_3^{-2}), \quad \hat{I} = Diag.(n_1^2, n_2^2, n_3^2), \tag{3.2.85a}$$

$$\hat{r}^{\hat{2}} = (x^2/n_1^2 + y^2/n_2^2 + z^2/n_3^2) \times \hat{I} \quad \in \quad \hat{R}. \tag{3.2.85b}$$

The isotopies of the conventional spherical coordinates in $E(r, \delta, R)$ then yields the following *isospherical coordinates* here presented in the projected form on $\hat{E}(r, \hat{\delta}, R)$

$$x = r \times n_1 \times \sin(\theta/n_3) \times \sin(\phi/n_1 \times n_2), \tag{3.2.86a}$$

$$y = r \times n_2 \times \sin(\theta/n_3) \times \cos(\phi/n_1 \times n_2), \tag{3.2.86b}$$

$$z = r \times n_3 \times \cos(\theta/n_3). \tag{3.2.86c}$$

Via the use oif the above general rules, the reader can now construct all needed isofunctions.

The reader should meditate a moment on the isotrigonometric functions. In fact, they provide a *generalization of the Pythagorean theorem to curvilinear*

[9]Note, again, that a different definition of isologarithm was assumed in Eq. (6.7.5) of Ref. [6].

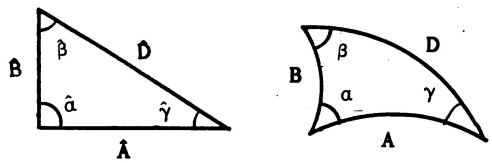

Figure 3.5. An intriguing application of isotrigonometric functions, the generalization of the conventional Pythagorean Theorem (left view) to triangles with curvilinear sides (right view). This is due to the fact that conventional triangles and the Pithagoream theorem are preserves identically on isospaces over isofields, but the projection on conventional Eyckudean spaces of straight lines in isospaces over isofields are curves. Therefore in isospace we have expressions such as $\hat{A} = \hat{D}\hat{\times}\text{isosinus}(\hat{\gamma})$ with projections in the conventional space for curvilinear sides $A = D \times \text{isosinus}(\gamma)$, where A and D are now the lengths of the curvilinear sides.

triangles. This is due to the fact that the projection of $\hat{E}(\hat{r}, \hat{\delta}, \hat{R})$ into the original space $E(r, \delta, R)$ characterizes indeed curvilinear triangles, trivially, because the n-characteristic quantities are functions.

However, the reader is suggested to verify that the *isotriangle*, that is, the image on $\hat{E}$ of an ordinary triangle on E coincides with the latter because the changes caused by the lifting are compensated by the inverse changes of the unit.

By noting that their value must be isonumbers, the *isointegral* can be defined by (here expressed for the simple case of isounits independent form the integration variable)

$$\hat{\int} \hat{d}\hat{r} = \hat{I} \times \int \hat{T} \times d(r \times \hat{I}) = \hat{I} \times \int dr, \tag{3.2.87}$$

whose extension to the case of isounits with an explicit functional dependence on the integration variables has a complexity that goes beyond the elementary level of this presentation.

Isointegrals and isoexponentiations then permit the introduction of the following *Fourier-Kadeisvili isotransforms*, first studied in Ref. [19,20] (also represented here to avoid excessive mathematical complexities for the simpler case of isounits without an explicit dependence on the integration variables)[10]

$$\hat{f}(\hat{x}) = (\hat{1}\hat{/}2\hat{\pi})\hat{\times}\hat{\int}_{-\infty}^{+\infty} \hat{g}(\hat{k})\hat{\times}\hat{e}^{\hat{i}\hat{\times}\hat{x}}\hat{d}\hat{k}, \tag{3.2.88a}$$

[10]The reader should be aware that in most applications of hadronic mechanics the isounits can be effectively approximated into constants, thus avoiding the complex mathematics needed for isointegrals and isotransforms with an explicit functional dependence on the integration variables.

$$\hat{g}(\hat{x}) = (\hat{1}\hat{/}2\hat{\pi})\hat{\times}\hat{\int}_{-\infty}^{+\infty} \hat{f}(\hat{k})\hat{\times}\hat{e}^{\hat{i}\hat{\times}\hat{x}}\hat{d}\hat{x}, \tag{3.2.88b}$$

with similar liftings for Laplace transforms, etc. Other transforms can be defined accordingly [6].

We confirm in this way a major feature of isomathematics, the fact that *Hamiltonian quantities preserve not only their axioms, but also their numerical value under isotopic lifting when defined on isospaces over isofields, and all deviations occur in the projection of the lifting into the original space.*

The explicit construction of the *isodual isofunctional analysis* is also instructive and intriguing because they reveal properties that have essentially remained unknown until recently, such as the fact that *the isofourier transforms are isoselfdual* (see also Refs. [6,7]).

3.2.5 Isodifferential Calculus and its Isodual

As indicated in Chapter 1, the delay to complete the construction of hadronic mechanics since its proposal in 1978 [5] was due to difficulties in identifying the origin of the non-invariance of its initial formulation, that is, the lack of prediction of the same numerical values for the same quantities under the same conditions, but at different times, a fundamental invariance property fully verified by quantum mechanics.

These difficulties were related to the lack of a consistent isotopic lifting of the familiar quantum mechanical momentum. More particular, *all* aspects of quantum mechanics could be consistently and easily lifted via a nonunitary transforms, except the eigenvalue equation for the linear momentum, as shown by the following lifting

$$p \times \psi(t,r) = -i \times \hbar \times \frac{\partial}{\partial r}\psi(t,r) = K \times \psi(t,r) \rightarrow$$

$$\rightarrow U \times [p \times \psi(t,r)] = (U \times p \times U^\dagger) \times (U \times U^\dagger)^{-1} \times [U \times \psi(t,r)] =$$

$$= \hat{p} \times \hat{T} \times \hat{\psi}(\hat{t},\hat{r}) = \hat{p}\hat{\times}\hat{\psi}(\hat{t},\hat{r}) =$$

$$= -i \times \hbar \times U[\frac{\partial}{\partial r}\psi(t,r)] = K \times U \times \psi(t,r) = \hat{K}\hat{\times}\hat{\psi}(\hat{t},\hat{r}), \tag{3.2.89}$$

where $\hat{K} = K \times \hat{I}$ is an isonumber.

As one can see, the initial and final parts of the lifting are elementary. The problem rested in the impossibility of achieving a consisting lifting of the intermediate step, that based on the partial derivative.

In the absence of a consistent isotopy of the linear momentum, the early studies of hadronic mechanics lacked consistent formulations of physical quantities depending on the isomomentum, such as the isotopies of angular momentum, kinetic energy, etc.

The origin of the above problem resulted in being where expected the least, in the *ordinary differential calculus,* and this explains the delay in the resolution of the impasse.

The above problem was finally resolved by Santilli in the second edition of Refs. [6,7] of 1995 (see Section 5.4.B of Vol. I and Section 8.4.A of Vol. II) with a mathematical presentation in memoir [10] of 1996. The resulting generalization of the ordinary differential calculus, today known as *Santilli's isodifferential calculus,* plays a fundamental role for these studies beginning with the first known structural generalization of Newton's equations in Newtonian mechanics, and then passing to the correct invariant formulation of all dynamical equations of hadronic mechanics.

For centuries, since its discovery by Newton and Leibnitz in the mid 1600, the ordinary differential calculus had been assumed to be independent from the basic unit and field, and the same assumption was kept in the earlier studies on hadronic mechanics, resulting in the lack of full invariance, inability to formulate physical models and other insufficiencies.

After exhausting all other possibilities, an inspection of the differential calculus soon revealed that, contrary to an erroneous belief kept in mathematics for about four centuries, the ordinary differential calculus is indeed dependent on the basic unit and related field.

In this section we review Santilli's isodifferential calculus in its version needed for applications and verifications of hadronic mechanics. This update is recommendable because of various presentations in whicb the role of $\hat{I}$ and $\hat{T}$ were interchanged, resulting in possible ambiguities that could cause loss of invariance even under the lifting of the differential calculus.

A main feature is that, *unlike all other aspects of hadronic mechanics, the isotopies of the differential calculus cannot be reached via the use of a non-canonical or nonunitary transform, and have to be built via different, yet compatible methods.*

Let $S(r, m, R)$ an N-dimensional metric or pseudo-metric space with *contravariant* coordinates $R = (r^k)$, metric $m = (m_{ij})$, $i, j, k = 1, 2, \ldots, N$, and conventional unit $I = Diag.(1, 1, \ldots, 1)$ on the reals R. Let $f(r)$ be an ordinary (sufficiently smooth) function on S, let dr^k be the differential in the local coordinates, and let $\partial f(r)/\partial r^k$ be its partial derivative.

As it is well known, the connection between covariant and contravariant coordinates is characterized by the familiar rules

$$r^k = m^{kj} \times r_j, \quad r_i = m_{ik} \times r^k, \tag{3.2.90a}$$

$$m^{ij} = [(m_{qw})^{-1}]^{ij}. \tag{3.2.90b}$$

Let $\hat{S}(\hat{r}, \hat{M}, \hat{R})$ be an isotope of S with N-dimensional isounit $\hat{I} = (\hat{I}^i_j)$, contravariant isocoordinates $\hat{r} = (r^k) \times \hat{I}$ and isometric $\hat{M} = (\hat{M}_{ij}) = (\hat{T}^s_i \times m_{sj}) \times \hat{I}$ on the isoreals $\hat{R}$.

The connection between covariant and contravariant isocoordinates is then given by

$$\hat{r}^k = \hat{M}^{kj} \hat{\times} \hat{r}_j, \quad \hat{r}_i = \hat{M}_{ik} \hat{\times} \hat{r}^k, \tag{3.2.91a}$$

$$\hat{M}^{ij} = [(\hat{M}_{qw})^{-1}]^{ij}. \tag{3.2.91b}$$

Therefore, on grounds of compatibility with the metric and subject to verifications later on on geometric grounds, we have the following:

LEMMA 3.2.5 [10]: Whenever the isounit of contravariant coordinates $\hat{r}^k$ on an isospace $\hat{S}(\hat{r}, \hat{M}, \hat{R})$ is given by

$$\hat{I} = (\hat{I}^i_j(t, r, \dots)) = 1/\hat{T} = (\hat{T}^j_i)^{-1}, \tag{3.2.92}$$

the isounit for the related covariant coordinates $\hat{r}_k$ is given by its inverse

$$\hat{T} = (\hat{T}^i_j(t, r, \dots)) = 1/\hat{I} = (\hat{I}^i_j)^{-1}, \tag{3.2.93}$$

and viceversa.

The *ordinary differential* of the contravariant isocoordinates is given by $d\hat{r}^k$ with covariant counterpart $d\hat{r}_k$ and they clearly do not constitute an isotopy. The condition for the preservation of the original axioms and value for constant isounits then leads to the following

DEFINITION 3.2.5 [6,7,10]: The isodifferentials of contravariant and covariant coordinates are given respectively by[11]

$$\hat{d}\hat{r}^k = \hat{d}(r^k \times \hat{I}) = \hat{T}^k_i \times d(r^i \times \hat{I}), \tag{3.2.94a}$$

$$\hat{d}\hat{r}_k = \hat{d}(r_k \times \hat{T}) = \hat{I}^i_k \times d(r_i \times \hat{T}). \tag{3.2.94b}$$

LEMMA 3.2.6 [loc. cit.]: For one-dimensional isounits independent from the local; variables, isodifferentials coincide with conventional differentials,

$$\hat{d}\hat{r}^k \equiv dr^k, \quad \hat{d}\hat{r}_k \equiv dr_k. \tag{3.2.95}$$

[11]It should be noted that the role of $\hat{I}$ and $\hat{T}$ in this definition and that of Ref. [10] are inverted. Also, the reader should keep in mind that, since they are assumed to be Hermitean, isounits can always be diagonalized. In fact, diagonal isounits are sufficient for the verifications and applications of hadronic mechanics, while leaving to the interested reader the formulation of hadronic mechanics according to the broader isodifferential calculus of Refs. [6,7,10].

Note that the above property constitutes a *new invariance of the differential calculus.* Its trivial character explains the reason isodifferential calculus escaped detection for centuries. Needless to say, the above triviality is lost for isounit with nontrivial functional dependence from the local variables as it is generally the case for hadronic mechanics.

The *ordinary derivative* of an isofunction oof contravariant coordinates is evidently given by

$$\frac{\partial \hat{f}(\hat{r}^k)}{\partial \hat{r}^k} = \lim_{\hat{d}\hat{r}^k \to 0} \frac{\hat{f}(\hat{r}^k + \hat{d}\hat{r}^k) - \hat{f}(\hat{r}^k)}{\hat{d}\hat{r}^k}. \tag{3.2.96}$$

with covariant version

$$\frac{\partial \hat{f}(\hat{r}_k)}{\partial \hat{r}_k} = \lim_{\hat{d}\hat{r}_k \to 0} \frac{\hat{f}(\hat{r}_k + \hat{d}\hat{r}_k) - \hat{f}(\hat{r}_k)}{\hat{d}\hat{r}_k}. \tag{3.2.97}$$

It is then simple to reach the following

DEFINITION 3.2.4 [loc. cit.]: The isoderivative of isofunctions on contravariant and covariant isocoordinates are given respectively by

$$\frac{\hat{\partial}\hat{f}(\hat{r}^k)}{\hat{\partial}\hat{r}^k} = \hat{I}^i_k \times \frac{\partial \hat{f}(\hat{r}^i)}{\partial \hat{r}^k}, \tag{3.2.98a}$$

$$\frac{\hat{\partial}\hat{f}(\hat{r}_k)}{\hat{\partial}\hat{r}_k} = \hat{T}^k_i \times \frac{\partial \hat{f}(\hat{r}_i)}{\partial \hat{r}^k}, \tag{3.2.98b}$$

where the isoquotient is tacitly assumed.[12]

A few examples are now in order to illustrate the axiom-preserving character of the isodifferential calculus. Assume that the isounit is not dependent on r. Then, for $\hat{f}(\hat{r}^k) = \hat{r}^k$ we have

$$\frac{\hat{d}\hat{r}^i}{\hat{d}\hat{r}^j} = \hat{\delta}^i_j = \delta^i_j \times \hat{I}. \tag{3.2.99}$$

Similarly we have

$$\frac{\hat{d}(\hat{r}^i)^{\hat{n}}}{\hat{d}\hat{r}^j} = \hat{\delta}^i_j \times \hat{(}r^i)^{\hat{n}-\hat{1}}. \tag{3.2.100}$$

[12]Note that the isofunction in the numerator contains an additional isounit, $\hat{f} = f \times \hat{I}$, that, however, cancels out with the isounit of the isoquotient, $\hat{/} = / \times \hat{I}$, resulting in expressions (3.2.98). Note also the lack of presence of a *factorized* isounit in the definition of the isodifferentials and isoderivatives, and this explains why the isodifferential calculus cannot be derived via noncanonical or nonunitary transforms.

It is instructive for the reader interested in learning Santilli isodifferential calculus to prove that *isoderivatives in different variables "isocommute" on isospace over isofields,*

$$\frac{\hat{\partial}}{\hat{\partial}\hat{r}^i}\frac{\hat{\partial}}{\hat{\partial}\hat{r}^j} = \frac{\hat{\partial}}{\hat{\partial}\hat{r}^j}\frac{\hat{\partial}}{\hat{\partial}\hat{r}^i}, \tag{3.2.101}$$

but their projections on ordinary spaces over ordinary fields do not necessarily "commute".

We are now sufficiently equipped to point out the completion of the construction of hadronic mechanics. First, let us verify the axiom-preserving character of the isoderivative of the isoexsponent in a contravariant coordinate for the simple case in which the isounit does not depend on the local variables. In fact, we have the expression

$$\frac{\hat{\partial}}{\hat{\partial}\hat{r}}\hat{e}^{\hat{r}} = \hat{I} \times \frac{\partial}{\partial \hat{r}}[\hat{I} \times e^{\hat{T}\times\hat{r}}] = \hat{I} \times \hat{T} \times [\hat{I} \times e^{\hat{T}\times\hat{r}}] = \hat{e}^{\hat{r}}. \tag{3.2.102}$$

Consider now the *isoplanewave* as a simply isotopy of the conventional planewave solution (again for the case in which the isounit does not depend explicitly on the local coordinates),

$$\hat{e}^{\hat{i}\hat{\times}\hat{r}\hat{\times}\hat{K}} = \hat{I} \times e^{i\times\hat{T}\times K\times\hat{r}}, \tag{3.2.103}$$

for which we have the isoderivatives

$$\frac{\hat{\partial}}{\hat{\partial}\hat{r}}\hat{e}^{\hat{i}\hat{\times}\hat{r}\hat{\times}\hat{K}} = \hat{I} \times \frac{\partial}{\partial \hat{r}}[\hat{I} \times e^{i\times\hat{T}\times K\times\hat{r}}] =$$

$$= -i \times K \times \hat{I} \times e^{i\times\hat{T}\times K\times\hat{r}} = \hat{i}\hat{\times}\hat{K}\hat{\times}\hat{e}^{\hat{i}\hat{\times}\hat{r}\hat{\times}\hat{K}}. \tag{3.2.104}$$

We reach in this way the following fundamental definition of *isomomentum*, first achieved by Santilli in Refs. [6,7] of 1995, that completed the construction of hadronic mechanics (its invariance will be proved later on in Section 3.5).

DEFINITION 3.2.7 [6,7,10]: The isolinear momentum on an iso-Hilbert space over the isofield of isocomplex numbers $\hat{C}$ (see Section 3.5 for details) is characterized by

$$\hat{p}_k\hat{\times}\hat{\psi}(\hat{t},\hat{r}) = -\hat{i}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{r}^k}\hat{\psi}(\hat{t},\hat{r}) =$$

$$= -i\hat{\times}\hat{I}^i_k \times \frac{\partial}{\partial \hat{r}^i}\hat{\psi}(\hat{t},\hat{r}) = \hat{K}\hat{\times}\hat{\psi}(\hat{t},\hat{r}). \tag{3.2.105}$$

Comparing the above formulation with Eq. (3.2.89), and in view of invariance (3.2.95),we reach the following

THEOREM 3.2.4 [6,7,10]: Planck's constant $\hbar$ is the fundamental unit of the differential calculus underlying quantum mechanics, i.e., quantum mechanical eigenvalue equations can be identically reformulated in terms of the isodifferential calculus with basic isounit $\hbar$,

$$p \times \psi(t,r) = -i \times \hbar \times \frac{\partial}{\partial r}\psi(t,r) \equiv$$

$$\equiv -i \times \frac{\hat{\partial}}{\hat{\partial} r}\psi(t,r). \tag{3.2.106}$$

In conclusion, *Santilli's isodifferential calculus establishes that the isounit not only is the algebraic unit of hadronic mechanics, but also replaces Planck's constant with an integro-differential operator $\hat{I}$, as needed to represent contact, nonlinear, nonlocal and nonpotential effects.*

More specifically, Santilli's isodifferential calculus establishes that, while in exterior dynamical systems such as atomic structures, we have the conventional quantization of energy, in interior dynamical systems such as in the structure of hadrons, nuclei and stars, we have a superposition of quantized energy level at atomic distances plus continuous energy exchanges at hadronic distances.

Needless to say, all models of hadronic mechanics will be restricted by the condition

$$\lim_{r\to\infty} \hat{I} \equiv \hbar, \tag{3.2.107}$$

under which hadronic mechanics recovers quantum mechanics uniquely and identically.

DEFINITION 3.2.8 [6,7,17]: The isodual isodifferentials are defined by

$$\hat{d}^d \hat{r}^d = (-\hat{d}^\dagger)(-\hat{r}^\dagger) = \hat{d}\hat{r}, \tag{3.2.108}$$

while isodual isoderivatives are given by

$$\hat{\partial}^d \hat{f}^d(\hat{r}^d)\hat{/}^d \hat{d}^d \hat{r}^d = -\hat{\partial}\hat{f}(\hat{r})\hat{/}\hat{d}\hat{r}. \tag{3.2.109}$$

THEOREM 3.2.5 [6,7,17]: Isodifferentials are isoselfduals.

The latter new invariance constitutes an additional, reason why the isodual theory of antimatter escaped attention during the 20-th century.

3.2.6 Isocontinuity and its Isodual

The notion of continuity on an isospace was first studied by Kadeisvili [19] in 1992 and it is today known as *Kadeisvili's isocontinuity.* A review up to 1995 was presented in monographs [6,7]. Rigorous mathematical study of isocontinuity has been done by Tsagas and Sourlas [22–23], R. M. Falcón Ganfornina and J. Núñez Valdés [24–26] and others. For mathematical studies we refer the interested reader to the latter papers. For the limited scope of this volume we shall present the notion of isocontinuity in its most elementary possible form.

Let $\hat{f}(\hat{r}) = f(\hat{T} \times \hat{r}) \times \hat{I}$ be an isofunction on an isospace $\hat{S}$ over the isofield $\hat{R}$. The *isomodulus* of said isofunction is defined by [19]

$$\hat{|}\hat{f}(\hat{r})\hat{|} = |f(\hat{T} \times \hat{r})| \times \hat{I}. \tag{3.2.110}$$

DEFINITION 3.2.9 [19,20]: An infinite sequence of isofunctions $\hat{f}_1(\hat{r})$, $\hat{f}_2(\hat{r})$, ... *is said to be "strongly isoconvergent" to the isofunction* $\hat{f}(\hat{r})$ *when*

$$\lim_{k\to\infty} \hat{|}\hat{f}_k(\hat{r}) - \hat{f}(\hat{r})\hat{|} \hat{=} \hat{0}. \tag{3.2.111}$$

while the "iso-Cauchy condition" can be defined by

$$\hat{|}\hat{f}_m(\hat{r}) - \hat{f}_n(\hat{r})\hat{|} \;\; < \;\; \hat{\delta} = \delta \times \hat{I}, \tag{3.2.112}$$

where δ *is a sufficiently small real number, and* m *and* n *are integers greater than a suitably chosen neighborhood of* δ.

The isotopies of other notions of continuity, limits, series, etc. can be easily constructed (see Refs. [6,7] for physical treatments and Refs. [22–26] for mathematical treatments).

Note that *functions that are conventionally continuous are also isocontinuous.* Similarly, *a series that is strongly convergent is also strongly isoconvergent.* However, a series that is strongly isoconvergent is not necessarily strongly convergent. We reach in this way the following important

THEOREM 3.2.6 [6,7]: Under the necessary continuity and regularity conditions, a series that is conventionally divergent can always be turned into a convergent isoform under a suitable selection of the isounit.

This mathematically trivial property has far reaching implications, e.g., the achievement, for the first time in physics, of convergent perturbative series for strong interactions, which perturbative treatments are conventionally divergent (see Section 3.4).

Similarly, the reader may be interested in knowing that, given a function which is not square-integrable in a given interval, there always exists an isotopy which turns the function into a square-integrable form [6,7]. The novelty is due to the fact that the underlying mechanism is not that of a weight function, but that of altering the underlying field.

The *isodual isocontinuity* is a simple isodual image of the preceding notions of continuity and will be hereon assumed.

3.2.7 Isotopology and its Isodual

Topology is the ultimate foundation of quantitative sciences because it identifies on rigorous mathematical grounds the limitations of the ensuing description.

Throughout the 20-th century, all quantitative sciences, including particle physics, nuclear physics, astrophysics, superconductivity, chemistry, biology, etc., have been restricted to the use of mathematics based on the conventional *local-differential topology,* with the consequence that the sole admitted representations are those dealing with a finite number of isolated point-like particles.

Since points are dimensionless, they cannot have contact interactions. Therefore, an additional consequence is that the sole possible interactions are those of action-at-a-distance type representable with a potential.

In conclusion, the very assumption of the conventional local-differential topology, such as the conventional topology for the Euclidean space, or the Zeeman topology for the Minkowski space, uniquely and unambiguously restrict the admitted systems to be local, differential and Hamiltonian.

This provided an approximation of systems that proved to be excellent whenever the mutual distances of particles are much greater than their size as it is the case for planetary and atomic systems.

However, the above conditions are the exception and not the rule in nature, because all particles have a well define extended wavepacket and/or charge distribution of the order of 10^{-13} cm. It is well known in pure and applied mathematics that the representation of the actual shape of particles is impossible with a local-differential topology.

Moreover, once particles are admitted as being extended, there is the emergence of the additional contact, zero-range nonpotential interactions that are nonlocal in the sense of occurring in a finite surface or volume that cannot be consistently reduced to a finite number of isolated points.

Consequently, it is equally know by experts that conventional local-differential topologies cannot represent extended particles at short distances and their nonlocal-nonpotential interactions, as expected in the structure of planets, strongly interacting particles, nuclei, molecules, stars and other interior dynamical systems.

The need to build a new topology, specifically conceived and constructed for hadronic mechanics was suggested since the original proposal [5] of 1978. It was not only until 1995 that the Greek mathematicians Gr. Tsagas and D. S. Sourlas [22,23] proposed the first *isotopology* on scientific record formulated on isospaces over ordinary fields. in 1996, the Italian-American physicist R. M. Santilli [10] extended the formulation to isospaces over isofields. Finally, comprehensive studies on isotopology were conducted by the Spanish Mathematicians R. M. Falcón Ganfornina and J. Núñez Valdés [24,25]. As a result, the new topology is hereon called the *Tsagas-Sourlas-Santilli-Falcón-Núñez isotopology* (or TSSFN Isotopology for short).

The author has no words to emphasize the far reaching implications of the new TSSFN isotopology because, for the first time in the history of science, mathematics can consistently represent the actual extended, generally nonspherical and deformable shape particles, their densities as well as their nonpotential and nonlocal interactions.

As an example, Newton's equations have remained unchanged in Newtonian mechanics since the time of their conception to represent point-particles. No consistent generalization was possible due to the underlying local-differential topology and related differential calculus. As we shall see in the next section, the isodifferential calculus and underlying isotopology will permit the first known structural generalization of Newton's equations in Newtonian mechanics for the representation of extended particles.

New coverings of quantum mechanics, quantum chemistry, special relativity, and other quantitative sciences are then a mere consequence. Perhaps more importantly, the new clean energies and fuels permitted by hadronic mechanics can see their origin precisely in the TSSFN isotopology, as we shall see later on in this chapter.

In their most elementary possible form accessible to experimental physicists, the main lines of the new isotopology can be summarized as follows. Being nowhere singular, Hermitean and positive-definite, N-dimensional isounits can always be diagonalized into the form

$$\hat{I} = Diag.(n_1^2, n_2^2, \dots, n_N^2), \quad n_k = n_k(t, r, v, \dots) > 0, \quad k = 1, 2, \dots, N. \tag{3.2.113}$$

Consider N isoreal isofields $\hat{R}_k(\hat{n}, \hat{+}, \hat{\times})$ each characterized by the isounit $\hat{I}_k = n_k^2$ with (ordered) Cartesian product

$$\hat{R}^N = \hat{R}_1 \times \hat{R}_2 \times \ldots \times \hat{R}_N. \tag{3.2.114}$$

Since each isofield $\hat{R}_k$ is isomorphic to the conventional field of real numbers $R(n, +, \times)$, it is evident that $\hat{R}^N$ is isomorphic to the Cartesian product of N ordinary fields

$$R^N = R \times R \times \ldots \times R. \tag{3.2.115}$$

Let

$$\tau = \{R^N, K_i\} \tag{3.2.116}$$

be the conventional *topology* on R^N (whose knowledge is here assumed for brevity), where K_i represents the subset of R^N defined by

$$K_i = \{P = (a_1, a_2, \dots, a_N)/n_i < a_1, a_2, \dots, a_N < m; \;\; n_i, m_i, a_i \in R\}. \tag{3.2.117}$$

We therefore have the following:

DEFINITION 3.2.8 [10,22-25]: The isotopology can be defined as the simple lifting on $\hat{R}^N$ of the conventional topology on R^N, and we shall simply write

$$\hat{\tau} = \{\hat{R}^N, \hat{K}_i\}, \tag{3.2.118a}$$

$$\hat{K}_i = \{\hat{P} = (\hat{a}_1, \hat{a}_2, \dots, \hat{a}_N)\hat{/}\hat{n}_i < \hat{a}_1, \hat{a}_2, \dots, \hat{a}_N < \hat{m}; \;\; \hat{n}_i, \hat{m}_i, \hat{a}_i \in \hat{R}\}. \tag{3.2.118b}$$

As one can see, the above isotopology coincides everywhere with the conventional topology of R^N except at the isounit $\hat{I}$. In particular, $\hat{\tau}$ is everywhere local-differential, except at $\hat{I}$ which can incorporate nonlocal integral terms.

It is evident that isotopology can characterize for the first time in scientific history, extended, nonspherical and deformable particles. In fact, for the case of three-dimensions in diagonal representation (3.2.113), we have the characterization of deformable spheroidal ellipsoids with variable semiaxes n_1^2, n_2^2, n_3^2 depending on local quantities, such as energy, density, pressure, etc. For the case of four-dimension the quantity n_4^2 represents, for the first time in scientific record, the density of the particle considered[13].

The reader should be aware that the above formulation of the isotopology is the simplest possible one, being restricted to the description of *one* isolated *isoparticle,* that is, an extended and nonspherical particle on isospace over isofields that, as such, has no interactions.

Consequently, numerous generalizations of the above formulations are possible and actually needed for hadronic mechanics. The first broadening is given by the case of *two* or more isoparticles in which case the basic isounit is given by the Cartesian product of two isounits of type (3.2.113). The second broadening is given by exponential factors incorporating nonlinear integral terms as in the general isounit (3.1.19). In the preceding formulation, these exponential factors have been incorporated in the n's since they are common factors.

[13]The reader is encouraged to inspect any desired textbook in particle physics and verify the complete lack of representation of the density of the particle considered.

A lesser trivial broadening of the above formulation of isotopology is given by *nondiagonal isounits* that are capable of representing nonspheroidal shapes and other complex geometric occurrences (see in Ref. [6], page 213 the case of a nondiagonal isotopy contracting the dimensions from three to one, also reviewed in the next section). The study of the latter more general formulations of isotopology is left to the interested reader.

DEFINITION 3.2.11 [22-25]: An isotopological isospace $\hat{\tau}(\hat{R}^N)$ is the isospace $\hat{R}^N$ equipped with the isotopology $\hat{\tau}$. An isocartesian isomanifold $\hat{M}(\hat{R}^N)$ is the isotopological isospace $\hat{M}(\hat{R}^N)$ equipped with a isovector structure, an isoaffine structure and the mapping

$$\hat{F} : \hat{R}^N \rightarrow \hat{R}^N; \quad \hat{a} \rightarrow \hat{f}(\hat{a}), \ \ \forall \hat{a} \in \hat{R}^N. \tag{3.2.119}$$

An iso-Euclidean isomanifold $\hat{M}(\hat{E}(\hat{r}, \hat{\delta}, \hat{R}))$ occurs when the N-dimensional isospace $\hat{E}$ is realized as the Cartesian product (3.2.106) and equipped with isotopology (3.2.118) with basic isounit (3.2.113).

The *isodual isotopology* and related notions can be easily constructed with the isodual map (3.2.15) and its explicit study is left as an instructive exercise for the interested reader.

3.2.8 Iso-Euclidean Geometry and its Isodual

The isotopies of the Euclidean space and geometry were introduced for the first time by Santilli in Ref. [14] of 1983 as a particular case of the broader isotopies of the Minkowski space and geometry treated in the next section.

The same isotopies were then studies in various works by the same author and a comprehensive treatment was presented in Chapter 5 of Vol. I [6]. These isotopies are today known as the *Euclid-Santilli isospace and isogeometry.* The presentation of Vol. I will not be repeated here for brevity. We merely limit ourselves to outline the main aspects for minimal self-sufficiency of this monograph.

Consider the fundamental isospace for nonrelativistic hadronic mechanics, the three-dimensional *Euclid-Santilli isospace* with contravariant isocoordinates $\hat{r}$, isometric $\hat{\delta}$ over the isoreals $\hat{R} = \hat{R}(\hat{n}, \hat{+}, \hat{\times})$ (see Section 3.3)

$$\hat{E}(\hat{r}, \hat{\delta}, \hat{R}) : \ \ \hat{r} = (\hat{r}^k) = (\hat{x}, \hat{y}, \hat{z}) = (r^k) \times \hat{I} = (x, y, z) \times \hat{I}, \ \ k = 1, 2, 3; \tag{3.2.120a}$$

$$\hat{I} = Diag.(n_1^2, n_2^2, n_3^2) = 1/\hat{T} > 0, \quad n_k = n_k(t, r, v, a, \mu, \tau, \dots) > 0, \tag{3.2.120b}$$

$$\hat{\Delta} = \hat{\delta} \times \hat{I}; \ \ \hat{\delta} = \hat{T} \times \delta = Diag.(n_1^{-2}, n_2^{-2}, n_3^{-2}), \tag{3.2.120c}$$

with basic isoinvarioant on $\hat{E}$

$$\hat{r}^{\hat{2}} = \hat{r}^i \hat{\times} \hat{\Delta}_{ij} \hat{\times} \hat{r}^j = \hat{r}^i \times \hat{\delta}_{ij} \times \hat{r}^j = \hat{r}^i \times (\hat{T}_i^k \times \delta_{kj}) \times \hat{r}^j =$$
$$= \hat{x}^{\hat{2}} + \hat{y}^{\hat{2}} + \hat{z}^{\hat{2}} == \frac{\hat{x}^2}{n_1^2} + \frac{\hat{y}^2}{n_2^2} + \frac{\hat{z}^2}{n_3^2} \in \hat{R}. \tag{3.2.121}$$

and projection on the conventional Euclidean space

$$r^2 = \frac{x^2}{n_1^2} + \frac{y^2}{n_2^2} + \frac{z^2}{n_3^2} \in R. \tag{3.2.121}$$

where the scalar functions n_k, besides being sufficiently smooth and positive-definite, have an unrestricted functional dependence on time t, coordinates r, velocities v, acceleration a, density μ, temperature τ, and any needed local variable.

The *Euclid-Santilli isogeometry* is the geometry of the above isospaces. A knowledge of the following main features is essential for an understanding of nonrelativistic hadronic mechanics.

Since the isospaces $\hat{E}$ are all locally isomorphic to the conventional Euclidean space $E(r, \delta, R)$, it is evident that *the Euclid-Santilli isogeometry verifies all axioms of the conventional geometry,* as proved in detail in Section 5.2 of Vol. I [6]. In fact, the conventional and isotopic geometries coincide at the abstract, realization free level to such an extent that they can be expressed with the same abstract symbols, the differences between the conventional and the isotopic geometries emerging only in the selected realizations of said abstract axioms.

Note that, while the Euclidean space and geometry are unique, there exist an infinite family of different yet isomorphic Euclid-Santilli isospaces and isogeometries, evidently characterized by different isometrics in three dimension and signature $(+, +, +)$.

Recall from Section 3.2.3 that the structure of the basic invariant is given by Eq. (3.2.66). Therefore, the *isosphere*, namely, the image on $\hat{E}$ of the perfect sphere on E remains a perfect sphere. However, the projection of the isosphere on the original space E is a spheroidal ellipsoid, as clearly indicated by invariant (3.2.121). Therefore, *the isosphere on isospace over isofields unifies all possible spheroidal ellipsoids on ordinary spaces over ordinary fields.* These features are crucial to understand later on the reconstruction of the *exact* rotational symmetry for *deformed* spheres (see Fig. 3.6).

Since the functional dependence of the isometric is unrestricted except verifying the condition of positive-definiteness, it is easy to see that *the Euclid-Santilli isogeometry unifies all possible three-dimensional geometries with the signature)* $(+, +, +)$, thus including as particular cases the Riemannian, Finslerian, non-Desarguesian and other geometries. As an example, the Riemannian metric $g_{ij}(r) = g^t$ is a trivial particular case of Santilli's isometric

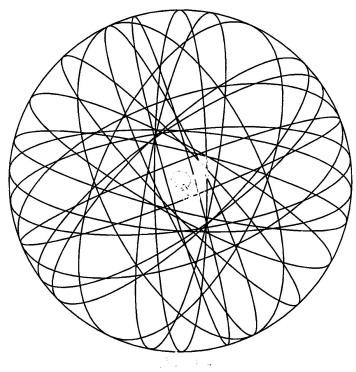

Figure 3.6. A schematic view of the "isosphere", namely, the perfect sphere on isospace over isofield represented by isoinvariant (3.2.121), that is assumed as the geometric representation of hadrons used in this monograph. The actual nonspherical and deformable shape of hadrons is obtained by projecting the isosphere in our Euclidean space, as illustrated in the last identify of Eq. (3.2.122).

$\hat{\delta}_{ij}(t, r, \dots)$. This occurrence has profound physical implications that will be pointed out in Section 3.5.

Yet another structural difference between conventional and isotopic geometries is that the former has the same unit for all three reference axes. In fact, the geometric unit $I = Diag.(1, 1, 1)$ is a dimensionless representation of the selected units, for instance, $I = diag.(1\,\text{cm}, 1\,\text{cm}, 1\,\text{cm})$. In the transition to the isospace, the units are different for different axes and we have, for instance, $\hat{I} = Diag.(n_1^2\,\text{cm}, n_2^2\,\text{cm}, n_3^2\,\text{cm})$. It then follows that *shapes detected by our sensory perception are not necessarily absolute, in the sense that they may appear basically different for an isotopic observer* (see Fig. 3.7).

Note that in the conventional space $E(r, \delta, R)$ there are two trivially different trivial units, namely, the unit $I = +1$ of the base field R and the unit $I = Diag.(1, 1, 1)$ of the space, related geometry and symmetries. The isotopies have identified for the first time the fact that *the unit of the space must coincide with the unit of the base field.*

In fact, the isounit of isospace $\hat{E}(\hat{r}, \hat{\delta}, \hat{R})$ must coincide with the isounit of the isofield $\hat{R}$. It is then evident that, at the limit $\hat{I} \rightarrow I = Diag.(1, 1, 1)$ the unit matrix $I = Diag.(1, 1, 1)$ must be the unit of both the Euclidean space and of the basic field. This implies a trivial reformulation of R that is ignored hereon.

Another important notion is that of *isodistance* between two points P_1 and P_2 on $\hat{E}$ that can be defined by the expression

$$\hat{D}_{1-2}^2 = (\hat{x}_1 - \hat{x}_2)^2/n_1^2 + (\hat{y}_1 - \hat{y}_2)^2/n_2^2 + (\hat{z}_1 - \hat{z}_2)^2/n_3^2. \tag{3.2.123}$$

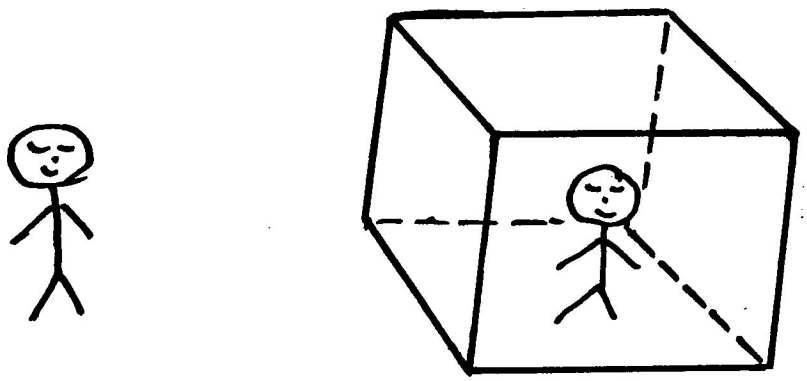

Figure 3.7. A schematic view of the "space isocube", namely, an ordinary cube inspected by two observers, an exterior observer in Euclidean space with basic units of measurements $I = Diag.(1 \text{ cm}, 1 \text{ cm}, 1 \text{ cm})$ and an interior observer on isospace with basic isounits $\hat{I} = Diag.(n_x^2 \text{ cm}, n_y^2 \text{ cm}, n_z^2 \text{ cm})$. It it then evident that, if the exterior observer measures, for instance, the sides of the cube to be $3m$, the interior observers measures different length that can be bigger or smaller than $3m$ depending on whether the isounit is smaller or bigger, respectively, than the original unit. Also, for the case of the Euclidean observer, the units in the three space directions are the same, while the corresponding isounits have different values for different directions. Therefore, the same object appears as a cube of a given size to the external observer, while having a completely different shape and size for the internal observer.

It then follows that *local alterations of the space geometry cause a change in the distance,* an occurrence first identified in Ref. [6] as originating from a lifting of the units, and today known as *isogeometric locomotion* studied in Chapter 13. We are here referring to a new form of non-Newtonian locomotion in which objects can move without the application of a force or, equivalently, without any application of the principle of action and reaction (see Figure 3.8).

Finally, it is important to point out that *the dimensionality of the original Euclidean space is not necessarily preserved under isotopies.* This occurrence constitutes another intriguing epistemological feature because isotopies are axiom-preserving. Therefore, our senses based on the three Eustachian lobes perceive no difference in dimension between a conventional and an isotopic shape.

The epistemological question raised by the isotopies is then whether our perception of space as three-dimensional is real, in the sense of being intrinsic, or it is a mere consequence of our particular sensory perception, with different dimensions occurring for other observers.[14]

The occurrence was discovered by Santilli in Ref. [6], page 213, via the following isotopic element

$$\hat{\mathbf{T}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{3.2.124}$$

that is positive definite since $\mathrm{Det}\,\hat{T} = 1$, thus being a fully acceptable isotopic element.

It is easy to see that the isoinvariant of the Euclid-Santilli isospace characterized by the above non-diagonal isotopy is given by

$$\hat{r}^{\hat{2}} = \hat{r}^i \times \hat{T}_i^k \times \delta_{kj} \times \hat{r}^j =$$
$$= \hat{x} \times \hat{z} + \hat{y} \times \hat{z} - \hat{z} \times \hat{y} = \hat{x} \times \hat{x}, \tag{3.2.125}$$

namely, in this case the *isotopic image of the three-dimensional Euclidean space is one dimensional.*

This occurrence provides another illustration of the fact that, despite their simplicity, the geometric implications of the isotopies are rather deep indeed.

The *isodual Euclid-Santilli isospace* in three dimension can be represented by the expressions

$$\hat{E}^d(\hat{r}^d, \hat{\Delta}^d, \hat{R}^d): \quad \hat{r}^d = (-\hat{x}, -\hat{y}, -\hat{z}); \tag{3.2.126a}$$

$$\hat{I}^d = Diag.(-n_1^2, -n_2^2, -n_3^2) = -1/\hat{T} > 0, \quad n_k = n_k(t, r, \dots) > 0,$$
$$\hat{\Delta}^d = \hat{\delta}^d \times \hat{I}, \quad \hat{\delta}^d = \hat{T}^d \times^d \delta^d = Diag.(-n_1^{-2}, -n_2^{-2}, -n_3^{-2}), \tag{3.2.126b}$$

with isodual isoinvariant on $\hat{R}^d$

$$\hat{r}^{d^{\hat{2}^d}} = \hat{r}^{di} \hat{\times}^d \hat{\Delta}^d_{ij} \hat{\times}^d \hat{r}^{dj} =$$
$$= -\hat{x}^{d^{\hat{2}^d}} - \hat{y}^{d^{\hat{2}^d}} - \hat{z}^{d^{\hat{2}^d}} \quad \in \quad \hat{R}^d. \tag{3.2.127}$$

and projection on the isodual Euclidean space

$$r^{d^2} = (-x^2/n_1^2 - y^2/n_2^2 - z^2/n_3^2) \times \hat{I} \quad \in \quad R^d. \tag{3.2.128}$$

A study of the *isodual Euclide-Santilli isogeometry* from Vol. I [6] is essential for a study of antimatter in interior conditions.

[14] As we shall see in Chapter 4, an even deeper epistemological issue emerges from our hyper-isotopies in which the unit is characterized by a *set* of values. In this case, space can be "three-dimensional" yet be "hyper-dimensional", in the sense that each dimension can be multi-valued.

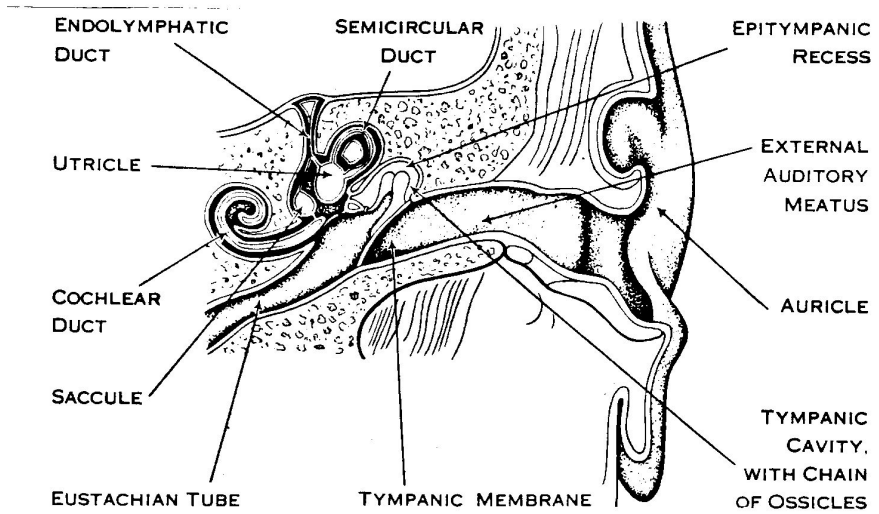

Figure 3.8. A view of the three Eustachian lobes allowing us to perceive three-dimensional shapes. The intriguing epistemological issue raised by the Euclid-Santilli isogeometry is whether living organisms with different senses perceive the same object with different shape and size than ours. As illustrated with the isobox of Figure 3.7, the same object can appear with dramatically different shapes and sizes to a conventional and an isotopic observer, as well as in dimension different than the original ones, as illustrated in the text. Another illustration of the meaning and importance of isotopies is that being axiom-preserving, different shapes, sizes and dimensions on isospaces are rendered compatible with our sensory perception.

3.2.9 Minkowski-Santilli Isogeometry and its Isodual

3.2.9A. Conceptual Foundations. The isotopies of the Minkowski space and geometry are the main mathematical methods of relativistic hadronic mechanics, because they are at the foundations of the Poincaré-Santilli isosymmetry, and related broadening of special relativity for relativistic interior dynamical systems.

The isotopies of the Minkowski space and geometry were first proposed by Santilli in Ref. [14] of 1983 and then studies in numerous papers (see monographs [6,7,14,15] and papers quoted therein) and are today known as *Minkowski-Santilli isospace and isogeometry.*

Due to their fundamental character, the new spaces and geometry were treated in great details in Refs. [6,7], particularly in the second edition of 1995, and that presentation is here assumed as known for brevity.

The primary purpose of this section is to identify the most salient advances occurred since the second edition of Refs. [6,7] with particular reference to the geometric treatment of gravitation.

In essence, the original efforts in the construction of relativistic hadronic mechanics were based on *two different isotopies,* the isotopies of the Minkowskian geometry for nongravitational profiles, and the isotopies of the Riemannian geometry for gravitational aspects. The presentation of Refs. [6,7] was based on this dual approach.

Subsequently, it became known that *the isotopies of the Riemannian geometry could not resolve the catastrophic inconsistencies of gravitation identified in Chapter 1 because they are inherent in the background Riemannian treatment itself, thus persisting under isotopies.*

The resolution of these catastrophic inconsistencies was finally reached by Santilli in Ref. [26] of 1998 via the *unification of the Minkowskian and Riemannian geometries into Minkowski-Santilli isogeometry.* In fact, the isometric of the latter geometry admit, as a particular cases, all possible Riemannian metrics.

Consequently, it became clear that the various methods used for the Riemannian geometry (such as covariant derivative, Christoffel, symbols, etc.) are inapplicable to the conventional Minkowski space evidently because flat, but the same methods are fully applicable to the Minkowski-Santilli isogeometry.

The achievement of a geometric unification of the Minkowskian and Riemannian geometries reached in memoir [26] permitted truly momentuous advances, such as the geometric unification of the special and general relativities, an axiomatically consistent grand unification of electroweak and gravitational interactions, the first known axiomatically consistent operator form of gravity, and other basic advances reviewed in Section 3.5.

3.2.9B. Minkowski-Santilli Isospaces. We now review in this subsection the foundations of the Minkowski-Santilli isospaces by referring interested readers to volumes [6,7] for details.

DEFINITION 3.2.12 [26]: Consider the conventional Minkowski space

$$M = M(x, \eta, R) : x = (x^\mu) = (r, c_o t), \tag{3.2.129a}$$

$$x^\mu = \eta^{\mu\nu} \times x_\nu, \;\; x_\mu = \eta_{\mu\nu} \times x^\nu, \tag{3.2.129}$$

where c_o is the speed of light in vacuum, metric

$$\eta = (\eta_{\mu\nu}) Diag.(+1, +1, +1, -1), \;\; \eta^{\mu\nu} = [(\eta_{\alpha,\beta})^{-1}]^{\mu\nu}, \tag{.3.2.130}$$

basic unit

$$I = Diag.(+1, +1, +1, +1), \tag{3.2.131}$$

and invariant on the reals

$$x^2 = x^\mu \times x_\mu = (x^\mu \times \eta_{\mu\nu} \times x^\nu) \times I \quad \in \quad R = R(n, +, \times), \tag{3.2.132}$$

$$\mu, \nu, \alpha, \beta = 1, 2, 3, 4.$$

Then, the Minkowski-Santilli isospaces can be defined by isotopies

$$\hat{M} = \hat{M}(\hat{x}, \hat{G}, \hat{R}) : \hat{x} = (\hat{x}^\mu) = (r, c_o t) \times \hat{I}, \tag{1.33a}$$

$$\hat{x}^\mu = \hat{G}^{\mu\nu} \hat{\times} \hat{x}_\nu, \;\; \hat{x}_\mu = \hat{G}_{\mu\nu} \hat{\times} \hat{x}^\nu, \tag{3.2.133}$$

with isometric on isospaces over isofields

$$\hat{G} = \hat{\eta} \times \hat{I} = (\hat{T}^\rho_\mu \times \eta_{\rho\nu}) \times \hat{I} =$$

$$= Diag.(\hat{T}_{11}, \hat{T}_{22}, \hat{T}_{33}, \hat{T}_{44}) \times \hat{I} \quad \in \quad \hat{R} = \hat{R}(\hat{n}, \hat{+}, \hat{\times}), \tag{3.2.134a}$$

$$\hat{G}^{\mu\nu} = [(\hat{G}_{\alpha,\beta})^{-1}]^{\mu\nu}, \tag{3.2.134b}$$

and isounit

$$\hat{I} = Diag.(\hat{T}^{-1}_{11}, \hat{T}^{-1}_{22}, \hat{T}^{-1}_{33}, \hat{T}^{-1}_{44}) \tag{3.2.135}$$

where $\hat{T}_{\mu\nu}$ are positive-definite functions of spacetime coordinates x, velocities v, accelerations a, densities μ, temperature τ, wavefunctions, their derivatives and their conjugates and any other needed quantity

$$\hat{T}_{\mu\nu} = \hat{T}_{\mu\nu}(x, v, a, \mu, \tau, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \dots) \quad > \quad 0 \tag{3.2.136}$$

isoinvariant on isospaces over the isofield of isoreal numbers

$$\hat{x}^{\hat{2}} = \hat{x}^\mu \hat{\times} \hat{x}_\mu = (\hat{x}^\mu \hat{\times} \hat{G}_{\mu\nu} \hat{\times} \hat{x}^\nu) \times I \quad \in \quad \hat{R} = R(\hat{n}, \hat{+}, \hat{\times}) \tag{3.2.137}$$

with projection in our spacetime

$$\hat{M}(x, \hat{\eta}, R) : x = (x^\mu) \times I, \tag{3.2.138a}$$

$$x^\mu = \hat{\eta}^{\mu\nu} \times x_\nu, \;\; x_\mu = \hat{\eta}_{\mu\nu} \times x^\nu, \tag{3.2.138b}$$

metric over the field of real numbers

$$\hat{\eta} = (\hat{\eta}_{\mu\nu}) =$$

$$= (\hat{T}^\rho_\mu \times \eta_{\rho\nu}) = Diag.(\hat{T}_{11}, \hat{T}_{22}, \hat{T}_{33}, \hat{T}_{44}) \quad \in \quad R = R(n, +, \times), \tag{3.2.139a}$$

$$\hat{\eta}^{\mu\nu} = [(\hat{\eta}_{\alpha,\beta})^{-1}]^{\mu\nu}, \tag{3.2.139b}$$

and invariant in our spacetime over the reals

$$x^2 = x^\mu \times x_\nu = x^\mu \times \hat{\eta}_{\mu\nu}(x, v, a, \mu, \tau, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \dots) \times x^\nu =$$

$$= T_{11} \times x_1^2 + \hat{T}_{22} \times x_2^2 + \hat{T}_{33} \times x_3^2 - \hat{T}_{44} \times x_4^2 \ \in \ R. \tag{3.2.140}$$

Note that all scalars on M must be lifted into *isoscalars* to have meaning for $\hat{M}$, i.e., they must have the structure of the isonumbers $\hat{n} = n \times \hat{I}$. This condition requires the re-definition $x \ \rightarrow \ \hat{x} = x \times \hat{I}$, $\eta_{\mu\nu} \ \rightarrow \ \hat{G}_{\mu\nu} = \hat{\eta}_{\mu\nu} \times \hat{I}$, $x^2 \rightarrow \hat{x}^{\hat{2}}$, etc.

The reader interested in learning in depth the new isogeometry should also study from the preceding sectyions the different realizations of the isometry whether realized in the original Minkowskian coordinates or in the isocoordinates, since the functional dependence is different in these two cases.

Note however the redundancy in practice for using the forms $\hat{x} = x \times \hat{I}$ and $\hat{G} = \hat{\eta} \times \hat{I}$ because of the identity $\hat{x}^{\hat{2}} = \hat{x}^{\mu} \hat{\times} \hat{G}_{\mu\nu} \hat{\times} \hat{x}^{\nu} \equiv (x^{\mu} \times \hat{\eta}_{\nu} \times x^{\nu}) \times \hat{I}$. For simplicity we shall often use the conventional coordinates x and the isometric will be referred to $\hat{\eta} = \hat{T} \times \eta$. The understanding is that the full isotopic formulations are needed for mathematical consistency.

A fundamental property of the infinite family of generalized spaces (3.2.133) is the lifting of the basic unit $I \ \rightarrow \ \hat{I}$ while the metric is lifted of the *inverse* amount, $\eta \ \rightarrow \ \hat{\eta} = \hat{T} \times \eta$, $\hat{I} = \hat{T}^{-1}$. This implies the preservation of all original axioms, and we have the following:

THEOREM 3.2.7 [26]: All infinitely possible isominkowski spaces $\hat{M}(\hat{x}, \hat{\eta}$,-$\hat{R})$ over the isofields $\hat{R}(\hat{n}, \hat{+}, \hat{\times})$ with a common positive-definite isounit $\hat{I}$ preserve all original axioms of the Minkowski space $M(x, \eta, R)$ over the reals $R(n, +, \times)$.

The nontriviality of the lifting is that *the Minkowskian axioms are preserved under an arbitrary functional dependence of the metric $\hat{\eta} = \hat{\eta}(x, v, a, \mu, \tau, \dots)$* for which the sole x-dependence of the Riemannian metric $g(x)$ is opnly a simple *particular case*. As a matter of fact, we have the following

THEOREM 3.2.8 [26]: Minkowski-Santilli isospaces are "directly universal" in spacetime, that is, they represent all infinitely possible spacetimnes with signature $(+, +, +, -)$ ("universality"), directly with the isometric and without any use of the transformation theory ("direct universality").

Note that all possible "deformations" of the Minkowski space are also particular cases of the above isospaces. However, the former are still referred to the old unit I, thus losing the isomorphic between deformed and Minkowski spaces, while the isotopies preserve the original axioms by construction.

A fundamental physical characteristic of the Minkowski-Santilli isospaces is that *it alters the units of space and time*. Recall that the unit

$$I = Diag.(\{1, 1, 1\}, 1)$$

of the Minkowski space represents in a dimensionless form the units of the three Cartesian axes and time, e.g., $I = (+1\,\text{cm}, +1\,\text{cm}, +1\,\text{cm}, +1\,\text{sec})$. Recall also that the Cartesian space-units are *equal for all axes*.

Consider now the isospaces, and recall that $\hat{I}$ is positive-definite. Consequently, we have the following lifting of the units in which the $\hat{T}_{\mu\mu}$ quantities are reinterpreted as constants

$$I = (+1\,\text{cm}, +1\,\text{cm}, +1\,\text{cm}, +1\,\text{sec}) \rightarrow$$

$$\rightarrow \hat{I} = Diag.(n_1^2, n_2^2, n_3^2, n_4^2) = 1/\hat{T},\ \hat{I}_\mu^\mu = n_\mu^2, n_\mu > 0. \tag{3.2.141}$$

This means that, not only the original units are now lifted into arbitrary positive values, but the *units of different space axes generally have different values*. Jointly, the components of the metric are lifted by the *inverse* amounts n_μ^{-2}. This implies the preservation on $\hat{M}$ over $\hat{R}$ of the original *numerical* values on M over R, including the crucial preservation of the maximal causal speed c_o, as we shall see in Section 3.5.

Note also the necessary condition that *the isospace and isofield have the same isounit* $\hat{I}$. This condition is absent in the conventional Minkowski space where the unit of the space is the unit *matrix* $I = Diag.(1, 1, 1, 1)$, while that of the underlying field is the *number* $I = +1$. Nevertheless, the latter can be trivially reformulated with the common unit matrix I, by achieving in this way the form admitted as a particular case by the covering isospaces

$$M(x, \eta, R) :\ x = \{x^\mu \times I\},\ x^2 = (x^\mu \times \eta_{\mu\nu} \times x^\nu) \times I \in R. \tag{3.2.142}$$

The structure of both the conventional and isotopic invariants is therefore given by Theorem 3.2.66, namely

$$\text{Basic Invariant} = (\text{Length})^2 \times (\text{Unit})^2. \tag{3.2.143}$$

which illustrates more clearly the preservation under the dual lifting $\eta \rightarrow \hat{\eta} = \hat{T} \times \eta$ and $I \rightarrow \hat{I} = 1/\hat{T}$ of the original axioms as well as numerical values.

THEOREM 3.2.9 [6,7,26]: Conventional and isotopic symmetries of space-time are 11-dimensional.

Proof. In addition to the 10-dimensionality of the Poincaré symmetry, there is an additional 11-th dimensionality characterized by the isotransform

$$\eta \rightarrow \hat{\eta} = \eta/n^2,\ \ I \rightarrow \hat{I} = n^2 \times I, \tag{3.2.144}$$

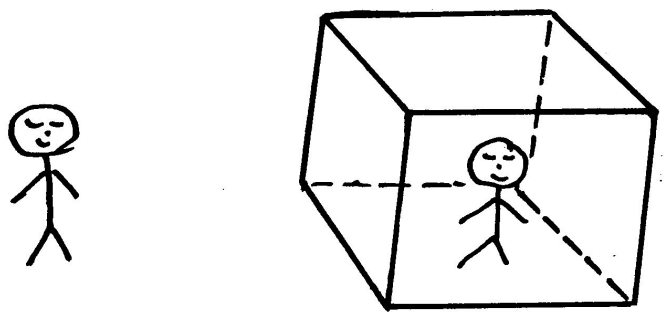

Figure 3.9. A view of the "spacetime isocube" characterized by the "space isocube" of Figure 3.7 now inspected in two spacetimes, the conventional Minkowski spacetime in the exterior and Santilli isospacetime in the interior. In addition to the variations of shape, size and dimensions indicated in Figure 3.7, the same object can be in different times for the two observers, all in a way fully compatible with our sensory perception. Consequently, seeing in a telescope a far away quasar or galaxy it does not mean that that astrophysical structure ie necessarily in our time, since it could be evolving far away in the future or in the past.

where n is a non-null constant. **q.e.d.**

Note the crucial role of Santilli's isonumbers in the above property. This explains why the 11-th dimensionality remained undiscovered throughout the 20-th century.

A significant difference between the conventional space M and its isotopes $\hat{M}$ is that the former admit only *one* formulation, the conventional one, while the latter admit *two* formulations: that on isospace itself (i.e., expressed with respect to the isounit $\hat{I}$) and its *projection* in the original space M (i.e., expressed with respect to the conventional unit I).

Note that the projection of $\hat{M}(\hat{x}, \hat{M}, \hat{R})$ into $M(x, \eta, R)$ is not a conformal map, but an *inverse isotopic map* because it implies the transition from generalized units and fields to conventional units and fields.

The axiomatic motivation for constructing the isotopies of the Minkowskian geometry is that any modification of the Minkowski metric requires the use of *noncanonical transforms* $x \rightarrow x'(x)$,

$$\eta_{\mu\nu} \rightarrow \hat{\eta}_{\mu\nu} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \eta_{\alpha\beta} \frac{\partial x'^{\beta}}{\partial x^{\nu}} \neq \eta_{\mu\nu}, \tag{3.2.145}$$

and this includes the case of the transition from the Minkowskian metric η to the Riemannian metric $g(x)$.

In turn, all noncanonical theories, thus including the Riemannian geometry, do not possess invariant units of space and time, thus having the catastrophic inconsistencies studied in Chapter 1. A primary axiomatic function of the isospace is that of restoring the invariance of the basic units, as established by the Poincaré-Santilli isosymmetry.

This is achieved by embedding all noncanonical content in the generalization of the unit. Invariant for noncanonical structures such as Riemannian metrics, is then assured by the fact indicated earlier that, whether conventional or generalized, the unit is the basic invariant of any theory.

Stated in different terms, a primary axiomatic difference between the special and general relativities is that the time evolution of the former is a *canonical transform*, thus implying the majestic mathematical and physical consistency of special relativity recalled in Chapter 1, while the time evolution of the latter is a *noncanonical transform*, thus implying a number of unresolved problematic aspects that have been lingering throughout this century.

The reformulation of the Riemannian geometry in terms of the Minkowskian axioms is the sole possibility known to this author for achieving axiomatic consistency under a nontrivial functional dependence of the metric.

In summary, Minkowski-Santilli isospaces have the following primary applications. First, they are used for a re-interpretation of the Riemannian metrics $g(x)$ for the particular case

$$\hat{\eta} = \hat{\eta}(x) = g(x) \tag{3.2.146}$$

characterizing *exterior gravitational problems in vacuum*. Second, the same isospaces are used for the characterization of *interior gravitational problems* with isometrics of unrestricted functional dependence

$$\hat{\eta} = \hat{\eta}(x, v, a, \mu, \tau, \dots) = g(x, v, a, \mu, \tau, \dots) \tag{3.2.147}$$

while preserving the original Minkowskian axioms.

Since the explicit functional dependence is inessential under isotopies, our studies will be generally referred to the interior gravitational problem. Unless otherwise stated, only diagonal realizations of the isounits will be used hereon for simplicity. An example of nondiagonal isounits inherent in a structure proposed by Dirac is indicated in Section 3.5. More general liftings of the Minkowski space of the so-called *genotopic and multivalued-hyperstructural type* will be indicated in Chapter 4.

3.2.9C. Isoderivative, Isoconnection, and Isoflatness. In the preceding subsections we have presented the *Minkowskian* aspects of the new isogeometry. We are now sufficiently equipped to present the novel part of the Minkowski-Santilli isogeometry, its *Riemannian* character as first derived in Ref. [26].

Our study is strictly in local coordinates representing the *fixed* frame of the observer without any un-necessary use of the transformation theory or abstract treatments. Our presentation will be as elementary as possible without reference to advanced topological requirements, such as Kadeisvili's isocontinuity (Section 3.2.6), isomanifolds and related TSSFN isotopology (Section 3.2.7) .

Also, our presentation is made, specifically, for the (3+1)-dimensional isospacetime, with the understanding that the extension to arbitrary dimensions and signatures or signatures different than the conventional one $(+,+,+,-)$ is elementary, and will be left to interested readers.

Let $\hat{M}(\hat{x},\hat{G},\hat{R})$ be a Minkowski-Santilli isospace and let $\hat{M}(x,\hat{\eta},R)$ be its projection in our spacetime as per Definition 3.2.12. To illustrate the transition from isocoordinates $\hat{x}$ to conventional spacetime coordinates x, we shall denote the projection $\hat{M} = \hat{M}(\hat{x},\hat{\eta},R)$. This notation emphasizes that the referral of the isospace to the conventional units and field causes the reduction of the isometric from the general form $\hat{G} = \hat{\eta}\times\hat{I}$ to $\hat{\eta} = \hat{T}\times\eta$, where, as now familiar, $\hat{I} = 1/\hat{T}$ and $\eta = Diag.(1,1,1,-1)$ is the familiar Minkowskian metric.

According to this notation the Riemannian content of the Minkowski-Santilli isogeometry can be unified in both its isospace formulation properly speaking and its projection in our spacetime. All differences in the interpretations whether occurring in isospace or in our spacetime are then deferred to the selection of the basic unit.

Consider now the infinitesimal version of isoinvariant (3.2.137) permitted by the isodifferential calculus

$$\hat{d}\hat{s}^{\hat{2}} = \hat{d}\hat{x}_\mu \hat{\times} \hat{d}\hat{x}^\mu \in \hat{R}. \tag{3.2.148}$$

The *isonormal coordinates* occur when the isometric $\hat{\eta}$ is reduced to the Minkowski metric η as in conventional Riemannian geometry. Consequently, isonormal coordinates coincide with the conventional normal coordinates, and the Minkowski-Santilli isogeometry verifies the *principle of equivalence* as for the conventional Riemannian geometry.

By using the isodifferential calculus, we now introduce the *isodifferential of a contravariant isovector field* on $\hat{M}$ over $\hat{R}$ [15]

$$\hat{d}\hat{X}^\beta = (\hat{\partial}_\mu \hat{X}^\beta)\hat{\times}\hat{d}\hat{x}^\mu = \hat{I}^\rho_\mu \times (\partial_\rho \hat{X}^\beta)\hat{\times}\hat{T}^\mu_\sigma \times d\hat{x}^\sigma \equiv$$

$$\equiv (\partial_\mu X^\beta) \times d\hat{x}^\mu = (\partial^\rho X^\beta) \times \hat{\eta}_{\rho\sigma} \times d\hat{x}^\sigma, \tag{3.2.149}$$

where the last expression is introduce to recall that the contractions are in isospace. The preceding expression then shows that *isodifferentials of isovec-*

[15] We should note that the role of the isounit and of the isoelement in this presentation and in that of Ref. [26] are interchanged for general compatibility with the various applications and developments.

tor fields coincide at the abstract level with conventional differentials for all isotopies of the class here admitted (that with $\hat{I} > 0$).

DEFINITION 3.2.13 [26]: The isocovariant differential are defined by

$$\hat{D}\hat{X}^{\beta} = \hat{d}\hat{X}^{\beta} + \hat{\Gamma}^{\beta}_{\alpha\gamma}\hat{\times}\hat{X}^{\alpha}\hat{\times}\hat{d}\hat{x}^{\gamma}, \tag{3.2.150}$$

with corresponding isocovariant derivative

$$\hat{X}^{\beta}_{\hat{|}\mu} = \hat{\partial}_{\mu}\hat{X}^{\beta} + \hat{\Gamma}^{\beta}_{\alpha\mu}\hat{\times}\hat{X}^{\hat{\alpha}}, \tag{3.2.151}$$

where the iso-Christoffel's symbols are given by

$$\hat{\Gamma}^{\beta}_{\alpha\gamma}(x, v, a, \mu, \tau, \dots) = \frac{\hat{1}}{\hat{2}}\hat{\times}(\hat{\partial}_{\alpha}\hat{\eta}_{\beta\gamma} + \hat{\partial}_{\gamma}\hat{\eta}_{\alpha\beta} - \hat{\partial}_{\beta}\hat{\eta}_{\alpha\gamma}) \times \hat{I} = \hat{\Gamma}_{\gamma\beta\alpha}, \tag{3.2.152a}$$

$$\hat{\Gamma}^{\beta}_{\alpha\gamma} = \hat{\eta}^{\beta\rho} \times \hat{\Gamma}_{\alpha\rho\gamma} = \hat{\Gamma}^{\beta}_{\gamma\alpha}. \tag{3.2.152b}$$

Note the unrestricted functional dependence of the connection which is notoriously absent in conventional treatments. Note also the abstract identity of the conventional and isotopic connections. Note finally that *local numerical values of the conventional and isotopic connections coincide when computed in their respective spaces.* This is due to the fact that in Eq.s (3.2.152) $\hat{\eta} \equiv g(x)$ for exterior problems, while the value of derivatives ∂_{μ} and isoderivatives $\hat{\partial}_{\mu}$ coincide when computed in their respective spaces.

Note however that, when projected in the conventional spacetime, the conventional and isotopic connections are different even in the exterior problem in which $\hat{\eta} = g(x)$,

$$\hat{\Gamma}_{\alpha\beta\gamma} = \frac{1}{2} \times (\hat{I}^{\mu}_{\alpha} \times \partial_{\mu} g_{\beta\gamma} + \hat{I}^{\rho}_{\gamma} \times \partial_{\rho}\hat{\eta}_{\alpha\beta} - \hat{I}^{\sigma}_{\beta} \times \partial_{\sigma} g_{\alpha\gamma}) \times \hat{I} \neq \Gamma_{\alpha\beta\gamma} \times \hat{I}. \tag{3.2.153}$$

The extension to covariant isovector fields and covariant or contravariant isotensor fields is consequential.

Without proof we quote the following important result from Ref. [26]:

LEMMA 3.2.7 (Iso-Ricci Lemma) [26]: Under the assumed conditions, the isocovariant derivatives of all isometrics on Minkowski-Santilli isospaces spaces are identically null,

$$\hat{\eta}_{\alpha\beta\hat{|}\gamma} \equiv 0, \quad \alpha, \beta, \gamma = 1, 2, 3, 4. \tag{3.2.154}$$

The novelty of the isogeometry is then illustrated by the fact that *the Ricci property persists under an arbitrary dependence of the metric, as well as under Minkowskian, rather than Riemannian axioms.*

The *isotorsion* on $\hat{M}$ is defined by

$$\hat{\tau}^{\beta}_{\alpha\gamma} = \hat{\Gamma}^{\beta}_{\alpha\gamma} - \hat{\Gamma}^{\beta}_{\gamma\alpha}, \tag{3.2.155}$$

and coincides again with the conventional torsion at the abstract level, although the two torsions have significant differences in their explicit forms when both projected in our space-time.

DEFINITION 3.2.14 [26]: The Minkowski-Santilli isogeometry is characterized by the following isotensor: the isoflatness isotensor

$$\hat{R}^{\beta}_{\alpha\gamma\delta} = \hat{\partial}_{\delta}\hat{\Gamma}^{\beta}_{\alpha\gamma} - \hat{\partial}_{\gamma}\hat{\Gamma}^{\beta}_{\alpha\delta} + \hat{\Gamma}^{\beta}_{\rho\delta}\hat{\times}\hat{\Gamma}^{\rho}_{\alpha\gamma} - \hat{\Gamma}^{\beta}_{\rho\gamma}\hat{\times}\hat{\Gamma}^{\rho}_{\alpha\delta}; \tag{3.2.156}$$

the iso-Ricci isotensor

$$\hat{R}_{\mu\nu} = \hat{R}^{\beta}_{\mu\nu\beta}; \tag{3.2.157}$$

the isoflatness isoscalar

$$\hat{R} = \hat{\eta}^{\alpha\beta} \times \hat{R}_{\alpha\beta}; \tag{3.2.158}$$

the iso-Einstein isotensor

$$\hat{G}_{\mu\nu} = \hat{R}_{\mu\nu} - \frac{\hat{1}}{2}\hat{\times}\hat{N}_{\mu\nu}\hat{\times}\hat{R},\ \hat{N}_{\mu\nu} = \hat{\eta}_{\mu\nu} \times \hat{I}; \tag{3.2.159}$$

and the isotopic isoscalar

$$\hat{\Theta} = \hat{N}^{\alpha\beta}\hat{\times}\hat{N}^{\gamma\delta}\hat{\times}(\hat{\Gamma}_{\rho\alpha\delta}\hat{\times}\hat{\Gamma}^{\rho}_{\gamma\beta} - \Gamma_{\rho\alpha\beta}\hat{\times}\hat{\Gamma}^{\rho}_{\gamma\delta}) =$$

$$= \hat{\Gamma}_{\rho\alpha\beta}\hat{\times}\hat{\Gamma}^{\rho}_{\gamma\delta}\hat{\times}(\hat{N}^{\alpha\delta}\hat{\times}\hat{N}^{\gamma\beta} - \hat{N}^{\alpha\beta}\hat{\times}\hat{N}^{\gamma\delta}). \tag{3.2.160}$$

the latter being new for the Minkowski-Santilli isogeometry.

Note the lack of use of the term "isocurvature" and the use instead of the term "isoflatness". This is due to the fact that the prefix "iso-" represents the preservation of the original axioms. The term "isocurvature" would then be inappropriate because the basic axioms of the geometry are flat.

In any case, the main problem underlying the studies herein reported is, as indicated in Chapter 1, that *curvature is the ultimate origin of the catastrophic inconsistencies of general relativity.* Consequently, all geometric efforts are here aimed at the replacement of the notion of curvature with a covering notion resolving the indicated catastrophic inconsistencies.

As we shall see better in Section 3.5, the notion of "isoflatness" does indeed achieve the desired objectives because flatness and its related invariance of gravitation under the Poincaré-Santilli isosymmetry is reconstructed on isospaces over isofields, while the ordinary curvature emerge as a mere projection in our spacetime.

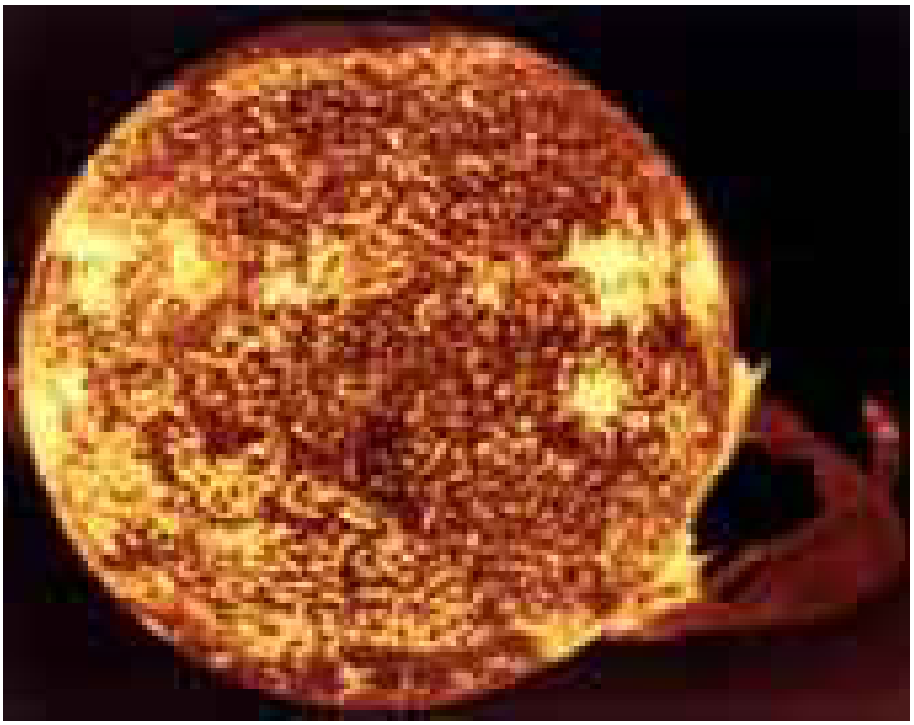

Figure 3.10. Primary objectives of the Minkowski-Santilli isogeometry are the resolution of the catastrophic inconsistencies of the Riemannian formulation of exterior gravitation (Section 1.4) and a representation of interior gravitation as occurring for the Sun depicted in this figure and any other massive object. These objectives are achieved via the isotopies of the Minkowskian geometry since they are flat in isospace, thus admitting a well defined invariance for all possible gravitation, by adding sources requested by the Freud identity and other reasons, and by unifying exterior and interior gravitational problem in a single formulation in isospace that formally coincides with that for the exterior problem, the interior effects being incorporated in the isounit (see Section 3.5).

3.2.9D. The Five Identities of the Minkowski-Santilli Isogeometry. By continuing our review of memoir [26], tedious but simple calculations yield the following *five basic identities of the Minkowski-Santilli isogeometry:*

Identity 1: *Antisymmetry of the last two indices of the isoflatness isotensor*

$$\hat{R}^{\beta}_{\alpha\gamma\delta} = -\hat{R}^{\beta}_{\alpha\delta\gamma}; \tag{3.2.161}$$

Identity 2: *Symmetry of the first two indices of the isoflatness isotensor*

$$\hat{R}_{\alpha\beta\gamma\delta} \equiv \hat{R}_{\beta\alpha\gamma\delta}; \tag{3.2.162}$$

Identity 3: *Vanishing of the totally antisymmetric part of the isoflatness isotensor*

$$\hat{R}^{\beta}_{\alpha\gamma\delta} + \hat{R}^{\beta}_{\gamma\delta\alpha} + \hat{R}^{\beta}_{\delta\alpha\gamma} \equiv 0; \tag{3.2.163}$$

Identity 4: *Iso-Bianchi identity*

$$\hat{R}^{\beta}_{\alpha\gamma\delta\hat{|}\rho} + \hat{R}^{\beta}_{\alpha\rho\gamma\hat{|}\delta} + \hat{R}^{\beta}_{\alpha\delta\rho\hat{|}\gamma} \equiv 0; \tag{3.2.164}$$

Identity 5: *Iso-Freud identity*

$$\hat{R}^{\alpha}_{\beta} - \frac{\hat{1}}{2}\hat{\times}\hat{\delta}^{\alpha}_{\beta}\hat{\times}\hat{R} - \frac{\hat{1}}{2}\hat{\times}\hat{\delta}^{\alpha}_{\beta}\hat{\times}\hat{\Theta} = \hat{U}^{\alpha}_{\beta} + \hat{\partial}_{\rho}\hat{V}^{\alpha\rho}_{\beta}, \tag{3.2.165}$$

where $\hat{\Theta}$ is the isotopic isoscalar and

$$\hat{U}^{\alpha}_{\beta} = -\frac{1}{2}\frac{\hat{\partial}\hat{\Theta}}{\hat{\partial}\hat{\eta}^{\alpha\beta}_{\hat{|}\alpha}}\hat{\eta}^{\alpha\beta}_{\hat{|}\beta}, \tag{3.2.166a}$$

$$\hat{V}^{\alpha\rho}_{\beta} = \frac{1}{2}[\hat{\eta}^{\gamma\delta}(\delta^{\alpha}_{\beta}\hat{\Gamma}^{\rho}_{\alpha\delta} - \delta^{\rho}_{\beta}\hat{\Gamma}^{\rho}_{\gamma\delta}) + \tag{3.2.166b}$$

$$+(\delta^{\rho}_{\beta}\hat{\eta}^{\alpha\gamma} - \delta^{\alpha}_{\beta}\hat{\eta}^{\rho\gamma})\hat{\Gamma}^{\delta}_{\gamma\delta} + \hat{\eta}^{\rho\gamma}\hat{\Gamma}^{\alpha}_{\beta\gamma} - \hat{\eta}^{\alpha\gamma}\hat{\Gamma}^{\rho}_{\beta\gamma}], \tag{3.2.166c}$$

Note that the conventional Riemannian geometry is generally thought to possess only *four* identities. In fact, the *fifth* identity (3.2.165) is generally unknown in the contemporary literature in gravitation as the reader is encouraged to verify in the specialized literature in the Riemannian geometry (that is so vast to discourage discriminatory listings).

The latter identity was introduced by Freud [27] in 1939, treated in detail by Pauli in his celebrated book [28] of 1958 and then generally forgotten for a half a century, apparently because of its evident incompatibility between Einstein's conception of exterior gravitation in vacuum as pure curvature without source (see Section 3.4)

$$G^{\alpha}_{\beta} = R^{\alpha}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}R = 0, \tag{3.2.167}$$

and the need for a source term also in exterior gravitation in vacuum mandated by the Freud identity and other reasons

$$R^{\alpha}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}R - \frac{1}{2}\delta^{\alpha}_{\beta}\Theta = U^{\alpha}_{\beta} + \hat{\partial}_{\rho}V^{\alpha\rho}_{\beta}. \tag{3.2.168}$$

Freud's identity was rediscovered by the author during his accurate study of Pauli's historical book and studied in detail in Refs. [6,7] of 1992. Additional studies of the Freud identity were done by Yilmaz [30]. Following a suggestion

by the author, the late mathematician Hanno Rund [29] studied the identity in one of his last papers and proved that:

LEMMA 3.2.8 (Rund's Lemma) [29]: Freud's identity is a bona fide identity for all Riemannian spaces irrespective of dimension and signature.

In this way, Rund confirming the general need of a source also in vacuum (see Sections 1.4 and 3.5).

Following Ref. [26], in this paper we have presented the isotopies of the Freud identity on Minkowski-Santilli isospaces, as characterized by the isodifferential calculus. Its primary functions for this monograph is to identify the geometric structure of the *interior* gravitational problem. The persistence of the source in vacuum as per the Freud identity, electrodynamics and other needs will then be consequential, thus confirming the inconsistency of Einstein's conception of gravity in vacuum as pure curvature without source.

Note that *all conventional and isotopic identities coincide at the abstract level.*

3.2.9E. Isoparallel Transport and Isogeodesics. An isovector field $\hat{X}^\beta$ on $\hat{M} = \hat{M}(\hat{x}, \hat{M}, \hat{R})$ is said to be transported by *isoparallel* displacement from a point $\hat{m}(\hat{x})$ on a curve $\hat{C}$ on $\hat{M}$ to a neighboring point $\hat{m}'(\hat{x} + \hat{d}\hat{x})$ on $\hat{C}$ if

$$\hat{D}\hat{X}^\beta = \hat{d}\hat{X}^\beta + \hat{\Gamma}^\beta_{\alpha\gamma} \hat{\times} \hat{X}^\alpha \hat{\times} \hat{d}\hat{x}^\gamma \equiv 0. \tag{3.2.169}$$

or in integrated form

$$\hat{X}^\beta(\hat{m}') - \hat{X}^\beta(m) = \hat{\int}_{\hat{m}}^{\hat{m}'} \frac{\hat{\partial}\hat{X}^\beta}{\hat{\partial}\hat{x}^\alpha} \frac{\hat{d}\hat{x}^\alpha}{\hat{d}\hat{s}} \hat{\times} \hat{d}\hat{s}. \tag{3.2.170}$$

where one should note the isotopic character of the integration. The isotopy of the conventional case then yields the following:

LEMMA 3.2.9 [26]: Necessary and sufficient condition for the existence of an isoparallel transport along a curve $\hat{C}$ on a $(3+1)$-dimensional Mionkowski-Santilli isospace is that all the following equations are identically verified along $\hat{C}$

$$\hat{R}^\beta_{\alpha\gamma\delta} \hat{\times} \hat{X}^\alpha = 0, \quad \alpha, \beta, \gamma, \delta = 1, 2, 3, 4. \tag{3.2.171}$$

Note, again, the abstract identity of the conventional and isotopic parallel transport. However, it is easy to see that the projection of the isoparallel transport in ordinary spacetime is structurally different than the conventional parallel transport.

Consider, as an example, an extended object in gravitational fall in atmosphere (see Figure 3.12). Its trajectory is evidently irregular and depends on

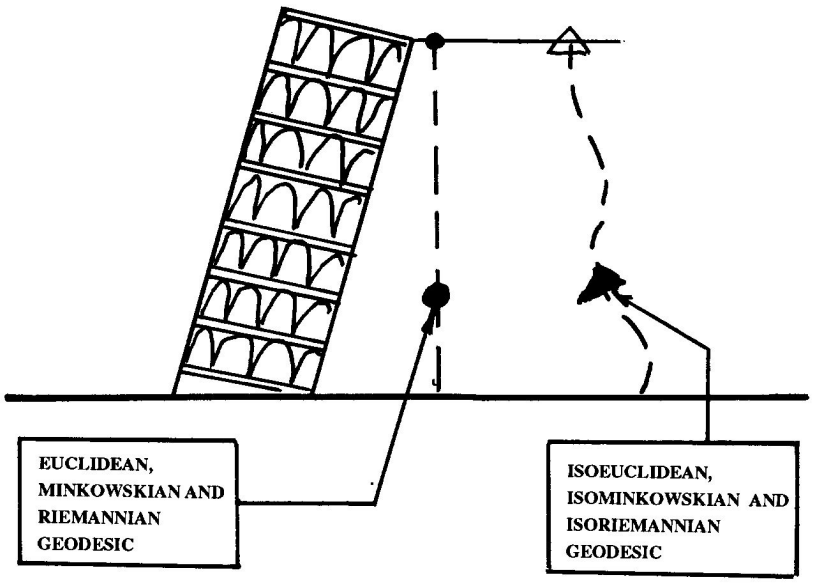

Figure 3.11. A schematic view of two objects released from the Pisa tower. The vertical trajectory represents the approximate geodesic considered by Galileo, used by Einstein and adopted until the end of the 20-th century, namely, the approximation under the lack of resistance due to our atmosphere. The Minkowski-Santilli isogeometry has been built to represent as isogeodesics actual trajectories within physical media.

the actual shape of the object, as well as its weight. The understanding of the new Minkowski-Santilli isogeometry requires the knowledge of the fact that said trajectory is represented on isospace over isofields as a *straight line,* that is, via the trajectory in the absence of the resistive medium. The actual, irregular trajectory appears only in the projection of said isotrajectory in our spacetime.

if the latter treatment is represented by a rocket, one would note a twisting action as occurring in the reality of motion within physical media, which is evidently absent in the exterior case.

Along similar lines, we say that a smooth isopath $\hat{x}_\alpha$ on $\hat{M}$ with isotangent $\hat{v}_\alpha = \hat{d}\hat{x}_\alpha/\hat{d}\hat{s}$ is an *isogeodesic* when it is solution of the isodifferential equations

$$\frac{\hat{D}\hat{v}^\beta}{\hat{D}\hat{s}} = \frac{\hat{d}\hat{v}}{\hat{d}\hat{s}} + \hat{\Gamma}_{\alpha\beta\gamma} \hat{\times} \frac{\hat{d}\hat{x}^\alpha}{\hat{d}\hat{s}} \hat{\times} \frac{\hat{d}\hat{x}^\gamma}{\hat{d}\hat{s}} = 0. \tag{3.2.172}$$

It is easy to prove the following:

LEMMA 3.2.10 [26]: The isogeodesics of a Minkowski-Santilli isospace $\hat{M}$ are the isocurves verifying the isovariational principle

$$\hat{\delta}\hat{\int}[\hat{G}_{\alpha\beta}(\hat{x},\hat{v},\hat{a},\mu,\tau,\dots)\hat{\times}\hat{d}\hat{x}^{\alpha}\hat{\times}\hat{d}\hat{x}^{\beta}]^{\hat{1/2}}=0. \tag{3.2.173}$$

where again isointegration is understood.

Finally, we point out the property inherent in the notion of isotopies according to which

COROLLARY 3.2.10A: [26]: Trajectories in an ordinary Riemannian space coincide with the corresponding isogeodesic trajectories in Minkowski-Santilli isospace, but not with the projection of the latter in the original space.

For instance, if a circle is originally a geodesic, its image under isotopy in isospace remains the perfect circle, the *isocircle* (Section 3.2.9), even though its projection in the original space can be an ellipse. The same preservation in isospace occurs for all other curves.

The differences between a geodesic and an isogeodesic therefore emerge only when projecting the latter in the space of the former.

An empirical but conceptually effective rule is that *interior physical media "disappear" under their isoteometrization*, in the sense that actual trajectories under resistive forces due to physical media (which are not geodesics of a Minkowski space) are turned into isogeodesics in isospace having the shape of the geodesics in the absence of resistive forces.

The simplest possible example is given by the isoeuclidean representation of a straight stick partially immersed in water. In conventional representations the stick penetrating in water with an angle α appears as bended at the point of immersion in water with an angle $\gamma = \alpha + \beta$, where β is the angle of refraction. In isoeuclidean representation the stick remains straight also in its immersion because the isoangle $\hat{\gamma} = \gamma \times \hat{I}_{\gamma}$ recovers the original angle α with $\hat{I}_{\gamma} = \alpha/(\alpha+\beta)$.

The situation is essentially the same for our representation of interior gravitation because the latter is represented in isospace over isofield via field equations (this time necessarily with sources) that formally coincide with conventional equations on a conventional Riemannian spacetime. Being noncanonical, all interior features are invariantly represented via generalized units.

3.2.9F. Isodual Minkowski-Santilli isospaces and isogeometry. The *isodual Minkowski-Santilli isospaces* were introduced for the first time by San-

tilli in Ref. [8] of 1985 and then studied in various works (see the references of Chapter 1), and can be written

$$\hat{M}^d = \hat{M}^d(\hat{x}^d, \hat{\eta}^d, \hat{R}^d) :$$

$$\hat{x}^d = \{x^{\mu d}\} \times^d \hat{I}^d = \{x^\mu\} \times (-\hat{I}) = \{r^d, c_o^d \times^d t^d\} \times^d \hat{I}^d, \tag{3.2.174a}$$

$$\hat{\eta}^d = -\hat{\eta}. \tag{3.2.174b}$$

The *isodual Minkowski-Santilli isogeometry* is the geometry of isodual isospaces M^d over R^d and was studied for the first time by Santilli in Ref. [26] of 1998.

The physically and mathematically most salient property of the latter geometry is that it is *characterized by negative units of space, time, etc., and negative norms.* Therefore, in addition to a change of the sign of the change, we also have change of sign of masses, energies, and other quantities normally positive for matter. Similarly, we have the *isodual isospace and isotime coordinates*

$$\hat{x}^d = \hat{x}^d \times^d \hat{I} = -\hat{x}, \;\; \hat{t}^d = t^d \times^d \hat{I}^d = -\hat{t}. \tag{3.2.175}$$

Thus, motion under isoduality is in a time direction *opposite* to the conventional motion. These features are necessary so as to have a classical representation of antimatter in interior conditions whose operator image yields indeed antiparticles (rather than particles with the worng sign of the charge).

We also have the following important

LEMMA 3.2.12 [17]: Isodualities are independent from spacetime inversions

$$r' = \pi \times r = -r, \;\; t' = \tau \times t = -t. \tag{3.2.176}$$

Proof. Inversions occur within the same original space and keep the unit fixed, while isodualities require a map to a different space, and change the sign of the unit. Therefore, in addition to maps in different spaces, isodualities have numerical value different than the inversions. **q.e.d.**

These are the conceptual roots for the isodual theory of antimatter to predict a *new photon*, the *isodual photon* emitted by antimatter [17]. When applied to the photon, charge conjugation and, more generally, the PCT theorem, do not yield a new photon, as well know. This is not the case under isoduality because all physical characteristics change in sign and numerical value. As a result, *the isodual photon is indistinguishable from the ordinary photon under all interactions except graviton.* In fact, as indicated in Chapter 1, the isodual photon is predicted to experience antigravity in the field of matter, thus offering, apparently for the first time, a possibility for the future study whether far away galaxies and quasars are made up of matter or of antimatter.

Another important property of isoduality is expressed by the following:

LEMMA 3.2.13 [26]: The intervals of conventional and isotopic Minkowskian spaces are invariant under the joint isodual maps $\hat{I}^d \rightarrow \hat{I}^d$ and $\hat{\eta} \rightarrow \hat{\eta}^d$,

$$\hat{x}^2 = (x^\mu \times \hat{\eta}_{\mu\nu} \times x^\nu) \times \hat{I} \equiv [x^\mu \times (-\hat{\eta}_{\mu\nu}) \times x^\nu] \times (-\hat{I}). \tag{3.2.177}$$

As a result, *all physical laws applying in conventional Minkowskian geometry for the characterization of matter also apply to its isodual image for the characterization of antimatter.*

Note that, strictly speaking, the intervals are not isoselfdual because

$$\hat{x}^{\hat{2}} = \hat{x}^\mu \hat{\times} \hat{M}_{\mu\nu} \hat{\times} \hat{x}^\mu \ \rightarrow \ \hat{x}^{d\hat{2}d} = \hat{x}^{\mu d} \times^d \hat{M}^d_{\mu\nu} \times^d \hat{x}^{\mu d} = \hat{x}^{d\hat{2}d} = -\hat{x}^{\hat{2}}. \tag{3.2.178}$$

To outline the *Riemannian* characteristics of the isodual Minkowski-Santilli isogeometry, we consider an *isodual isovector isofield* $\hat{X}^d(\hat{x}^d)$ on $\hat{M}^d$ which is explicitly given by $\hat{X}^d(\hat{x}^d) = -X^t(-x^t \times \hat{I}) \times \hat{I}$. The *isodual exterior isodifferential* of $\hat{X}^d(\hat{x}^d)$ is given by

$$\hat{D}^d \hat{X}^{\mu d}(\hat{x}^d) = \hat{d}^d \hat{X}^{\mu d}(\hat{x}^d) \hat{+}^d \hat{\Gamma}^{d\,\mu}_{\alpha\beta} \hat{\times}^d \hat{X}^{\alpha d} \hat{\times}^d \hat{d}^d \hat{x}^{\beta d} = \hat{D} \hat{X}^{t\mu}(-\hat{x}^t), \tag{3.2.179}$$

where the $\hat{\Gamma}^d$'s are the components of the *isodual isoconnection*. The *isodual isocovariant isoderivative* is then given by

$$\hat{X}^{\mu d}(\hat{x}^d)_{\hat{|} d\nu} = \hat{\partial}^d \hat{X}^{\mu d}(\hat{x}^d) \hat{/}^d \hat{\partial}^d \hat{x}^{\nu d} \hat{+} \hat{\Gamma}^{d\,\mu}_{\alpha\nu} \hat{\times}^d \hat{X}^{\alpha d}(\hat{x}^d) = -\hat{X}^{t\mu}(-\hat{x}^t)_{\hat{|} k}. \tag{3.2.180}$$

The interested reader can then easily derive the remaining notions of the new geometry. It is an instructive exercise for the interested reader to prove of the following isodualities:

Isodual isounit	$\hat{I} \ \rightarrow \ \hat{I}^d = -\hat{I}$,
Isodual isometric	$\hat{\eta} \ \rightarrow \ \hat{\eta}^d = -\eta$,
Isodual isoconnection coefficients	$\hat{\Gamma}_{\alpha\beta\gamma} \ \rightarrow \ \hat{\Gamma}^d_{\alpha\beta\gamma} = \hat{\Gamma}_{\alpha\beta\gamma}$,
Isoflatness isotensor	$R_{\alpha\beta\gamma\delta} \ \rightarrow \ R^d_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\gamma\delta}$,
Isodual iso-Ricci isotensor	$\hat{R}_{\mu\nu} \ \rightarrow \ \hat{R}^d_{\mu\nu} = \hat{R}_{\mu\nu}$,
Isodual iso-Ricci isoscalar	$\hat{R} \ \rightarrow \ \hat{R}^d = \hat{R}$,
Isodual iso-Freud isoscalar	$\hat{\Theta} \ \rightarrow \ \hat{\Theta}^d = -\hat{\Theta}$,
Isodual Iso-Einstein isotensor	$\hat{G}_{\mu\nu} \ \rightarrow \ \hat{G}^d_{\mu\nu} = -\hat{G}_{\mu\nu}$,
Isodual electromagnetic potentials	$A_\mu \ \rightarrow \ A^d_\mu = -A_\mu$,
Isodual electromagnetic field	$F_{\mu\nu} \ \rightarrow \ F^d_{\mu\nu} = -F_{\mu\nu}$,
Isodual elm energy-mom. isotensor	$T_{\mu\nu} \ \rightarrow \ T^d_{\mu\nu} = -T_{\mu\nu}$.

(3.2.181)

More detailed isogeometric studies are left to interested readers. Specific applications to gravitational treatments of matter and antimatter are presented in Section 3.5.

3.2.10 Isosymplectic Geometry and its Isodual

As it is well known, the *symplectic geometry* had an important role in the construction of quantum mechanics because it permitted the mathematically rigorous verification, known as *symplectic quantization,* that original quantization procedures, known also as *naive quantization,* were correct.

No broadening of quantum mechanics can be considered mature unless it admits fully equivalent procedures in the map from classical to operator forms known as *isoquantization* also called *hadronization* (rather than quantization).

For this purpose. Santilli [31] presented in 1988 the first known *isotopies of the symplectic geometry,* subsequently studied in various works, with a general presentation available in Vols. I, II of this series (see in particular Chapter 5 of Vol. I [6]). The new geometry is today known as *Santilli's isosymplectic geometry.*

We cannot possibly review here the isosymplectic geometry in detail and have to suggest interested readers to study Refs. [6,7]. Nevertheless, an indication of the basic lines is important for the self-sufficiency of this monograph.

Let us ignore the global (also called abstract) formulation of the symplectic geometry and consider for clarity and simplicity only its realization in a local chart (or coordinates).[16] A *topological manifold* $M(R)$ on the reals R admits the local realization as an Euclidean space $E(r, \delta R)$ with local contravariant coordinates $r = (r^i),\ i = i, 2, \ldots, N$. The *cotangent bundle* T^*M then becomes the ordinary phase space with local coordinates $(r, p) = (r^i, p_i)$, where p_i represents the tangent vectors (physically the linear momentum). The *canonical one-form* then admits the local realization

$$\theta = p_i \times dr^i. \tag{3.2.182}$$

The *fundamental (canonical) symplectic form* is then given by the exterior derivative of the preceding one form

$$\omega = d\theta = p_i \wedge dr^i, \tag{3.2.183}$$

and one can easily prove that it is closed, namely, that $d\omega \equiv 0$.

Consider now the *isotopological isomanifold* (introduced earlier) $\hat{M}(\hat{R})$ on the isoreals $\hat{R}$ with basic isounit $\hat{I}$. Its realization on local coordinates is

[16] Again, the literature on the conventional symplectic geometry is so vast to discourage discriminatory quotations.

given by the Euclid-Santilli isospace $\hat{E}(\hat{r}, \hat{\Delta}, \hat{R})$ with local contravariant isocoordinates $\hat{r} = (r^i) \times \hat{I}$. Then, the *isocotangent isobundle* $\hat{T}^* \hat{M}$ admits as local realization the *isophase isospace* with local coordinates $(\hat{r}^i, \hat{p}_i)$, where $\hat{p}$ is again a tangent isovector. The novelty is given by the fact that the unit of $\hat{p}$ is the *inverse* of that of $\hat{r}$ and we shall write

$$\hat{r} = r \times \hat{I}, \;\; \hat{p} = p \times \hat{T}, \;\; \hat{I} = 1/\hat{T}. \tag{3.2.184}$$

This property was identified for the first time by Santilli [31] (for a mathematical treatment see also Ref. [10]) because not identifiable in the conventional symplectic geometry due to the use of the trivial unit for ehich $I^{-1} \equiv I = +1$.

Consequently, we have the isodifferentials

$$\hat{d}\hat{r} = \hat{T} \times d(r \times I), \;\; \hat{d}\hat{p} = \hat{I} \times d(p \times \hat{T}). \tag{3.2.185}$$

The *isocanonical one-isoform* is then given by

$$\hat{\theta} = \hat{p} \hat{\times} \hat{d}\hat{r} = (p \times \hat{T}) \times \hat{I} \times \hat{d}(\hat{r}) = p \times \hat{T} \times d(r \times \hat{I}). \tag{3.2.186}$$

The *fundamental isocanonical two-isoform* is then given by

$$\hat{\omega} = \hat{d}\hat{\theta} = \hat{p} \hat{\wedge} \hat{d}\hat{r} = dp_i \wedge dr^i \equiv \omega \tag{3.2.187}$$

from which the preservation of closure under isotopy, $\hat{d}\hat{\omega} \equiv \hat{0} = 0$ trivially follows.

LEMMA 3.2.14 [31,10]: The fundamental symplectic and isosymplectic two-forms coincide.

The identity of the fundamental isocanonical and canonical two-forms explains why isosymplectic geometry escaped detection by mathematicians for centuries.

It is evident that, in view of the positive-definiteness of the isounit, *the symplectic and isosymplectic geometries coincide at the global (abstract) realization-free level* to such an extent that there is not even the need of changing formulae in the literature of the symplectic geometry because the isosymplectic geometry can be expressed with the pre-existing formalism and merely subject it to a broader realization.

Despite this simplicity, the physical implications are by far non-trivial. In fact, unlike the conventional two-form, and thanks to the background TSSFN isotopology, the fundamental isocanonical two-form is universal for all possible (sufficiently smooth and regular but otherwise arbitrary) nonlocal and non-Hamiltonian systems. To illustrate this feature, let us consider a *vector field* of the cotangent bundle that must be strictly *local-differential* to avoid

catastrophic inconsistencies with the underlying local-differential Euclidean topology, T^*M

$$X(r,p) = A_i(r,p) \times \frac{\partial}{\partial r^i} + B^i(r,p) \times \frac{\partial}{\partial p_i}, \tag{3.2.188}$$

or in unified notations

$$b = (b^\mu) = (r^i, p_j), \quad \mu = 1, 2, \ldots, 2N \tag{3.2.189}$$

$$X(b) = X_\mu(b) \times \frac{\partial}{\partial b^\mu}, \tag{3.2.190}$$

is said to be a *Hamiltonian vector field* when there exists a function $H(r,p) = H(b)$ on T^*M, called the *Hamiltonian,* verifying the identity

$$A_i \times dr^i + B^i \times dp_i = -dH(r,p) \tag{3.2.191}$$

or in unified notation

$$X \rfloor \omega = dH, \tag{3.2.192}$$

that is

$$\omega_{\mu\nu} \times X^\mu \times db^\nu = -dH, \tag{3.2.193}$$

where the fundamental symplectic form has the components

$$\omega = dp_i \wedge dr^i = \frac{1}{2} \times \omega_{\mu\nu} \times db^\mu \wedge db^\nu, \tag{3.2.194}$$

$$(\omega_{\mu\nu}) = \begin{pmatrix} O_{N\times N} & -I_{N\times N} \\ I_{N\times N} & O_{N\times N} \end{pmatrix}. \tag{3.2.195}$$

Eq. (3.2.192) can hold if and only if

$$\omega_{\mu\nu} \times \frac{db^\nu}{dt} = \frac{\partial H}{\partial b^\mu}, \tag{3.2.196}$$

from which one recovers the familiar truncated Hamilton's equations

$$\frac{dr^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial r^i}. \tag{3.2.197}$$

The main physical limitations is that *the condition for a vector field to be Hamiltonian constitutes a major restrictions because vector fields in the physical reality are generally non-Hamiltonian, besides existing from the limitations of the topology underlying the symplectic geometry.*

As we shall see in Section 3.3, the above restrictions is removed for Santilli isosymplectic geometry that acquire the character of *direct universality,* that is, the capability of representing all sufficiently smooth and regular but

but otherwise arbitrary vector fields (universality) in the local chart of the experimenter (direct universality).

In fact, expression (3.2.192) is lifted into the form

$$\hat{\omega}_{\mu\nu} \hat{\times} \frac{\hat{d}\hat{b}^\nu}{\hat{d}\hat{t}} = \frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{b}^\mu}, \tag{3.2.198}$$

that, under the assumption for simplicity that $\hat{t} = t$, and by removing common factors, reduces to

$$\frac{dr^i}{dt} = \frac{\hat{\partial}H}{\hat{\partial}p_i} = \hat{T}^i_j(r,p) \times \frac{\partial H}{\partial p_j}; \tag{3.2.199}$$

$$\frac{dp_i}{dt} = -\frac{\hat{\partial}H}{\hat{\partial}r^i} = -\hat{I}^j_i \times \frac{\partial H}{\partial r^j}. \tag{3.2.200}$$

As we shall see better in Section 3.3, direct universality then follows from the number of free functions $\hat{T}^j_i$ as well as the arbitrariness of their functional dependence.

We shall also show that the achievement of a direct isogeometric representation of nonlocal and non-Hamiltonian vector fields representing interior dynamical problems permits their consistent map into an operator form, by therefore reaching hadronic mechanics in a mathematically rigorous, unique and unambiguous way.[17]

The construction of the *isodual isosymplectic geometry* [6] is an instructive exercise for readers interested in serious studies of antimatter in interior dynamical conditions.

3.2.11 Isolinearity, Isolocality, Isocanonicity and Their Isodualities

In Section 3.1 we pointed out that the primary physical characteristics of particles and antiparticles in interior conditions (such as a neutron in the core of a neutron star) are nonlinear, nonlocality and noncanonicity due to the mutual penetration-overlapping of their wavepackets with those of the surrounding medium.

In the preceding subsections we have identified isotopic means for mapping linear, local and canonical systems into their most general possible nonlinear, nonlocal and noncanonical form. In this section we show how the isotopies permit the reconstruction of linearity, locality and canonicity on isospaces over isofields, called *isolinearity, isolocality and isocanonicity* for the case of particles, with their isodual counterpart for antiparticles.

[17]Note the crucial role of the isodifferential calculus for the isosymplectic geometry and its implications.

The understanding of this seemingly impossible task requires the knowledge that conventional methods have only one formulation. By contrast, all isotopic methods have a dual formulation, the first in isospace over isofields, and the second when projected in ordinary spaces over ordinary fields. Deviations from conventional properties can only occur in the latter formulation because in the former all original axiomatic properties are preserved by construction.

Let $S(r, R)$ be a conventional real vector space with local coordinates r over the reals $R = R(n, +, \times)$, and let

$$r' = A(w) \times r, \;\; r'^t = r^t \times A^t(w), \;\; w \in R. \tag{3.2.201}$$

be a conventional right and left linear, local and canonical transformation on S, where t denotes transpose.

The isotopic lifting $S(r, R) \to \hat{S}(\hat{r}, \hat{R})$ requires a corresponding necessary isotopy of the transformation theory. In fact, it is instructive for the interested reader to verify that the application of conventional linear transformations to the isospace $\hat{S}(\hat{r}, \hat{R})$ causes the loss of linearity, transitivity and other basic properties.

For these and other reasons, Santilli submitted in the original proposals [4,5] of 1978 (see monographs [6,7] for comprehensive treatments and applications) the isotopy of the transformation theory, called *isotransformation theory,* which is characterized by isotransforms (where we make use of the notion of isofunction of Section 3.2.4)

$$\hat{r}' = \hat{A}(\hat{w})\hat{\times}\hat{r} = \hat{A}(\hat{w}) \times \hat{T} \times \hat{r} = [A(\hat{T} \times w) \times \hat{I}] \times \hat{T} \times (r \times \hat{I}) =$$

$$= A[\hat{T}(r, \dots) \times w] \times \hat{r}, \tag{3.2.202a}$$

$$\hat{r}'^{\hat{t}} = \hat{r}^{\hat{t}}\hat{\times}\hat{A}^{\hat{t}}\hat{w} = \hat{r}^{\hat{t}} \times A^t[\hat{T}(r) \times w]. \tag{3.2.202b}$$

The most dominant aspect in the transition from the conventional to the isotopic transforms is that, while the former are linear, local and canonical, the latter are nonlinear in the coordinates as well as other quantities and their derivatives of arbitrary order, nonlocal-integral in all needed quantities, and noncanonical when projected in the original spaces $S(r, R)$. This is due to the unrestricted nature of the functional dependence of the isotopic element $\hat{T} = \hat{T}(r, \dots)$.

But the conventional and isotopic transforms coincide at the abstract level where we have no distinction between the modular action $A(w) \times r$ and $\hat{A}(\hat{w})\hat{\times}\hat{r}$. Therefore, Isotransforms(3.2.202) are are *isolinear* when formulated on isospace $\hat{S}$ over the isofield $\hat{R}$, because they verify the conditions

$$\hat{A}\hat{\times}(\hat{n}\hat{\times}\hat{r} + \hat{m}\hat{\times}\hat{p}) = \hat{n}\hat{\times}\hat{A}\hat{\times}\hat{r} + \hat{m}\hat{\times}\hat{A}\hat{\times}\hat{p}, \;\; \hat{r}, \hat{p} \in \hat{S}, \;\; \hat{n}, \hat{m} \in \hat{R}. \tag{3.2.203}$$

Note that conventional transforms are characterized by a *right modular associative action* $A \times r$. Isotransforms are then characterized by the *right*

isomodular isoassociative action $\hat{A}\hat{\times}\hat{r}$. Therefore, we do have the preservation of the original axiomatic structure and isotransforms are indeed an isotopy of conventional transforms.

The situation for locality and canonicity follows the same lines [4,5,6,7]. Conventional methods are local in the sense that they are defined at a finite set of isolated points. The isotopic methods are *isolocal* in the sense that they verify the condition of locality in isospaces over isofields. However, their projection on conventional space is nonlocal-integral, because that is the general characteristic of the isotopic element $\hat{T}$, as illustrated, e.g., in Eq. (3.1.202).

Similarly, conventional methods are canonical in the sense that they cvan be characterized via a first-order canonical action in phase space (or cotangent bundle). The isotopic methods are *isocanonical* in the sense that, as we shall see in Section 3.3, they are derivable from an *isoaction* that is first-order and canonical on isospaces over isofields, although, when projected on ordinary spaces over ordinary fields, such an isoaction is of arbitrary order.

LEMMA 3.2.15 [6,7]: All possible nonlinear, nonlocal and noncanonical transforms on a vector space $S(r, R)$

$$r' = B(w, r, \dots) \times r, \;\; r \in S, \;\; w \in R, \tag{3.2.204}$$

can always be rewritten in an identical isolinear, isolocal and isocanonical form, that is, there always exists at least one isotopy of the base field, $R \to \hat{R}$, *and a corresponding isotopy of the space* $S(r, R) \to \hat{S}(\hat{r}, \hat{R})$, *such as*

$$B(w, r, \dots) \equiv A(\hat{T} \times w), \tag{3.2.205}$$

under which

$$r' = B(w, r, \dots) \times r \equiv A(\hat{T} \times w) \times r \equiv \hat{A}(\hat{w})\hat{\times} r, \tag{3.2.206}$$

from which the isolinear form (3.2.202) follows.

COROLLARY 3.2.15A [6,7]: Under sufficient continuity and regularity conditions, all possible ordinary differential equations that are nonlinear in ordinary spaces over ordinary fields can always be turned into an identical form that is isolinear on isospaces over isofields,[18]

$$\dot{r} - E(\dot{r}, w, \dots) \to \hat{\dot{r}} - A[\hat{T}(\dot{r}, w, \dots) \times \dot{r} - B[\hat{T}(\dot{r}, w, \dots] \equiv$$

[18]The author has proposed for over a decade that mathematicians use the property of this Corollary 4.3.15A to identify simpler methods for the solution of nonlinear differential equations, but the request has not been met as yet, to our best knowledge.

$$\equiv \hat{\dot{r}} - \hat{A}(\hat{w})\hat{\times}\hat{\dot{r}} - \hat{B}(w) = 0. \tag{3.2.207}$$

The above properties are at the foundation of the *direct universality* of isotopic methods, that is, their applicability to all possible (sufficiently smooth and regular) nonlinear, nonlocal and noncanonical systems (universality) in the frame of the experimenter (direct universality).

In order to apply isotopic methods to a nonlinear, nonlocal and noncanonical system, one has merely to identity one of its possible isolinear, isolocal and isocanonical identical reformulations in the same system of coordinates. The applicability of the methods studied in this monograph then follows.

The *isodual isotransforms* are given by the image of isotransforms (3.2.202) under isoduality, and, as such, are defined on the isodual isospace $\hat{S}^d(\hat{r}^d, \hat{R}^d)$ over the isodual isofield $\hat{R}^d$ with isodual isounit $\hat{I}^d = 1/\hat{T}^d = -\hat{I}^\dagger$. [6,7] with evident properties

$$\hat{A}^d\hat{\times}^d(\hat{n}^d\hat{\times}^d\hat{r}^d + \hat{m}^d\hat{\times}^d\hat{p}^d) =$$
$$= \hat{n}^d\hat{\times}^d\hat{A}^d\hat{\times}^d\hat{r}^d + \hat{m}^d\hat{\times}^d\hat{A}^d\hat{\times}^d\hat{p}^d, \;\; \hat{r}^d, \hat{p}^d \in \hat{S}^d, \;\; \hat{n}^d, \hat{m}^d \in \hat{R}^d. \tag{3.2.208}$$

The definition of *isodual isolinearity, isolocality and isocanonicity* then follows.

From now on, we shall use isotransforms for the study of interior dynamical systems of particles and their isodual for interior systems of antiparticles.

3.2.12 Isotopies and isodualities of Lies theory

3.2.12A. Statement of the Problem. As it is well known, Lie's theory has permitted outstanding achievements in various disciplines throughout the 20-th century. Nevertheless, in its current conception and realization, Lie's theory is linear, local-differential and canonical-Hamiltonian.[19]

As such, Lie's theory is exactly valid for exterior dynamical systems, but possesses clear limitations for interior dynamical systems since the latter are nonlinear, nonlocal and noncanonical. This occurrence mandates a suitable revision of Lie's theory such to be exactly valid for interior dynamical systems without approximations.

Independently from that, Lie's theory in its current formulation is solely applicable to matter, evidently because there exists no antiautomorphic version of the conventional Lie's theory as necessary for the correct treatment of antimatter beginning at the classical level, as shown in Chapters 1 and 2.

Another central problem addressed in these studies is the construction of the universal *symmetry* (and not "covariance") of gravitation for matter and, independently, for antimatter, that is, a symmetry for all possible exterior

[19]The literature on Lie's theory is also vast to discourage discriminatory listings. In any case, its knowledge is a necessary pre-requisite for the understanding of this section.

and interior gravitational line elements of matter and, under antiautomorphic image, of antimatter.

Yet another need in physics is the identification of the exact symmetry that can effectively replace broken Lie symmetries, which exact symmetry cannot possibly be a conventional Lie symmetry due to the need of preserving the original dimensions so as to avoid the prediction on unphysical effects and/or hypothetical new particles.

It is evidently that Lie's theory in its current formulation is unable to solve the above identified problems. In a memoir of 1978, Santilli [4] proposed a step-by-step generalization of the conventional Lie theory specifically conceived for nonlinear, nonlocal-integral and nonpotential-noncanonical systems.

The generalized theory was subsequently studied by Santilli in a variety of papers (see monographs [1,2,6,7,14,15] and references quoted therein). The theory was also studied by a number of mathematicians and theoreticians, and it is today called the *Lie-Santilli isotheory* (see, e.g., monographs [32–37] and references quoted therein, as well as specialized papers [38–43]).

A main characteristic of the Lie-Santilli isotheory, that distinguishes it from other possible generalizations, is its isotopic character, that is, the preservation of the original Lie axioms when formulated on isospaces over isofields, despite its nonlinear, nonlocal and noncanonical structure when projected in ordinary spaces. This basic feature is evidently permitted by the reconstruction of linearity, locality and canonicity on isospaces over isofields studied in the preceding section.

To begin, let us recall that Lie's theory is centrally dependent on the basic N-dimensional unit $I = Diag.(1, 1, \dots, 1)$ of the enveloping algebra. The main idea of the Lie-Santilli isotheory [4] is the reformulation of the entire conventional theory with respect to the most general possible isounit $\hat{I}(x, \dot{x}, \ddot{x}, \dots)$.

One can therefore see from the very outset the richness and novelty of the isotopic theory since isounits with different topological features (such as Hermiticity, non-Hermiticity, positive-definiteness, negative-definiteness, etc.) characterize different generalized theories.

In this section we outline the rudiments of the Lie-Santilli isotheory properly speaking, that with positive-definite isounits and its isodual with negative-definite isounits. A knowledge of Lie's theory is assumed as a pre-requisite. A true technical knowledge of the Lie-Santilli isotheory can only be acquired from the study of mathematical works such as monographs [2,6,14,36,37].

In inspecting the literature, the reader should be aware that Santilli [4] constructed the isotopies of Lie's theory as a particular case of the broader Lie-admissible theory studied in Chapter 4 occurring for non-Hermitean generalized units, and known as *Lie-Santilli genotheory.* As a matter of fact, a number of aspects of the isotheory can be better identified within the context of the broader genotheory.

The extension to non-Hermitean isounits (that was the main object of the original proposal [4]) requires the exiting of Lie's theory in favor of the covering Lie-admissible theory, and will be studied in Chapter 4.

The isotopies of Lie's theory were proposed by Santilli from first axiomatic principles without the use of any map or transform. It is today known that the isotheory cannot be entirely derived via the use of noncanonical-nonunitary transforms since some of the basic structures (such as the isodifferential calculus) are not entirely derivable via noncanonical-nonunitary transforms.

3.2.12B. Universal Enveloping Isoassociative Algebras. Let ξ be an *associative algebra* over a field $F = F(a, +, \times)$ of characteristic zero with generic elements $A, B, C, \ldots$, trivial associative product $A \times B$ and unit I. The infinitely possible isotopes $\hat{\xi}$ of ξ were first introduced in Ref. [4] under the name of *isoassociative algebras*. In the original proposal $\hat{\xi}$ coincides with ξ as vector spaces but is equipped with Santilli's isoproduct so as to admit the isounit as the correct left and right unit

$$\hat{I}(x, \dot{x}, \ddot{x}, \ldots) = 1/\hat{T} > 0, \tag{3.2.209a}$$

$$\hat{A}\hat{\times}\hat{B} = \hat{A} \times \hat{T} \times \hat{B}, \;\; \hat{A}\hat{\times}(\hat{B}\hat{\times}\hat{C}) = (\hat{A}\hat{\times}\hat{B})\hat{\times}\hat{C}, \tag{3.2.209b}$$

$$\hat{I}\hat{\times}\hat{A} = \hat{A}\hat{\times}\hat{I} \equiv \hat{A}, \;\; \forall \hat{A} \in \hat{\xi}, \tag{3.2.209c}$$

where $\hat{A}, \hat{B}, \ldots$ denote the original elements $A, B, \ldots$ formulated on isospace over isofields.

Let $\xi = \xi(L)$ be the *universal enveloping associative algebra* of an N-dimensional Lie algebra L with ordered basis $X_k, \; k = 1, 2, \ldots, N$, and attached antisymmetric algebra isomorphic to the Lie algebras, $[\xi(L)]^- \approx L$ over F, and let the infinite-dimensional basis $I, X_k, \; X_i \times X_j, \; i \leq j, \ldots$ of $\xi(L)$ be characterized by the *Poincaré-Birkhoff-Witt theorem.*

A fundamental property submitted in the original proposal [4] (see also [2], pp. 154–163) is the following

THEOREM 3.2.11 (Poincaré-Birkhoff-Witt-Santilli isotheorem): Isocosets of the isounit and the standard, isomonomials

$$\hat{I}, \;\; X_k, \;\; \hat{X}_i\hat{\times}\hat{X}_j, \; i \leq j, \;\; \hat{X}_i\hat{\times}\hat{X}_j\hat{\times}\hat{X}_k, \; i \leq j \leq k, \ldots, \tag{3.2.210}$$

form a basis of universal enveloping isoassociative algebra $\hat{\xi}(L)$ of a Lie algebra L (also called isoenvelope for short).

The first application of the above infinite-dimensional basis ie a rigorous characterization of the isoexponentiation, Eq. (3.2.72), i.e.,

$$\hat{e}^{\hat{i}\hat{\times}\hat{w}\hat{\times}\hat{X}} = \hat{e}^{i \times w \times \hat{X}} =$$

$$= \hat{I} + \hat{i}\hat{\times}\hat{w}\hat{\times}\hat{X}\hat{/}\hat{1}! + (\hat{i}\hat{\times}\hat{w}\hat{\times}\hat{X})\hat{\times}(\hat{i}\hat{\times}\hat{w}\hat{\times}\hat{X})\hat{/}\hat{2}! + \ldots =$$

$$= \hat{I} \times (e^{i\times w\times\hat{T}\times\hat{X}}) = (e^{i\times w\times\hat{X}\times\hat{T}}) \times \hat{I}, \;\; \hat{i} = i \times \hat{I}, \hat{w} = w \times \hat{I} \in \hat{F}. \quad (3.2.211)$$

The nontriviality of the Lie-Santilli isotheory is illustrated by the emergence of the nonlinear, nonlocal and noncanonical isotopic element $\hat{T}$ directly in the exponent, thus ensuring the desired generalization.

The implications of Theorem 3.2.11 also emerge at the level of isofunctional analysis because all structures defined via the conventional exponentiation must be suitably lifted into a form compatible with Theorem 3.2.11, as illustrated by the *iso-Fourier transforms,* Eq. (3.2.88).

It is today known that the main lines of isoenvelopes can indeed be derived via the use of noncanonical-nonunitary transforms, such as

$$U \times U^\dagger \neq I, \quad (3.2.212a)$$

$$I \rightarrow \hat{I} = U \times I \times U^\dagger, \quad (3.2.212b)$$

$$X_i \times X_j \rightarrow U \times (X_i \times X_j) \times U^\dagger = \hat{X}_i \hat{\times} \hat{X}_j, \quad (3.2.212c)$$

$$X_i \times X_j \times X_k \rightarrow U \times (X_i \times X_j \times X_k) \times U^\dagger = \hat{X}_i \hat{\times} \hat{X}_j \hat{\times} \hat{X}_k, \;\; \text{etc.} \quad (3.2.212d)$$

Nevertheless, the uncontrolled use of the above transforms may lead to misrepresentations. In fact, a primary objective of the Lie-Santilli isotheory is that of preserving the original generators and parameters and change instead the associative and Lie products in an axiom-preserving way to accommodate the treatment of nonlinear, nonlocal and noncanonical interactions.

The preservation of the generators is, in particular, necessary for physical consistency because they represent conserved total quantities (such as the total energy, total angular momentum, etc.). These total quantities remain unchanged in the transition from closed Hamiltonian and non-Hamiltonian systems see Section 3.1.2). Equivalently, the generators of Lie's theory cannot be altered by non-Hamiltonian effects.

This physical requirement can only be achieved by preserving conventional generators X_k and lifting instead their product $X_i \times X_j \rightarrow X_i \hat{\times} X_j = X_i \times \hat{T} \times X_j$, which is the original formulation of the Lie-Santilli isotheory [4] and remain the formulation needed for applications to this day. It is essentially given by the projection of the isotopic formulation on conventional spaces over conventional fields.

3.12C. Lie-Santilli Isoalgebras. As it is well known, Lie algebras are the antisymmetric algebras $L \approx [\xi(L)]^-$ attached to the universal enveloping algebras $\xi(L)$. This main characteristic is preserved although enlarged under isotopies (see [4,2] for details). We therefore have the following

DEFINITION 3.2.15 [4]: A finite-dimensional isospace $\hat{L}$ with generic elements $\hat{A}, \hat{B}, \dots$, over the isofield $\hat{F}$ with isounit $\hat{I} = 1/\hat{T} > 0$ is called a "Lie-Santilli isoalgebra" over $\hat{F}$ when there is a composition $[\hat{A}\hat{,}\hat{B}]$ in $\hat{L}$, called "isocommutator", that is isolinear as an isovector space and such that all the following axioms are satisfied

$$[\hat{A}\hat{,}\hat{B}] = -[\hat{B}\hat{,}\hat{A}], \tag{3.2.213a}$$

$$[\hat{A}\hat{,}[\hat{B}\hat{,}\hat{C}]] + [\hat{B}\hat{,}[\hat{C}\hat{,}\hat{A}]] + [\hat{C}\hat{,}[\hat{A}\hat{,}\hat{B}]] \equiv 0, \tag{3.2.213b}$$

$$[\hat{A}\hat{\times}\hat{B}\hat{,}\hat{C}] = \hat{A}\hat{\times}[\hat{B}\hat{,}\hat{C}] + [\hat{A}\hat{,}\hat{C}]\hat{\times}\hat{B}, \quad \forall \hat{A}, \hat{B}, \hat{C} \in \hat{L}. \tag{3.2.213c}$$

The isoalgebras are said to be: *isoreal, isocomplex or isoquaternionic* depending on the assumed isofield and *isoabelian* when $[\hat{A}\hat{,}\hat{B}] \equiv \forall \hat{A}, \hat{B} \in \hat{L}$. A subset $\hat{L}^o$ of $\hat{L}$ is said to be an *isosubalgebra* of $\hat{L}$ when $[\hat{L}^o\hat{,}\hat{L}^o] \subseteq \hat{L}^o$. $\hat{L}^o$ is called an *isoideal* of $\hat{L}$ when $[\hat{L}^o\hat{,}\hat{L}] \subseteq \hat{L}^o$. A *maximal isoideal* verifying the property $[\hat{L}^o\hat{,}\hat{L}^o] = 0$ is called the *isocenter* of $\hat{L}$.

For the isotopies of additional conventional notions, theorems and properties of Lie algebras, one may see monograph [2,6,36,37].

We merely recall the *isotopic generalizations of the celebrated Lie's First, Second and Third Theorems* introduced in the original proposal [4], but which we do not review here for brevity. For instance, the *Lie-Santilli Second Isotheorem* reads

$$[\hat{X}_i\hat{,}\hat{X}_j] = \hat{X}_i\hat{\times}\hat{X}_j - \hat{X}_j\hat{\times}\hat{X}_i = \tag{3.2.214a}$$

$$= \hat{X}_i \times \hat{T}(x, \dot{x}, \ddot{x}, \dots) \times \hat{X}_j - \hat{X}_j \times \hat{T}(x, \dot{x}, \ddot{x}, \dots) \times \hat{X}_i = \hat{C}^k_{ij}(x, \dot{x}, \ddot{x}, \dots)\hat{\times}\hat{X}_k, \tag{3.2.214b}$$

where the C's, called the *structure isofunctions*, generally have an explicit dependence on the underlying isovariable (see the examples later on), and verify certain restrictions from the Isotopic Third Theorem.

It is today known that Lie-Santilli isoalgebras can be reached via a noncanonical-nonunitary transform of conventional Lie algebras. In fact, we have

$$[X_i, X_j] = C^k_{ij} \times X_k \rightarrow$$

$$U \times [X_i, X_j] \times U^\dagger = [\hat{X}_i\hat{,}\hat{X}_j] =$$

$$U \times (C^k_{ij} \times X_k) \times U^\dagger = \hat{C}^k_{ij}(x, \dot{x}, \ddot{x}, \dots)\hat{\times}\hat{X}_k. \tag{3.2.215}$$

However, again, this type of derivation of the isotheory may be misleading in physical applications due to the need to preserve the original generators unchanged, in accordance with the original formulation [4] of 1978. In this

case we shall use the following projection of the isoalgebras on the original space over the original field

$$[X_i \hat{,} X_j] = X_i \times \hat{T} \times X_j - X_j \times \hat{T} \times X_i = C_{ij}^k(x, \dot{x}, \dots) \times X_k. \qquad (3.2.216)$$

It has been proved (see, e.g., [2,4,6] for details) that *Lie-Santilli isoalgebras $\hat{L}$ are isomorphic to the original algebra L.* In other words, the isotopies with $\hat{I} > 0$ cannot characterize any new algebra because all possible Lie algebras are known from Cartan classification. Therefore, Lie-Santilli isoalgebras merely provide new nonlinear, nonlocal and noncanonical realizations of existing algebras. It should be stresses that the above isomorphism is lost for more general liftings as shown in the next chapter.

3.2.12D. Lie-Santilli Isogroups. Under certain integrability conditions hereon assumed, Lie algebras L can be "exponentiated" to their corresponding *Lie transformation groups* G and, vice-versa, Lie transformation groups G admit their corresponding Lie algebra L when computed in the neighborhood of the identity I.

These basic properties are preserved under isotopies although broadened to the most general possible nonlinear, nonlocal and noncanonical transformations groups.

DEFINITION 3.2.16 [4]: A right isomodular Lie-Santilli isotransformation group $\hat{G}$ on an isospace $\hat{S}(\hat{x}, \hat{F})$ over an isofield $\hat{F}$ with common isounit $\hat{I} = 1/\hat{T} > 0$ is a group mapping each element $\hat{x} \in \hat{S}$ into a new element $\hat{x}' \in \hat{S}$ via the isotransformations

$$\hat{x}' = \hat{g}(\hat{w}) \hat{\times} \hat{x}, \;\; \hat{x}, \hat{x}' \in \hat{S}, \;\; \hat{w} \in \hat{F}, \qquad (3.2.217)$$

such that:
1) The map $\hat{g} \hat{\times} \hat{S}$ into $\hat{S}$ is isodifferentiable $\forall \hat{g} \in \hat{G}$;
2) $\hat{I}$ is the left and right unit

$$\hat{I} \hat{\times} \hat{g} = \hat{g} \hat{\times} \hat{I} \equiv \hat{g}, \;\; \forall \hat{g} \in \hat{G}; \qquad (3.2.218)$$

3) the isomodular action is isoassociative, i.e.,

$$\hat{g}_1 \hat{\times} (\hat{g}_2 \hat{\times} \hat{x}) = (\hat{g}_1 \hat{\times} \hat{g}_2) \hat{\times} \hat{x}, \;\; \forall \hat{g}_1, \hat{g}_2 \in \hat{G}; \qquad (3.2.219)$$

4) in correspondence with everyt element $\hat{(}\hat{w}) \in \hat{G}$ there is the inverse element $\hat{g}^{-\hat{I}} = \hat{g}(-\hat{w})$ such that

$$\hat{g}(\hat{0}) = \hat{g}(\hat{w}) \hat{\times} \hat{g}(-\hat{w}) = \hat{I}; \qquad (3.2.220)$$

5) following composition laws are verified

$$\hat{g}(\hat{w})\hat{\times}\hat{g}(\hat{w}') = \hat{g}(\hat{w}')\hat{\times}\hat{g}(\hat{w}) = \hat{g}(\hat{w}+\hat{w}'), \forall \hat{g} \in \hat{G},\ \hat{w} \in \hat{F}. \tag{3.2.221}$$

The I left isotransformation group is defined accordingly.

The notions of *connected or simply connected transformation groups* carry over to the isogroups in their entirety.

The most direct realization of the (connected) isotransformation groups is that via isoexponentiation,

$$\hat{g}(w) = \prod_k \hat{e}^{\hat{i}\hat{\times}\hat{w}_k\hat{X}_k} = \Big(\prod_k e^{i\times w_k\times X_k\times\hat{T}(x,\dot{x},\ddot{x},\dots)}\Big)\times\hat{I}, \tag{3.2.222}$$

where the X's and w's are the infinitesimal generators and parameters, respectively, of the original algebra L, with corresponding connected isotransformations

$$\hat{x}' = \hat{g}(\hat{w})\hat{\times}\hat{x} = \Big(\prod_k \hat{e}^{\hat{i}\hat{\times}\hat{w}_k\hat{X}_k}\Big)\times\hat{I}\times\hat{T}\times x\times\hat{I} =$$

$$= \Big(\prod_k e^{i\times w_k\times X_k\times\hat{T}(x,\dot{x},\ddot{x},\dots)}\Big)\times x\times\hat{I}. \tag{3.2.223}$$

Equations (3.2.223) hold in some open neighborhood N of the isoorigin of $\hat{L}$ and, in this way, characterize some open neighborhood of the isounit of $\hat{G}$. Consequently, under the assumed continuity and connectivity properties, Lie-Santilli isoalgebras can be obtained as infinitesimal versions of finite Lie-Santilli isogroups, as illustrated by the following finite isotransform

$$\hat{A}(\hat{w}) = (\hat{e}^{\hat{i}\hat{\times}\hat{w}\hat{\times}\hat{X}})\hat{\times}\hat{A}(\hat{0})\hat{\times}(\hat{e}^{-\hat{i}\hat{\times}\hat{w}\hat{\times}\hat{X}}) =$$

$$= (e^{i\times w\times\hat{X}\times\hat{T}})\times\hat{A}(\hat{0})\times(e^{-i\times w\times\hat{T}\times\hat{X}}) \tag{3.2.224}$$

with infinitesimal version in the neighborood of $\hat{I}$

$$\hat{A}(\hat{d}\hat{w}) = (\hat{I}+\hat{i}\hat{\times}\hat{d}\hat{w}\hat{\times}\hat{X}+\dots)\hat{\times}\hat{A}(0)\hat{\times}(\hat{I}-\hat{i}\hat{\times}\hat{d}\hat{w}\hat{\times}\hat{X}+\dots) =$$

$$= \hat{A}(\hat{0}) + \hat{i}\hat{\times}\hat{d}\hat{w}\hat{\times}\hat{X}\hat{\times}\hat{A}(\hat{0}) - \hat{i}\hat{\times}\hat{d}\hat{w}\hat{\times}\hat{A}(\hat{0})\hat{\times}\hat{X}, \tag{3.2.225}$$

that can be written

$$\hat{i}\hat{\times}\frac{\hat{d}\hat{A}(\hat{w})}{\hat{d}\hat{w}} = \hat{A}\hat{\times}\hat{X} - \hat{X}\hat{\times}\hat{A} = [\hat{A}\hat{,}\hat{X}]. \tag{3.2.226}$$

Note the crucial appearance of the isotopic element $\hat{T}(x,\dot{x},\ddot{x},\dots)$ in the exponent of the isogroup. This ensures a structural generalization of Lies theory of the desired nonlinear, nonlocal and noncanonical form.

Still another important property is that conventional group composition laws admit a consistent isotopic lifting, resulting in the following *Baker-Campbell-Hausdorff-Santilli Isotheorem* [4]

$$(\hat{e}^{\hat{X}_1})\hat{\times}(\hat{e}^{X_2}) = \hat{e}^{\hat{X}_3}, \tag{3.2.227a}$$

$$\hat{X}_3 = \hat{X}_1 + \hat{X}_2 + [\hat{X}_1\hat{,}\hat{X}_2]\hat{/}\hat{2} + [(\hat{X}_1 - \hat{X}_2)\hat{,}[\hat{X}_1\hat{,}\hat{X}_2]]\hat{/}\hat{12} + \dots . \tag{3.2.227b}$$

Let $\hat{G}_1$ and $\hat{G}_2$ be two isogroups with respective isounits $\hat{I}_1$ and $\hat{I}_2$. The *direct isoproduct* $\hat{G}_1\hat{\times}\hat{G}_2$ is the isogroup of all ordered pairs

$$(\hat{g}_1, \hat{g}_2), \quad \hat{g}_1 \in \hat{G}_1, \hat{g}_2 \in \hat{G}_2, \tag{3.2.228}$$

with isomultiplication

$$(\hat{g}_1, \hat{g}_2)\hat{\times}(\hat{g}_1', \hat{g}_2') = (\hat{g}_1\hat{\times}\hat{g}_1', \hat{g}_2\hat{\times}\hat{g}_2'), \tag{3.2.229}$$

total isounit $(\hat{I}_1, \hat{I}_2)$ and inverse $(\hat{g}_1^{-\hat{I}_1}, \hat{g}_2^{-\hat{I}_2})$.

The following particular case is important for the isotopies of inhomogeneous groups. Let $\hat{G}$ be an isogroup and $\hat{G}_{\hat{a}}$ the isogroup of all its inner isoautomorphisms. Let $\hat{G}^o_{\hat{a}}$ be a subgroup of $\hat{G}_{\hat{a}}$, and let $\Lambda(\hat{g})$ be the image of $\hat{g} \in \hat{G}$ under $\hat{G}_{\hat{a}}$. The *semidirect isoproduct* $\hat{G}\hat{\times}\hat{G}^o_{\hat{a}}$ is the isogroup of all ordered pairs

$$(\hat{g}, \hat{\Lambda})\hat{\times}(\hat{g}^o, \hat{\Lambda}^o) = (\hat{g}, \hat{\Lambda}(\hat{g}^o), (\hat{\Lambda}, \hat{\Lambda}^o), \tag{3.2.230}$$

with total isounit given by $\hat{I}_{tot} = \hat{I} \times \hat{I}^o$.

The the studies of the isotopies of the remaining aspects of the structure theory of Lie groups is then consequential.

It is hoped that the reader can see from the above elements that the entire conventional Lie theory does indeed admit a consistent and nontrivial lifting into the covering Lie-Santilli formulation.

3.2.12E. Isorepresentations of Lie-Santilli Isoalgebras. Despite considerable research on the Lie-Santilli isotheory over the past 26 years, the study of the *isorepresentations* of the Lie-Santilli isoalgebras remains vastly unknown at this writing (summer 2004), with the sole exception of the *fundamental (or regular) isorepresentations* that were also identified by Santilli in the original proposal [4].

In this monograph we shall primarily use in the applications of hadronic mechanics the fundamental isorepresentations or other isorepresentations reducible to the latter.

Let L be an N-dimensional Lie algebra with N-dimensional unit $I = Diag.$-$(1, 1, \dots, 1)$. Let R be the fundamental, N-dimensional matrix representation of L. Let $\hat{L}$ be the isotope of L characterized by the N-dimensional isounit

$\hat{I} = U \times U^\dagger > 0$. It is then evident that the it fundamental isorepresentation of $\hat{L}$ is given by

$$\hat{R} = U \times R \times U^\dagger, \quad U \times U^\dagger = \hat{I} \neq I, \hat{I} > 0. \tag{3.2.231}$$

Interested colleagues are encouraged to study the isorepresentation theory because, as we shall see in the next sections, the fundamental notion of hadronic mechanics, that of *isoparticles,* is characterized by an irreducible isorepresentation of the Poincaré-Santilli isosymmetry.

3.2.12F. Isodual Lie-Santilli Isotheory. As indicated Chapters 1 and 2, the contemporary formulation of Lie's theory is one of the most serious obstacles for a consistent *classical* representation of antimatter, because it lacks an appropriate conjugate formulation that, after quantization, is compatible with charge conjugation.[20]

It is easy to verify that the isotheory presented above admits a consistent antiautomorphic image under isoduality, thus thus permitting the treatment of antimatter under nonlinearity, nonlocality and noncanonicity as occurring in interior conditions, such as for the structure of an antimatter star.

In fact, we have the *isodual universal enveloping isoassociative isoalgebra* $\hat{\xi}^d$ characterized by the *isodual Poincaré-Birkhoff-Witt-Santilli isotheorem* with infinite dimensional basis

$$\hat{I}^d, \;\; X_k^d, \;\; \hat{X}_i^d \hat{\times}^d \hat{X}_j^d, \; i \leq j, \;\; \hat{X}_i^d \hat{\times}^d \hat{X}_j^d \hat{\times}^d \hat{X}_k^d, \; i \leq j \leq k, \; \ldots . \tag{3.2.232}$$

The *isodual Lie-Santilli isoalgebra* $\hat{L}^d \approx (\hat{\xi}^d)^-$ attached to $\hat{\xi}^d$ is characterized by the *isodual Lie-Santilli Second Isotheorem*

$$[\hat{X}_i^d \hat{,} \hat{X}_j^d] = \hat{X}_i^d \hat{\times}^d \hat{X}_j^d - \hat{X}_j^d \hat{\times}^d \hat{X}_i^d = \hat{C_{ij}^d}^k \hat{\times}^d \hat{X}_k^d. \tag{3.2.233}$$

Under the needed continuity and connectivity property, the *isodual exponentiation* of $\hat{L}^d$ characterizes the *connected isodual Lie-Santilli transformation isogroup*

$$\hat{x}'^d = (\hat{g}^d(\hat{w}^d) = \hat{\prod}^k \hat{e}^{\hat{d}^{\hat{i}^d \hat{\times}^d \hat{w}_k^d \hat{X}_k^d}}) \hat{\times}^d \hat{x}^d. \tag{3.2.234}$$

Interested readers can then easily derive any additional needed isodual property.

[20]The reader is urged to verify that the classical treatment of antimatter via the so-called *dual Lie algebras* does not achieve antiparticles under quantization, trivially, because of the uniqueness of the quantization channel for both particles and antiparticles.

3.2.13 Unification of All Simple Lie Algebras into Lie-Santilli Isoalgebras

The original proposal [4] of 1978 included the *conjecture that all simple Lie algebras of dimension N can be unified into a single Lie-Santilli isoalgebra of the same dimension,* and gave an explicit example. The conjecture was subsequently proved by the late mathematicians Gr. Tsagas [42] in 1996 for all simple Lie algebras of type A, B, C and D. The premature departure of Prof. Tsagas while working at the problem prevented him to complete the proof of the conjecture for the case of all exceptional Lie algebras. As a result, the proof of the indicated conjecture remain incomplete at this writing.

For the unification here considered it is important to eliminate the restriction that the isounits are necessarily positive definite, while preserving all other characteristics, such as nowhere singularity and Hermiticity. As a result, in its simple possible form, the isounit can be diagonalized into the form whose elements can be either positive or negative,

$$\hat{I} = Diag.(\pm n_1^2, \pm n_2^2, \dots, \pm n_N^2) = 1/\hat{T},\ n_k \in R,\ n_k \neq 0,\ k = 1, 2, \dots, N. \tag{3.2.235}$$

The example provided in the original proposal [4], subsequently studied in detail in Refs. [8], consisted in the *classification of all possible simple Lie algebra of dimension 3.* In this case, Cartan's classification produces two non-isomorphic Lie algebras, the compact rotational algebra in three dimension $SO(3)$ and the noncompact algebra $SO(2.1)$.

The distinction between compact and noncompact algebras are lost under the class of isotopies here considered. In fact, the *classification of all possible, simple, three-dimensional Lie-Santilli isoalgebras $\hat{L}_3$ for the case of diagonal isounits* is characterized by the isounit itself and can be written

$$\hat{I} = Diag.(+1, +1, +1),\ \ \hat{L}_3 \approx SO(3), \tag{3.2.236a}$$

$$\hat{I} = Diag.(+1, +1, -1),\ \ \hat{L}_3 \approx SO(2.1), \tag{3.2.236b}$$

$$\hat{I} = Diag.(+1, -1, +1),\ \ \hat{L}_3 \approx SO(2.1), \tag{3.2.236c}$$

$$\hat{I} = Diag.(-1, +1, +1),\ \ \hat{L}_3 \approx SOI(2.1), \tag{3.2.236d}$$

$$\hat{I} = Diag.(-1, -1, -1),\ \ \hat{L}_3 \approx SO(3)^d, \tag{3.2.236e}$$

$$\hat{I} = Diag.(-1, -1, +1),\ \ \hat{L}_3 \approx SO(2.1)^d, \tag{3.2.236f}$$

$$\hat{I} = diag.(-1, +1, -1),\ \ \hat{l}_3 \approx SO(2.1), \tag{3.2.236g}$$

$$\hat{I} = Diag.(+1, -1, -1),\ \ \hat{L}_3 \approx SO(2.1)^d, \tag{3.2.236h}$$

$$\hat{I} = Diag.(+n_1^2, +n_2^2, +n_3^2),\ \ \hat{L}_3 \approx SO(3), \tag{3.2.236i}$$

$$\hat{I} = Diag.(+n_1^2, +n_2^2, -n_3^2), \quad \hat{L}_3 \approx SO(2.1), \tag{3.2.236j}$$

$$\hat{I} = Diag.(+n_1^2, -n_2^2, +n_3^2), \quad \hat{L}_3 \approx SO(2.1), \tag{3.2.236k}$$

$$\hat{I} = Diag.(-n_1^2, +n_2^2, +n_3^2), \quad \hat{L}_3 \approx SOI(2.1), \tag{3.2.236l}$$

$$\hat{I} = Diag.(-n_1^2, -n_2^2, -n_3^2), \quad \hat{L}_3 \approx SO(3)^d, \tag{3.2.236m}$$

$$\hat{I} = Diag.(-n_1^2, -b_2^2, +n_3^2), \quad \hat{L}_3 \approx SO(2.1)^d, \tag{3.2.236n}$$

$$\hat{I} = diag.(-n_1^2, +n_2^2, -n_3^2), \quad \hat{L}_3 \approx SO(2.1), \tag{3.2.236o}$$

$$\hat{I} = Diag.(+n_1^2, -n_2^2, -n_3^2), \quad \hat{L}_3 \approx SO(2.1)^d, \tag{3.2.236p}$$

In conclusion, when studying simple algebras from the viewpoint of the covering Lie-Santilli isoalgebras, there exist *only one single isoalgebra in three dimensions,* $\hat{L}_3$ without any distinction between compact and noncompact algebras.

The *realization* of the simple isoalgebra $\hat{L}_3$ with diagonal isounits consists of 21 different Lie-Santilli isoalgebras in three dimension that can be reduced to 4 topologically different Lie algebras, namely $SO(3)$, $SO(2.1)$, $SO(3)^d$ and $SO(2.1)^d$.

All distinctions between these 21 different realizations are lost at the level of abstract Lie-Santilli isoalgebra $\hat{L}_3$.

It should be stressed that, by no means, the 21 realizations (3.2.236) exhaust all possible forms of Lie-Santilli simple isoalgebras in three dimensions because in realizations (3.2.236) we have excluded nondiagonal realizations of the isounit, as well as imposed additional restrictions on the isounit, such as single valuedness and Hermiticity.

Essentially the same results hold for the unification of the Lie Algebras of type A, B, C, and D studied by Tsagas [42].

It is hoped that interested mathematicians can complete the proof of Santilli's conjecture for the remaining exceptional algebras. In considering the problem, mathematicians are suggested to keep in mind that Hermitean and diagonal realizations of the isounit (3.2.135) are expected to be insufficient, thus implying the possible use of *nowhere singular, Hermitean, nondiagonal isounits,* or *nowhere singular, Hermitean, nondiagonal and multivalued isounits,* or *nowhere singular, non-Hermitean, nondiagonal and multivalued isounits.*

3.2.14 The Fundamental Theorem for Isosymmetries and Their Isoduals

The fundamental symmetries of the 20-th century physics deal with point-like abstractions of particles in vacuum under linear, local and potential interactions, and are the *Galilei symmetry* $G(3.1)$ for nonrelativistic treatment or the *Poincaré symmetry* for relativistic formulations.

A central objective of hadronic mechanics is the broadening of these fundamental spacetime symmetry to represent extended, nonspherical and deformable particles under linear and nonlinear, local and nonlocal and potential as well as nonpotential interactions.

In fact, as we shall see, all novel industrial applications of hadronic mechanics are crucially dependent on the admission of the extended character of particles or of their wavepackets in conditions of deep mutual penetration. In turn, the latter conditions imply new effects permitting basically new energies and fuels that are completely absent for conventional spacetime and other symmetries.

Alternatively and equivalent a central problem of hadronic mechanics is the *the construction in an explicit form of the symmetries of all possible nonsingular, but otherwise arbitrary deformations of conventional spacetime and internal invariants.*

All these problems and others are resolved by the following important:

THEOREM 3.2.12 [6]: Let G be an N-dimensional Lie symmetry group of a K-dimensional metric or pseudo-metric space $S(x, m, F)$ over a field F,

$$G: \quad x' = \Lambda(w) \times x, \quad y' = \Lambda(w) \times y, \quad x, y \in \hat{S}, \tag{3.2.237a}$$

$$(x' - y')^\dagger \times \Lambda^\dagger \times m \times \Lambda \times (x - y) \equiv (x - y)^\dagger \times m \times (x - y), \tag{3.2.237b}$$

$$\Lambda^\dagger(w) \times m \times \Lambda(w) \equiv m. \tag{3.2.237c}$$

Then, all infinitely possible isotopies $\hat{G}$ of G acting on the isospace $\hat{S}(\hat{x}, \hat{M}, \hat{F})$, $\hat{M} = \hat{m} \times \hat{I} = (\hat{T}_i^k \times m_{kj}) \times \hat{I}$ characterized by the same generators and parameters of G and new isounits $\hat{I} = 1/\hat{T} > 0$ leave invariant the isocomposition on the the projection $\hat{S}(x, \hat{m}, F)$ of $\hat{S}(\hat{x}, \hat{M}, \hat{F})$ on the original space $S(x, m, F)$

$$\hat{G}: \quad x' = \hat{\Lambda}(w) \times x, \quad y' = \hat{\Lambda}(w) \times y, \quad x, y \in \hat{S}, \tag{3.2.238a}$$

$$(x' - y')^\dagger \times \hat{\Lambda}^\dagger \times \hat{m} \times \hat{\Lambda} \times (x - y) \equiv (x - y)^\dagger \times \hat{m} \times (x - y), \tag{3.2.238b}$$

$$\hat{\Lambda}^\dagger(\hat{w}) \times \hat{m} \times \hat{\Lambda}(\hat{w}) \equiv \hat{m}. \tag{3.2.238c}$$

Similarly,all infinitely possible isodual isotopies $\hat{G}^d$ of $\hat{G}$ actiing on the isodual isospace $\hat{S}^d(\hat{x}^d, \hat{M}^d, \hat{F}^d)$, $\hat{M}^d = (\hat{T}^d \times m^d) \times \hat{I}^d$ characterized by the isodual generators $\hat{X}_k^d$ parameters $\hat{w}^d$ and isodual isounit $\hat{I}^d = 1/\hat{T}^d < 0$ leave invariant the isodual isocomposition on the projection $\hat{S}^d(x^d, \hat{m}^d, F^d)$

$$\hat{G}^d: \quad x'^d = \hat{\Lambda}^d \times^d x^d, \quad y'^d = \hat{\Lambda}^d \times^d y^d, \quad x^d, y^d \in \hat{S}^d, \tag{3.2.239a}$$

$$(x' - y')^{\dagger^d} \times^d \hat{\Lambda}^{\dagger^d} \times^d \hat{m}^d \times^d \hat{\Lambda}^d \times^d (x - y)^d \equiv (x - y)^{\dagger^d} \times^d \hat{m}^d \times^d (x - y)^d, \tag{3.2.239b}$$

$$\hat{\Lambda}^{\dagger^d} \times^d \hat{m}^d \times^d \hat{\Lambda}^d \equiv \hat{m}^d. \tag{3.2.239c}$$

Proof. Assume that $N = K$ and the representation Λ is the fundamental one. Recall that metrics, isometrics and isounits are diagonal. Then on $\hat{S}(x, \hat{m}, F)$ we have the identities

$$\hat{I} = U \times U^\dagger \neq I, \;\; \hat{T} = (U \times U^\dagger)^{-1}, \tag{3.2.240a}$$

$$U \times (\Lambda \times m \times \Lambda) \times U^\dagger =$$
$$= (U \times \Lambda \times U^\dagger) \times (U^{\dagger^{-1}} \times m \times U^{-1}) \times (U \times \Lambda \times U^\dagger) =$$
$$= \hat{\Lambda} \times (\hat{T} \times m) \times \hat{\Lambda} = \hat{\Lambda} \times \hat{m} \times \hat{\Lambda} = \hat{m}. \tag{3.2.240b}$$

The proof of the remaining cases are equally trivial. **q.e.d.**

Note that the isotopic symmetries and their isoduals can be uniquely and explicitly constructed with the methods summarized in this section via the sole use of the original symmetry and the isounit characterizing the deformation of the original metric m.

Under our assumptions, the isosymmetries can be constructed in the needed, explicit, nonlinear, nonlocal and noncanonical forms. In fact, the existence of the original symmetry transformations plus the condition $\hat{I} > 0$ ensure the convergence of the infinite isoseries of the isoexponentiation, resulting in the needed explicit form, as we shall see in various examples in the next sections.

3.3 CLASSICAL LIE-ISOTOPIC MECHANICS FOR MATTER AND ITS ISODUAL FOR ANTIMATTER

3.3.1 Introduction

One of the reasons for the majestic consistency of quantum mechanics is the existence of axiomatically consistent and invariant classical foundations, given by *classical Lagrangian and Hamiltonian mechanics,* namely, the discipline based on the *truncated analytic equations*

$$\frac{d}{dt}\frac{\partial L(t,r,v)}{\partial v_a^k} - \frac{\partial L(t,r,v)}{\partial r_a^k} = 0, \tag{3.3.1a}$$

$$\frac{dr_a^k}{dt} = \frac{\partial H(t,r,p)}{\partial p_{ak}}, \quad \frac{dp_{ak}}{dt} = -\frac{\partial H(t,r,p)}{\partial r_a^k}, \tag{3.3.1b}$$
$$k = 1,2,3; \;\; a = 1,2,3,\ldots,N,$$

with a unique and unambiguous map into operator forms.

Following the original proposal [5] of 1978 to build hadronic mechanics, this author did not consider the new discipline sufficiently mature for experimental verifications and industrial applications until the new discipline had equally consistent and invariant classical foundations with an equally unique and unambiguous map into operator formulations.

Intriguingly, the *operator* foundations of hadronic mechanics were sufficiently identified in the original proposal [5], as we shall see in the next section. However, the identification of the *classical* counterpart turned out to be a rather complex task that required decades of research.

The objective, fully identified in 1978, was the construction of a covering of classical Lagrangian and Hamiltonian mechanics, namely, a covering of Eqs. (3.3.1), admitting a unique and unambiguous map into the already known Lie-isotopic equations of hadronic mechanics.

The mandatory starting point was the consideration of the *true Lagrange and Hamilton equations,* those with external terms

$$\frac{d}{dt}\frac{\partial L(t,r,v)}{\partial v_a^k} - \frac{\partial L(t,r,v)}{\partial r_a^k} = F_{ak}(t,r,v), \tag{3.3.2a}$$

$$\frac{dr_a^k}{dt} = \frac{\partial H(t,r,p)}{\partial p_{ak}}, \quad \frac{dp_{ak}}{dt} = -\frac{\partial H(t,r,p)}{\partial r_a^k} + F_{ak}(t,r,p), \tag{3.3.2b}$$

since they were conceived, specifically, for the interior dynamical systems treated by hadronic mechanics.

In fact, the legacy of Lagrange and Hamilton is that classical systems *cannot* be entirely represented with one single function today called a Lagrangian or a Hamiltonian used for the representation of forces derivable from a potential, but require additional quantities for the representation of contact nonpotential forced represented precisely by the external terms.

As such, the true Lagrange and Hamilton equations constitute excellent candidates for the classical origin of hadronic mechanics.

3.3.2 Insufficiencies of Analytic Equations with External Terms

It was indicated by Santilli [4] also in 1978 (see the review in Chapter 1 for more details) that the true analytic equations cannot be used for the construction of a consistent covering of conventional analytic equations because the new algebraic brackets of the time evolution of a generic quantity $A(r,p)$ in phase space

$$\frac{dA}{dt} = (A,H,F) = [A,H] + \frac{\partial A}{\partial r^k} \times F_k =$$
$$= \frac{\partial A}{\partial r^k} \times \frac{\partial H}{\partial p_k} - \frac{\partial H}{\partial r^k} \times \frac{\partial A}{\partial p_k} + \frac{\partial A}{\partial r^k} \times F_k, \tag{3.3.3}$$

violate the right distributive and scalar laws, Eqs. (3.2.5) and (3.2.6). Consequently, *the true analytic equations in their original formulation lose "all" possible algebras, let alone all possible Lie algebras.* No axiomatically consistent covering can then be build under these premises.[21]

The above insufficiency essentially established the need of rewriting the true analytic equations into a form admitting a consistent algebra in the brackets of the time evolution laws and, in addition, achieves the same invariance possessed by the truncated analytic equations.

Even though its main lines were fully identified in 1978, the achievement of the new covering mechanics resulted to require a rather long and laborious scientific journey.

This section is intended to outline the final formulation of the classical mechanics underlying hadronic mechanics in order to distinguish it from the numerous attempts that were published with the passing of time.

As a brief guide to the literature, the reader should be aware that the true analytic equations (3.3.2) are generally set for *open nonconservative systems.* These systems require the broader *Lie-admissible branch of hadronic mechanics* that will be studied in the next chapter.

Therefore, the reader should be aware that several advances in Lie-isotopies have been obtained and can be originally identified as particular cases of the broader Lie-admissible theories.

This chapter is dedicated to the study of classical and operator closed-isolated systems verifying conventional total conservation laws while having linear and nonlinear, local and nonlocal as well as potential and nonpotential internal forces.

The verification of conventional total conservation law requires classical brackets that, firstly, verify the right and left distributive and scalar laws (as a condition to characterize an algebra), and, secondly, the brackets are necessarily antisymmetric.

The brackets of conventional Hamiltonian ,mechanics are Lie. Therefore, a necessary condition to build a true covering of Hamiltonian mechanics is the search of brackets that are of the broader Lie-isotopic type. As a matter of fact, this feature, fully identified in 1978 [4,5], was the very motivation for the construction of the isotopies of the Lie theory reviewed in Section 3.2.12.

In summary, the construction of a covering of the conventional Hamiltonian mechanics as the classical foundations of the Lie-isotopic branch of hadronic mechanics must be restricted to a reformulation of the true analytic equations (3.3.2) in such a way that the underlying brackets are Lie-isotopic, and the resulting mechanics is invariant.

[21]For additional problematic aspects of the true analytic equations, one may consult Ref. [4] or the review in Chapter 1.

3.3.3 Insufficiencies of Birkhoffian Mechanics

Santilli dedicated the second volume of *Foundations of Theoretical Mechanics* published by Springer-Verlag [2] in 1982 to the construction of a covering of classical Hamiltonian mechanics along the above indicated requirement. The resulting new mechanics was released under the name of *Birkhoffian mechanics* to honor G. D. Birkhoff who first discovered the underlying analytic equations in 1927.[22]

Conventional Hamiltonian mechanics is based on the *canonical action principle*

$$\delta A^o = \delta \int (p_k \times dr^k - H \times dt) = 0, \tag{3.3.4}$$

and, via the use of the unified notation

$$b = (b^\mu) = (r^i, p_j), \tag{3.3.5a}$$

$$R^o = (R^o_\mu) = (p_k, 0), \quad \mu = 1, 2, \ldots, 6, \tag{3.3.5b}$$

can be written

$$\delta A^o = \delta \int (R^o_\mu \times db^\mu - H \times dt) \equiv$$

$$\equiv \delta \int (p_k \times dr^k - H \times dt) = 0. \tag{3.3.6}$$

from which the conventional Hamilton's equations (3.3.1b) acquire the unified form

$$\omega_{\mu\nu} \times \frac{db^\nu}{dt} = \frac{\partial H}{\partial b^\mu}, \tag{3.3.7}$$

where

$$\omega_{\mu\nu} = \frac{\partial R^o_\nu}{\partial b^\mu} - \frac{\partial R^o_\mu}{\partial b^\nu} \tag{3.3.8}$$

is the fundamental (canonical) symplectic tensor (3.2.187).

The fundamental (conventional Poisson) brackets of the time evolution then acquire the unified form

$$\frac{dA}{dt} = [A, H] = \frac{\partial A}{\partial b^\mu} \times \omega^{\mu\nu} \times \frac{\partial H}{\partial b^\nu}, \tag{3.3.9}$$

where

$$\omega^{\mu\nu} = [(\omega_{\alpha\beta})^{-1}]^{\mu\nu} \tag{3.3.10}$$

is the fundamental (canonical) Lie tensor.

[22]Interested readers should consult, for brevity, the historical notes of Ref. [2].

Santilli [2] based the construction of a covering isotopic (that is, axiom-preserving) mechanics on the most general possible *Pfaffian action principle*

$$\delta A = \delta \int (R_\mu \times db^\mu - B \times dt) = 0, \tag{3.3.11}$$

where the $R_\mu(b)$ functions are now arbitrary functions in phase space, e.g., of the type

$$R(b) = (R_\mu) = (E_i(r,p), D^j(r,p)), \tag{3.3.12}$$

verifying certain regularity conditions [2].

It is easy to see that principle (3.2.10) characterizes the following analytic equations[23]

$$\Omega_{\mu\nu} \times \frac{db^\nu}{dt} = \frac{\partial B}{\partial b^\mu}, \tag{3.3.13a}$$

$$\Omega_{\mu\nu} = \frac{\partial R_\nu}{\partial b^\mu} - \frac{\partial R_\mu}{\partial b^\nu} \tag{3.3.13b}$$

is the most general possible symplectic tensor in local coordinates. Eqs. (3.3.12) were called *Birkhoff's equations* because, following a considerable research, they resulted to have been first identified by D. G. Birkhoff in 1927. The function B was called the *Birkhoffian* in order to distinguish it from the conventional Hamiltonian, since the latter represent the total energy, while the former does not.

The fundamental brackets of the time evolution then acquire the unified form

$$\frac{dA}{dt} = \frac{\partial A}{\partial b^\mu} \times \Omega^{\mu\nu} \times \frac{\partial B}{\partial b^\nu}, \tag{3.2.14a}$$

$$\Omega^{\mu\nu} = [(\Omega_{\alpha\beta})^{-1}]^{\mu\nu}. \tag{3.3.14b}$$

The covering nature of Eqs. (3.3.11)–(3.2.14) over the conventional Eqs. (3.3.4)–(3.3.10) is evident. In particular, brackets (3.3.14) are antisymmetric and verify the Lie axioms, although in the generalized Lie-Santilli isotopic form.

Moreover, Birkhoffian mechanics was proved in Ref. [2] to be "directly universal", that is, capable of representing "all" possible (sufficiently smooth and regular) Newtonian systems directly in the "frame of the observer" without any need for the transformation theory.

Therefore, at the time of releasing monograph [2] in 1982, the Birkhoffian mechanics appeared to have all the necessary pre-requisites to be the classical foundation of hadronic mechanics.

[23]The equations are called "analytic" in the sense of being derivable from a variational principle.

Unfortunately, subsequent studies established that *Birkhoffian mechanics cannot not be used for consistent physical applications* because it is afflicted by the catastrophic inconsistencies studied in Section 1.4.1, with particular reference to the lack of invariance, namely, the inability to predict the same numbers for the same physical conditions at different times owing to the non-canonical character of the time evolution.

Moreover, canonical action (3.3.4) is independent from the momenta, $A^o = A^o(r)$, while this is not the case for the Pfaffian action (3.3.11) for which we have $A = A(r, p)$. Consequently, any map into an operator form implies "wave-functions" dependent on both coordinates and momenta, $\psi(r, p)$. Therefore, the operator image of Birkhoffian mechanics is beyond our current knowledge, and its study is deferred to future generations.

The above problems requested the resumption of the search for the consistent classical counterpart of hadronic mechanics from its beginning.

Numerous additional generalized classical mechanics were identified but they still missed the achievement of the crucial invariance (for brevity, see monographs [15,16] of 1991 and the first edition of monograph [6,7] of 1993).

By looking in retrospect, the origin of all the above difficulties resulted to be where one would expect them the least, in the use of the ordinary differential calculus.

Following the discovery in 1995 (see the second edition of monographs [6,7] and Ref. [10]) of the isodifferential calculus, the identification of the final, axiomatically consistent and invariant form of the classical foundations of hadronic mechanics emerged quite rapidly.

3.3.4 Newton-Santilli Isomechanics for Matter and its Isodual for Antimatter

The fundamental character of *Newtonian Mechanics* for all scientific inquiries is due to the preservation at all subsequent levels of treatment (such as Hamiltonian mechanics, Galileo's relativity, special relativity, quantum mechanics, quantum chemistry, quantum field theory, etc.) of its main structural features, such as:

1) The underlying local-differential Euclidean topology;
2) The ordinary differential calculus; and
3) The consequential point-like approximation of particles.

Nevertheless, Newton's equations have well known notable limitations to maintain such a fundamental character for the entirely of scientific knowledge without due generalization for so many centuries.

As indicated in Chapter 1, the point-like approximation is indeed valid for very large mutual distances among particles compared to their size, as occurring for planetary and atomic systems (*exterior dynamical systems*). However, the same approximation is excessive for systems of particles at short mutual

distances, as occurring for the structure of planets, hadrons, nuclei and stars (*interior dynamical systems*).

Also, dimensionless particles cannot experience any contact or resistive interactions. Consequently, dissipative or, more generally, nonconservative forces used for centuries in Newtonian mechanics are a mere *approximation* of contact nonpotential nonlocal-integral interactions among extended constituents, the approximation being generally achieved via power series expansion in the velocities.

It should be finally recalled on historical grounds that *Newton had to construct the differential calculus as a pre-requisite for the formulation of his celebrated equations.*

No genuine structural broadening of the disciplines of the 20-th century is possible without a consistent structural generalization of their foundations, Newton's equations in Newtonian mechanics.

Santilli's isomathematics has been constructed to permit *the first axiomatically consistent structural generalization of Newton's equations in Newtonian mechanics since Newton's time, for the representation of extended, nonspherical and deformable particles under linear and nonlinear, local and nonlocal and potential as well as nonpotential interactions as occurring in the physical reality of interior dynamical systems.*

By following Newton's teaching, the author has dedicated primary efforts to the isotopic lifting of the conventional differential calculus, topology and geometries [6,10] as a pre-requisite for the indicated structural generalization of Newton's equations.

To outline the needed isotopies, let us recall that Newtonian mechanics is formulated on a 7-dimensional representation space characterized by the following Kronecker products of Euclidean spaces

$$S_{tot} = E(t, R_t) \times E(r, \delta, R_r) \times E(v, \delta, R_v), \tag{3.3.15}$$

of the one dimensional space $E(t, R_t)$ representing time t, the tree dimensional Euclidean space $E(r, \delta, R_r)$ of the coordinates $r = (r_a^k)$ (where $k = 1, 2, 3$ are the Euclidean axes and $a = 1, 2, \ldots, N$ represents the number of particles), and the velocity space $E(v, \delta, R_v), v = dr/dt$.

It is generally assumed that all variables t, r, and v are defined on the same field of real numbers R. However, the unit of time is the *scalar* $I = +1$, while the unit of the Euclidean space is the *matrix*, and the same happens for the velocities, $I_r = I_v = Diag.(1, 1, 1)$.

Therefore, on rigorous grounds, the representation space of Newtonian mechanics must be defined on the Kronecker product of the corresponding fields

$$R_{tot} = R_t \times R_r \times R_v \tag{3.3.16}$$

with total unit

$$I_{Tot} = 1_t \times Diag.(1,1,1)_r \times Diag.(1,1,1)_v. \tag{3.3.17}$$

The above total unit can be factorized into the production of seven individual units for time and the two sets of individual Euclidean axes a, y, a with corresponding factorization of the fields

$$I_{tot} = 1_t \times 1_{rx} \times 1_{ry} \times 1_{rz} \times 1_{vx} \times 1_{vy} \times 1_{vz}, \tag{3.3.18a}$$

$$R_{tot} = R_t \times R_{rx} \times R_{ry} \times R_{rz} \times R_{vx} \times R_{vy} \times R_{vz}, \tag{3.3.18b}$$

that constitute the foundations of the conventional *Euclidean topology* here assumed as known.

Via the use of Eqs. (3.1.5), *Newton's equations for closed-non-Hamiltonian systems* can then be written

$$m_a \times a_{ka} = m_a \times \frac{dv_{ka}}{dt} = F_{ka}(t,r,v) = F_{ka}^{SA} + F_{ka}^{NSA}, \tag{3.3.19a}$$

$$\sum_a \mathbf{F}_a^{NSA} = 0, \tag{3.3.19b}$$

$$\sum_a \mathbf{r}_a \bigodot \mathbf{F}_a^{NSA} = 0, \tag{3.3.19c}$$

$$\sum_a \mathbf{r}_a \bigwedge \mathbf{F}_a^{NSA} = 0, \tag{3.3.19d}$$

where SA (NSA) stands for *variational selfadjointness (variational nonselfadjointness),* namely, the verification (violation) of the integrability conditions for the existence of a potential [1], and conditions (3.3.xx), (3.3.xx) and (3.3.xx) assure the verification of conventional total conservation laws.

The isotopies of Newtonian mechanics, today known *Newton-Santilli isomechanics*, were first submitted in the second edition of monograph [5] and in the mathematical treatment [10].

They require the use of: the *isotime* $\hat{t} = t \times \hat{I}_t$ with isounit $\hat{I}_t = 1/\hat{T}_t > 0$ and related isofield $\hat{R}_t$; the *isocoordinates* $\hat{r} = (\hat{r}_a^k) = r \times \hat{I}_r$, with isounit $\hat{I}_r = 1/\hat{T}_> 0r$ and related isofield $\hat{R}_r$; and the *isovelocities* $\hat{v} = (v_{ka}) = v \times \hat{I}_v$ with isounit $\hat{I}_v = 1/\hat{T}_v > 0$ and related isofield $\hat{R}_v$.

The Newton-Santilli isomechanics is then formulated on the 7-dimensional isospace

$$\hat{S}_{tot} = \hat{E}(\hat{t}, \hat{R}_{\hat{t}}) \times \hat{E}(\hat{r}, \hat{\delta}_r, \hat{R}_{\hat{r}}) \times \hat{E}(\hat{v}, \hat{\delta}_v, \hat{R}_{\hat{v}}), \tag{3.3.20}$$

with isometrics

$$\hat{\delta}_r = \hat{T}_r \times \delta = (\hat{T}_{ir}^k \times \delta_{kj}), \quad \hat{\delta}_v = \hat{T}_v \times \delta = (\hat{T}_{iv}^k \times \delta_{kj}), \tag{3.3.21}$$

over the Kronecker product of isofields

$$\hat{R}_{tot} = \hat{R}_t \times \hat{R}_r \times \hat{R}_v, \tag{3.3.22}$$

with total isounit

$$\hat{I}_{tot} = \hat{I}_t \times \hat{I}_r \times \hat{I}_v =$$
$$= n_t^2 \times Diag.(n_{rx}^2, n_{ry}^2, n_{rz}^2) \times Diag.(n_{vx}^2, n_{vy}^2, n_{vz}^2). \tag{3.3.23}$$

Consequently, the isounit can also be factorized into the product of the following seven distinct isounits, with related product of seven distinct isofields

$$\hat{I}_{tot} = n_t^2 \times n_{rx}^2 \times n_{ry}^2 \times n_{rz}^2 \times n_{vx}^2 \times n_{vy}^2 \times n_{vz}^2, \tag{3.3.24a}$$

$$\hat{R}_{tot} = \hat{R}_t \times \hat{R}_{rx} \times \hat{R}_{ry} \times \hat{R}_{rz} \times \hat{R}_{vx} \times \hat{R}_{vy} \times \hat{R}_{vz}, \tag{3.3.24b}$$

and consequential applicability of the fundamental *Tsagas-Sourlas-Santilli-Falcón-Núñez isotopology* (or TSSFN Isotopology) that allows, for the first time to the author's best knowledge, an consistent representation of extended, nonspherical and deformable shapes of particles in newtonian mechanics, here represented via the semiaxes $n_\alpha^2 = n_\alpha^2(t, r, v, \dots)$, $\alpha = t, r, v$.

Note that the isospeed is the given by

$$\hat{v} = \frac{\hat{d}\hat{r}}{\hat{d}\hat{t}} = \hat{I}_t \times \frac{d(r \times \hat{I}_r)}{dt} = v \times \hat{I}_t \times \hat{I}_r + r \times \hat{I}_t \times \frac{d\hat{I}_r}{dt} = v \times \hat{I}_v, \tag{3.3.25}$$

thus illustrating that the isounit of the isospeed cannot be the same as that for the isocoordinates, having in particular the value

$$\hat{I}_v = \hat{I}_t \times \hat{I}_r \times \left(1 + \frac{r}{v} \times \frac{1}{\hat{I}_r} \times \frac{d\hat{I}_r}{dt}\right). \tag{3.3.26}$$

The *Newton-Santilli isoequation* [6,10] can be written

$$\hat{m}_a \hat{\times} \frac{\hat{d}\hat{v}_{ka}}{\hat{d}\hat{t}} = -\frac{\hat{\partial}\hat{V}(\hat{r})}{\hat{\partial}\hat{r}_a^k}, \tag{3.3.27}$$

namely, *the equations are conceived in such a way to formally coincide with the conventional equations for selfadjoint forces when formulated on isospace over isofields, while all nonpotential forces are represented by the isounits or, equivalently, by the isodifferential calculus.*

Such a conception is the only one known permitting the representation of extended particles with contact interactions that is invariant, thus avoiding the catastrophic inconsistencies of Section 1.4.1 and, in addition, achieves closure, namely, the verification of all conventional total conservation laws.

An inspection of Eqs. (3.3.27) is sufficient to see that *the Newton-Santilli isomechanics reconstructs linearity, locality and canonicity on isospaces over isofields,* as studied in Section 3.2.11. Note that this would not be the case if nonselfadjoint forces appear in the right hand side of Eqs. (3.27) as in Eqs. (3.3.2).

Note the truly crucial role of the isodifferential calculus for the above structural generalization of Newtonian mechanics (as well as of the subsequent mechanics), that justifies *a p[posteriory* its construction.

The verification of conventional total conservation laws is established by a visual inspection of Eqs. (3.3.27) since their symmetry is the *Galileo-Santilli isosymmetry* [14,15] that is isomorphic to the conventional Galilean symmetry, only formulated on isospace over isofields. By recalling that conservation laws are represented by the generators of the underlying symmetry, conventional total conservation laws then follow from the indicated invariance.

When projected in the conventional representation space S_{tot}, Eqs. (3.3.27) can be explicitly written

$$\hat{m}\hat{\times}\frac{\hat{d}\hat{v}}{\hat{d}\hat{t}} = m \times \hat{I}_t \times \frac{d(v \times \hat{I}_v)}{dt} =$$

$$= m \times \frac{dv}{dt} \times \hat{I}_t \times \hat{I}_v + m \times v \times \hat{I}_t \times \frac{d\hat{I}_v}{dt} = -\frac{\hat{\partial}\hat{V}(\hat{r})}{\hat{\partial}\hat{r}} = -\hat{I}_r \times \frac{\partial V}{\partial r}, \quad (3.3.28)$$

that is

$$m \times \frac{dv}{dt} = -\hat{T}_t \times \hat{T}_v \times \hat{I}_r \times \frac{\partial V}{\partial r} - m \times v \times \hat{T}_v \times \frac{d\hat{I}_v}{dt}. \quad (3.3.29)$$

The necessary and sufficient conditions for the representation of all possible SA and NSA forces are given by

$$\hat{I}_r = \hat{T}_t \times \hat{T}_r, \quad (3.3.30a)$$

$$m \times v \times \hat{T}_v \times \frac{d\hat{I}_v}{dt} = F^{NSA}, \quad (3.3.30b)$$

and they always admit a solution, since they constitute a system of $6n$ *algebraic* (rather than differential) equations in the $6N + 1$ unknowns given by $\hat{I}_t$, and the diagonal $3N$-dimensional matrices $\hat{I}_r$ and $\hat{I}_v$.

Note that for $\hat{T}_t = 1$ we recover from a dynamical viewpoint the condition $\hat{I}_r = 1/\hat{I}_v$ obtained in Section 3.2.4 and 3.2.10 on geometric grounds.

As a simple illustration among unlimited possibilities, we have the following equations of motion of an *extended* particle with the ellipsoidal shape experiencing a resistive force $F^{NSA} = -\gamma \times v$ because moving within a physical medium

$$m \times \frac{dv}{dt} = \int d\sigma\, \Gamma(\sigma, \mathbf{r}, p, \dots) \approx -\gamma \times v, \quad (3.3.31a)$$

$$\hat{I}_v = Diag.(n_1^2, n_2^2, n_3^2) \times e^{\gamma \times t/m}, \tag{3.3.31b}$$

where the nonlocal-integral character with respect to a kernel Γ is emphasized. Interested readers can then construct the representation of *any* desired non-Hamiltonian Newtonian system (see also memoir [10] for other examples).

Note the natural appearance in the NSA forces of the velocity dependence, as typical of resistive forces. Note also that the representation of the extended character of particles occurs only in isospace because, when Eqs. (3.3.xx) are projected in the conventional Newtonian space, factorized isounits cancel out and the point characterization of particles is recovered.

Note finally the *direct universality* of the Newton-Santilli isoequations, namely, their capability of representing all infinitely possible Newton's equations in the frame of the observer.

As now familiar earlier, Eqs. (3.3.27) can only describe a system of *particles.* The *isodual Newton-Santilli isoequations* for the treatment of a system of *antiparticles* are given by [6,10]

$$\hat{m}_a^d \hat{\times}^d \frac{\hat{d}^d \hat{v}_{ka}^d}{\hat{d}^d \hat{t}^d} = -\frac{\hat{\partial}^d \hat{V}^d(\hat{r}^d)}{\hat{\partial}^d \hat{r}_a^{kd}}. \tag{3.3.32}$$

The explicit construction of the remaining isodualities of the above isomechanics are instructive for the reader seriously interested in a classical study of antimatter under interior dynamical conditions.

3.3.5 Hamilton-Santilli Isomechanics for Matter and its Isodual for Antimatter

3.3.5A. Isoaction Principle and its Isodual. The isotopies of classical Hamiltonian mechanics were first introduced by Santilli in various works (see monographs [6,7] and references quoted therein), and are today known as the *Hamilton-Santilli isomechanics.*

To identify its representation space, recall that the conventional Hamiltonian mechanics is represented in a 7-dimensional space of time, coordinates and momenta (rather than velocity), the latter characterizing phase space (or cotangent bundle of the symplectic geometry).

Correspondingly, the new isomechanics is formulated in the 7-dimensional isospace of isotime $\hat{t}$, isocoordinates $\hat{r}$ and isomomenta $\hat{p}$

$$\hat{S}_{tot} = \hat{E}(\hat{t}, \hat{R}_{\hat{t}}) \times \hat{E}(\hat{r}, \hat{\delta}_r, \hat{R}_{\hat{r}}) \times \hat{E}(\hat{p}, \hat{\delta}_p, \hat{R}_{\hat{p}}), \tag{3.3.33}$$

with isometrics

$$\hat{\delta}_{\hat{r}} = \hat{T}_{\hat{r}} \times \delta = (\hat{T}_{ir}^k \times \delta_{kj}), \hat{\delta}_{\hat{p}} = \hat{T}_{\hat{p}} \times \delta = (\hat{T}_{ip}^k \times \delta_{kj}), \tag{3.3.34}$$

over the Kronecker product of isofields and related isounits

$$\hat{R}_{tot} = \hat{R}_t \times \hat{R}_r \times \hat{R}_p, \tag{3.3.35a}$$

$$\hat{I}_{tot} = \hat{I}_{\hat{t}} \times \hat{I}_{\hat{r}} \times \hat{I}_{\hat{p}} =$$
$$= n_t^2 \times Diag.(n_{rx}^2, n_{ry}^2, n_{rz}^2) \times Diag.(n_{px}^2, n_{py}^2, n_{pz}^2). \tag{3.3.35b}$$

The following new feature now appears. The *isophasespace,* or, more technically, the *isocotangent bundle* of the isosymplectic geometry in local isochart $(\hat{r}, \hat{p})$ requires that the isounits of the variables $\hat{r}$ and $\hat{p}$ are *inverse* of each others (Section 3.2.3 and 3.2.10)

$$\hat{I}_{\hat{r}} = 1/\hat{T}_{\hat{r}} = \hat{I}_{\hat{p}}^{-1} = \hat{T}_{\hat{p}} > 0. \tag{3.3.36}$$

Consequently, by ignoring hereon for notational simplicity the indeces for the N particles, the total isounit of the isophase space can be written

$$\hat{I}_{tot} = \hat{I}_{\hat{t}} \times \hat{I}_{\hat{r}} \times \hat{T}_{\hat{r}} = \hat{I}_{\hat{t}} \times \hat{I}_6, \tag{3.3.37a}$$

$$\hat{I}_6 = (\hat{I}_\mu^\nu) = \hat{I}_{\hat{r}} \times \hat{T}_{\hat{r}}. \tag{3.3.37b}$$

The fundamental *isoaction principle* for the classical treatment of matter in interior conditions can be written in the explicit form in the $\hat{r}$ and $\hat{p}$ isovariables

$$\hat{\delta}\hat{A}^o = \hat{\delta} \int_{t_1}^{t_2} (\hat{p}_k \hat{\times} \hat{d}\hat{r}^k - \hat{H} \hat{\times} \hat{d}\hat{t}) =$$

$$= \hat{\delta} \int_{t_1}^{t_2} [p_k \times \hat{T}_{\hat{r}}^{k_i(t,r,p,\dots)} \times \hat{d}\hat{r}^i - \hat{H} \times \hat{T}_{\hat{t}} \times d\hat{t}] = 0, \tag{3.3.38}$$

where

$$\hat{H} = \hat{p}^{\hat{2}} \hat{/} \hat{2} \hat{\times} \hat{m} - \hat{V}(\hat{r}), \tag{3.3.39}$$

is the *isohamiltonian* or simple the Hamiltonian because its projection on conventional spaces represents the orginary total energy except an inessential multiplicative factor.

By using the unified notation

$$\hat{b} = (\hat{b}^\mu) = (\hat{r}^i, \hat{p}_j) = (r^i, p_j) \times \hat{I}_6 = b \times \hat{I}_6, \tag{3.3.40}$$

and the isotopic image of the canonical R^o functions, Eqs. (3.3.xx),

$$\hat{R}^o = (\hat{R}_\mu^o) = (\hat{r}, \hat{0}), \tag{3.3.41}$$

the fundamental isoaction principle can be written in unified notation

$$\hat{\delta}\hat{A}^o = \hat{\delta} \int_{t_1}^{t_2} (\hat{p}_k \hat{\times} \hat{d}\hat{r}^k - \hat{H} \hat{\times} \hat{d}\hat{t}) \equiv$$

$$\equiv \hat{\delta} \int_{t_1}^{t_2} (\hat{R}^o_\mu \hat{\times} \hat{d}\hat{b}^\mu - \hat{H} \hat{\times} \hat{d}\hat{t}) =$$

$$= \hat{\delta} \int_{t_1}^{t_2} (R^o_\mu \times \hat{T}^\mu_{6\nu} \times d\hat{b}^\nu - H \times \hat{T}_{\hat{t}} \times d\hat{t}) = 0. \qquad (3.3.42)$$

A visual inspection of principle (3.3.38) establishes the *isocanonicity* of Hamilton-Santilli isomechanics (Section 3.2.11), namely, the reconstruction of canonicity on isospaces over isofield that is crucial for the consistency of hadronic mechanics.

In fact, the conventional action principle (3.3.4) and isoprinciple (3.3.38) coincide at the abstract, realization-free level by conception and construction.

The direct universality of classical isomechanics can be seen from the arbitrariness of the integrand of isoaction functional (3.3.38) once projected on conventional spaces over conventional fields.

An important property of the isoaction is that its functional dependence on isospaces over isofields is restricted to that on isocoordinates only, i.e., $\hat{A} = \hat{A}(\hat{r})$. However, when projected on conventional spaces, the functional dependence is arbitrary, i.e., $\hat{A}(\hat{r}) = \hat{A}(r \times \hat{I}) = \hat{A}(t, r, p, \dots)$. This feature will soon have a crucial role for the operator image of the classical isomechanics.

It should finally be noted that isoprinciple (3.3.38) essentially eliminates the entire field of *Lagrangian and action principles of orders higher than the first,* e.,g., $L = L(t, r, \dot{r}, \ddot{r}, \dots)$ because of these higher order formulations can be easily reduced to the isotopic first-order form (3.3.38).

Recall that the action principle has the important application via the use of the *optimal control theory* of optimizing dynamical systems, However, the latter can have only been Hamiltonian until now due to the lack of a universal action functional for non-Hamiltonian systems (that constitute, by far, the system most significant for optimization. Recall also that the optimal control theory can only be applied for local-differential systems due to the underlying Euclidean topologic, thus secluding from the optimization process the most important systems, those of extended, and, therefore, of nonlocal type.

Note that isoaction principle (3.3.38) occurs for all possible non-Hamiltonian as well as nonlocal-integral; systems, thanks also to the underlying TSSFN isotopology (Section 3.2.7). We, therefore, have the following important:

THEOREM 3.3.1 [6,10]: Isoaction principle (3.3.38) permits the (first known) optimization of all possible nonpotential/non-Hamiltonian and nonlocal-integral systems.

The *isodual isoaction principle* [10] for the classical treatment of antimatter in interior conditions is given by

$$\hat{\delta}^d \hat{A}^d = \hat{\delta}^d \int_{t_1}^{t_2}{}^d (\hat{p}_k^d \hat{\times}^d \hat{d}\hat{r}^{kd} - \hat{H}^d \hat{\times}^d \hat{d}^d \hat{t}^d) =$$

$$= \hat{\delta}^d \int_{t_1}^{t_2}{}^d (\hat{R}_\mu^{od} \hat{\times}^d \hat{d}^d \hat{b}^\mu - \hat{H}^d \hat{\times} \hat{d}^d \hat{t}^d) = 0. \tag{3.3.43}$$

Additional isodual treatments are left to the interested reader.

3.3.5B. Hamilton-Santilli Isoequations and their Isoduals. The discovery of the isodifferential calculus in 1995 permitted Santilli [6,10] the identification of the following classical dynamical equations for the treatment of matter at the foundations of hadronic mechanics, today known as the *Hamilton-Santilli isoequations*. They are easily derived via the isovariational principle and can be written from isoprinciple (3.3.38) in disjoint notation

$$\frac{\hat{d}\hat{r}^k}{\hat{d}\hat{t}} = \frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{p}_k}, \quad \frac{\hat{\partial}\hat{p}_k}{\hat{d}\hat{t}} = -\frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{r}^k}. \tag{3.3.44}$$

The same equations can be written in unified notation from principle (3.3.40)

$$\hat{\omega}_{\mu\nu} \hat{\times} \frac{\hat{d}\hat{b}^\mu}{\hat{d}\hat{t}} = \frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{b}^\mu}, \tag{3.3.45}$$

where

$$\hat{\omega}_{\mu\nu} = \omega_{\mu\nu} \times \hat{I}_6 \tag{3.3.46}$$

is the *isocanonical isosymplectic tenso* that coincides with the conventional canonical symplectic tensor $\omega_{\mu\nu}$ except for the factorization of the isounit (Section 3.2.10).

To verify the latter property from an analytic viewpoint, it is instructive for the reader to verify the following identify under isounits (3.3.37)

$$\hat{\omega}_{\mu\nu} = \frac{\hat{\partial}\hat{R}_\nu^o}{\hat{\partial}\hat{b}^\mu} - \frac{\hat{\partial}\hat{R}_\mu^o}{\hat{\partial}\hat{b}^\nu} = \omega_{\mu\nu} \times \hat{I}_6. \tag{3.3.47}$$

A simple comparison of the above isoanalytic equations with the isotopic and conventional Newton's equations established the following:

THEOREM 3.3.2: Hamilton-Santilli isoequations (3.3.5) are "directly universal" in Newtonian mechanics, that is, capable of representing all possible,

conventional or isotopic, hamiltonian and non-Hamiltonian Newtonian systems directly in the fixed coordinates of the experimenter.

It is now important to show that Eqs. (3.3.45) provide an identical reformulation of the true analytic equations (3.3.2). For this purpose, we assume the simple case in which isotime coincide with the conventional time, that is, $\hat{t} = t$, $\hat{I}_t = +1$ and we write isoequations (3.3.45) in the explicit form

$$\begin{aligned}(\omega) \times \begin{pmatrix} dr^k/dt \\ dp_k/dt \end{pmatrix} = \begin{pmatrix} 0_{3\times3} & -I_{3\times3} \\ I_{3\times3} & 0_{3\times3} \end{pmatrix} \times \begin{pmatrix} dr^k/dt \\ dp_k/dt \end{pmatrix} = \\ \begin{pmatrix} -dp_k/dt \\ dr^k/dt \end{pmatrix} = \begin{pmatrix} \hat{\partial}\hat{H}/\hat{\partial}r^k \\ \hat{\partial}\hat{H}/\hat{\partial}p_k \end{pmatrix} = \begin{pmatrix} \hat{I}^i_k \times \partial\hat{H}/\partial r^i \\ \hat{T}^k_i \times \partial\hat{H}/\partial p_i \end{pmatrix}.\end{aligned} \tag{3.3.48}$$

It is easy to see that Eqs. (3.3.xx) *coincide* with the true analytic equations (3.3.2) under the trivial algebraic identification

$$\hat{I}_{\hat{r}} = Diag.[I - F/(\partial H/\partial r)], \tag{3.3.49}$$

As one can see, the main mechanism of Eqs. (3.3.45) is that of *transforming the external terms* $F = F^{NSA}$ *into an explicit realization of the isounit* $\hat{I}_3$. As a consequence, *reformulation (3.3.45) constitutes direct evidence on the capability to represent non-Hamiltonian forces and effects with a generalization of the unit of the theory.*

Note in particular that *the external terms are embedded in the isoderivatives.* However, when written down explicitly, Eqs. (3.3.2) and (3.3.45) coincide. Note that $\hat{I}_3$ as in rule (3.49) is fully symmetric, thus acceptable as the isounit of isomathematics. Note also that all nonlocal and nonhamiltonian effects are embedded in $\hat{I}$.

The reader should note the extreme simplicity in the construction of a representation of given non-Hamiltonian equations of motion, due to the algebraic character of identifications (3.3.49).

Recall that *Hamilton's equations with external terms are not derivable from a variational principle.* In turn, such an occurrence has precluded the identification of the operator counterpart of Eqs. (3.3.2) throughout the 20-th century.

We now learn that *the identical* reformulation (3.3.45) of Eqs. (3.3.2) becomes fully derivable from a variational principle. In turn, this will soon permit the identification of the unique and unambiguous operator counterpart.

It should be noted that *the Hamilton-Santilli isoequations are generally irreversible* due to the general irreversibility of the external forces,

$$F(t, \dots) \neq F(-t, \dots), \quad \text{or} \tag{3.3.50a}$$

$$\hat{I}(t,\dots) = Diag.[I - F(t,\dots)/(\partial H/\partial t)] \neq \hat{I}(-t,\dots). \tag{3.3.50b}$$

In particular, we have irreversibility under the conservation of the total energy (see next chapter for full treatment). This feature is important to achieve compatibility with thermodynamics, e.g., to have credible analytic methods for the representation of the internal increase of the entropy for closed-isolated systems such as Jupiter.

The study of these thermodynamical aspects is left to the interested reader. In this chapter we shall solely consider *reversible closed-isolated systems* that occur for external forces not explicitly dependent on time and verify other restrictions.

An important aspect is that *the Hamilton-Santilli isoequations coincide with the Hamilton equations without external terms at the abstract level.* In fact, all differences between I and $\hat{I}$, $\times$ and $\hat{\times}$, ∂ and $\hat{\partial}$, *etc.*, disappear at the abstract level. This proves the achievement of a central objective of isomechanics, the property that the analytic equations with external terms can indeed be *identically* rewritten in a form equivalent to the analytic equations without external terms, provided, however, that the reformulation occurs via the broader isomathematics.

The it isodual Hamilton-Santilli isoequations for the classical treatment of antimatter, also identified soon after the discovery of the isodifferential calculus, are given by

$$\hat{\omega}^d_{\mu\nu} \hat{\times}^d \frac{\hat{d}^d \hat{b}^{d\mu}}{\hat{d}^d \hat{t}^d} = \frac{\hat{\partial}^d \hat{H}^d}{\hat{\partial}^d \hat{b}^{d\mu}}, \tag{3.3.51}$$

where

$$\hat{\omega}^d_{\mu\nu} = \omega^d_{\mu\nu} \times \hat{I}_6 \tag{3.3.52}$$

is the it isodual isocanonical isosymplectic tensor. The derivation of other isodual properties is instructed for the interested reader.

3.3.5C. Classical Lie-Santilli Brackets and their Isoduals. It is important to verify that Eqs. (3.3.44) or (3.3.45) resolve the problematic aspects of external terms indicated in Section 3.3.2 [4]. In fact, the isobrackets of the time evolution of matter are given by

$$\frac{d\hat{A}}{\hat{d}\hat{t}} = [\hat{A}\hat{,}\hat{H}] = \frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{r}^k} \hat{\times} \frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{p}_k} - \frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{r}^k} \hat{\times} \frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{p}_k}, \tag{3.3.53}$$

and they verify the left and right distributive and scalar laws, thus characterizing a consistent algebra. Moreover, that algebra results to be Lie-isotopic, for which reasons the above brackets are known as the *Lie-Santilli isobrackets.*

When explicitly written in our spacetime, brackets (3.3.53) recover the brackets (3.3.3) of the true analytic equations (3.3.2)

$$\frac{dH}{dt} = \frac{\partial H}{\partial r^k} \times \frac{\partial H}{\partial p_k} - \frac{\partial H}{\partial p_k} \times \frac{\partial H}{\partial r^k} + \frac{\partial H}{\partial p_k} \times F^k = \frac{\partial H}{\partial p_k} \times F^k \equiv 0, \qquad (3.3.54)$$

where the last identity holds in view of Eqs. (3.3.49). Therefore, the Hamilton-Jacobi isoequations do indeed constitute a reformulation of the true analytic equations with a consistent Lie-isotopic algebraic brackets, as needed (Section 3.3.3).

Note that, in which of their anti-isomorphic character, isobrackets (3.3.53) represent the conservation of the Hamiltonian,

$$\frac{d\hat{H}}{dt} = [\hat{H}\hat{,}\hat{H}] = \frac{\hat{\partial}\hat{H}}{\hat{\partial}r^k} \hat{\times} \frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{p}_k} - \frac{\hat{\partial}\hat{H}}{\hat{\partial}r^k} \hat{\times} \frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{p}_k} \equiv 0, \qquad (3.3.55)$$

This illustrates the reason for assuming closed-isolated Newtonian systems (3.3.19) at the foundations of this chapter.

Basic isobrackets (3.3.53) can be written in unified notation

$$[\hat{A}\hat{,}\hat{B}] = \frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{b}^\mu} \hat{\times} \hat{\omega}^{\mu\nu} \hat{\times} \frac{\hat{\partial}\hat{B}}{\hat{\partial}\hat{b}^\nu}. \qquad (3.3.56)$$

where $\hat{\omega}_{\mu\nu}$ is the Lie-Santilli isotensor. By using the notation $\hat{\partial}^\mu = \hat{\partial}/\hat{\partial}\hat{b}^\mu$, the isobrackets can be written

$$[\hat{A}\hat{,}\hat{B}] = \hat{\partial}_\mu \hat{A} \times \hat{T}^\mu_\rho \times \omega^{\rho\nu} \hat{\partial}_\nu \hat{\partial}_\nu \hat{B}, \qquad (3.3.57)$$

and, when projected in our spacetime, the isobrackets can be written

$$[A\hat{,}B] = \partial_\mu A \times \omega^{\mu\rho} \times \hat{I}^\nu_\rho \times \partial_\nu B, \qquad (3.3.58)$$

where $\omega^{\mu\nu}$ is the canonical Lie tensor.

The *isodual Lie-Santilli isobrackets* for the characterization of antimatter can be written

$$[\hat{A}^d\hat{,}\hat{B}^d] = \hat{\partial}^d_\mu \hat{A}^d \hat{\times}^d \hat{\omega}^{d\rho\nu} \hat{\partial}^d_\nu \hat{\partial}^d_\nu \hat{B}, \qquad (3.3.59)$$

where $\hat{\omega}^{d\mu\nu}$ is the *isodual Lie-Santilli isotensor*. Other algebraic properties can be easily derived by the interested reader.

3.3.5D. Hamilton-Jacobi-Santilli Isoequations and their isoduals. Another important consequence of isoaction principle (3.3.38) is the characterization of the following *Hamilton-Jacobi-Santilli isoequations* for matter [6,10]

$$\frac{\hat{\partial}\hat{A}^o}{\hat{\partial}\hat{t}} + \hat{H} = 0, \qquad (3.3.60a)$$

$$\frac{\hat{\partial}\hat{A}^o}{\hat{\partial}\hat{r}^k} - \hat{p}_k = 0, \tag{3.3.60b}$$

$$\frac{\hat{\partial}\hat{A}^o}{\hat{\partial}\hat{p}_k} \equiv 0, \tag{3.3.60c}$$

which will soon have basic relevance for isoquantization.

Note the *independence of the isoaction* $\hat{A}^o$ *from the isomomenta* that will soon be crucial for consistent isoquantization.

The isodual equations for antimatter are then given by

$$\frac{\hat{\partial}^d \hat{A}^{od}}{\hat{\partial}^d \hat{t}^d} + \hat{H}^d = 0, \tag{3.3.61a}$$

$$\frac{\hat{\partial}^d \hat{A}^{od}}{\hat{\partial}^d \hat{r}^{kd}} - \hat{p}^d_k = 0, \tag{3.3.61b}$$

$$\frac{\hat{\partial}^d \hat{A}^{od}}{\hat{\partial}^d \hat{p}^d_k} \equiv 0, \tag{3.3.61c}$$

The latter equations will soon result to be essential for the achievement of a consistent operator image of the classical treatment of antimatter in interior conditions.

3.3.5E. Connection Between Isotopic and Birkhoffian Mechanics. Since the Hamilton-Santilli isoequations are directly universal, they can also represent Birkhoff's equations (3.3.13) in the fixed b-coordinates. In fact, by assuming for simplicity that the isotime is the ordinary time, we can write the identities

$$\frac{db^\mu}{dt} = \Omega^{\mu\mu}(b) \times \frac{\partial H(b)}{\partial b^\nu} \equiv$$

$$\equiv \omega^{\mu\nu} \times \frac{\hat{\partial} H(b)}{\hat{\partial} b^\nu} = \omega^{\mu\rho} \times \hat{I}^\nu_{6\rho} \times \frac{\partial H}{\partial b^\nu}. \tag{3.3.62}$$

Consequently, we reach the following *decomposition of the Birkhoffian tensor*

$$\Omega^{\mu\nu}(b) = \omega^{\mu\rho} \times \hat{I}^\nu_{6\rho}(b). \tag{3.3.63}$$

Consequently, Birkhoff's equations can indeed be identically rewritten in the isotopic form, as expected. In the process, the reformulation provides additional insight in the isounit.

The reformulation also carries intriguing geometric implications since it confirms the *direct universality in symplectic geometry of the canonical two-form,* since a general symplectic two-form can always be identically rewritten in the

isocanonical form via decomposition of type (3.3.xx) and then the embedding of the isounit in the isodifferential of the exterior calculus.

As an incidental note, the reader should be aware that the construction of an analytic representation via Birkhoff's equations is rather complex, inasmuch as it requires *the solution of nonlinear partial differential equations* or integral equations [2].

By comparison, the construction of the same analytic equations via Hamilton-Santilli isoequations (3.3.44) or (3.3.45) is truly elementary, and merely requires the identification of the isounit according to *algebraic* rule (3.3.49) for arbitrarily given external forces $F_k(t, r, p)$.

3.3.6 Simple Construction of Classical Isomechanics

The above classical isomechanics can be constructed via a simple method which does not need any advanced mathematics, yet it is sufficient and effective for practical applications.

In fact, *the Hamilton-Santilli isomechanics can be constructed via the systematic application of the following noncanonical transform to all quantities and operations of the conventional Hamiltonian mechanics*

$$U = \begin{pmatrix} \hat{I}_3^{1/2} & 0 \\ 0 & \hat{T}_3^{1/2} \end{pmatrix}, \tag{3.3.64a}$$

$$U \times U^t = \hat{I}_6 \neq I, \tag{3.3.64b}$$

$$\hat{I}_3 = I - \frac{F}{\partial H/\partial p} = I - \frac{F}{p/m}. \tag{3.3.64c}$$

The success of the construction depends on the application of the above noncanonical transform to the *totality* of Hamiltonian mechanics, with no exceptions. We have in this way the lifting of: the 6-dimensional unit of the conventional phase space into the isounit

$$I_6 \rightarrow \hat{I}_6 = U \times I_6 \times U^t; \tag{3.3.65}$$

numbers into the isonumbers,

$$n \rightarrow \hat{n} = U \times n \times U^t = n \times (U \times U^t) = n \times \hat{I}_6; \tag{3.3.66}$$

associative product $A \times B$ among generic quantities A, B into the isoassociative product with the correct expression and property for the isotopic element,

$$A \times B \rightarrow A \hat{\times} B = U \times (A \times B) \times U^t = A' \times \hat{T} \times B', \tag{3.3.67a}$$

$$A' = U \times A \times U^t, \quad B' = U \times B \times U^t, \quad \hat{T} = (U \times U^t)^{-1} = T^t; \tag{3.3.67b}$$

Euclidean into isoeuclidean spaces (where we use only the space component of the transform)

$$\begin{gathered} x^2 = x^t \times \delta \times x \rightarrow \hat{x}^{\hat{2}} = U \times x^2 \times U^t = \\ = (x^t \times U^t) \times (U^{t-1} \times \delta \times U^{-1}) \times (U \times x) \times (U \times U^t) = \\ = [x'^t \times (\hat{T} \times \delta) \times x'] \times \hat{I}; \end{gathered} \tag{3.3.68}$$

and, finally, we have the following isotopic lifting of Hamilton's into Hamilton-Santilli isoequations (here derived for simplicity for the case in which the transform does not depend explicitly on the local coordinates),

$$\begin{gathered} db/dt - \omega \times \partial H/\partial b = 0 \rightarrow \\ \rightarrow U \times db/dt \times U^t - U \times \omega \times \partial/\partial b \times U^t = \\ = db/dt \times (U \times U^t) - (U \times \omega \times U^t) \times (U^{t\ -1} \times U^{-1}) \times \\ \times (U \times \partial H/\partial b \times U^t) \times (U \times U^t) = \\ = db/dt \times \hat{I} - \omega \times (\hat{\partial} H/\hat{\partial}\hat{b}) \times \hat{I} = 0, \end{gathered} \tag{3.3.69}$$

where we have used the important property the reader is rugend to verify

$$U \times \omega \times U^t \equiv \omega. \tag{3.3.70}$$

As one can see, the seemingly complex isomathematics and isomechanics are reduced to a truly elementary construction. e its universality.

3.3.7 Invariance of Classical Isomechanics

A final requirement is necessary for a physical consistency, and that is, *the invariance of isomechanics under its own time evolution*, as it occurs for conventional Hamiltonian mechanics.

Recall that a transformation $b \rightarrow b'(b)$ us called a *canonical transformations* when all the following identities hold

$$\frac{\partial b^\mu}{\partial b'^\alpha} \times \omega_{\mu\nu} \times \frac{\partial b^\nu}{\partial b'^\beta} = \omega_{\alpha\beta}. \tag{3.3.71}$$

The invariance of Hamiltonian mechanics follows from the property that its time evolution constitutes a canonical transformation, as well known.

The proof of the invariant of isomechanics is elementary. In fact, an isotransformation $\hat{b} \rightarrow \hat{b}'(\hat{b})$ constituted an *isocanonical isotransform* when all the following identities old

$$\frac{\hat{\partial}\hat{b}^\mu}{\hat{\partial}\hat{b}'^\alpha} \hat{\times} \hat{\omega}_{\mu\nu} \hat{\times} \frac{\hat{\partial}\hat{b}^\nu}{\hat{\partial}\hat{b}'^\beta} = \hat{\omega}_{\alpha\beta} = \omega_{\alpha\beta} \times \hat{I}_6. \tag{3.3.72}$$

But the above expression can be written

$$(\hat{I}_6^{\mu_\rho} \times \frac{\partial \hat{b}^\rho}{\partial \hat{b}^\alpha} \times \omega_{\mu\nu} \times \hat{I}_6^{\xi_\nu} \times \frac{\partial \hat{b}^\xi}{\partial \hat{b}'^\beta}) \times \hat{I}_6 = \omega_{\mu\nu} \times \hat{I}_6, \tag{3.3.73}$$

and they coincide with conditions (3.3.xx) in view of the identities

$$\hat{I}_6^{\mu_\rho} \times \omega_{\mu\nu} \times \hat{I}_6^{\nu_\xi} = \omega_{\rho\xi}. \tag{3.3.74}$$

Consequently, we have the following important

THEOREM 3.3.3 [6,10]: Following factorization of the isounit, isocanonical transformations are canonical.

The desired invariance of the Hamilton-Santilli isomechanics then follows.

It is an instructive exercise for the reader interested in learning isomechanics to verify that *all* catastrophic mathematical and physical inconsistencies of noncanonical theories pointed out in Chapter 1 (see Section 1.4.1 in particular) are indeed resolved by isomechanics as presented in this section.

3.4 OPERATOR LIE-ISOTOPIC MECHANICS FOR MATTER AND ITS ISODUAL FOR ANTIMATTER

3.4.1 Introduction

We are finally equipped to present the foundations of *the Lie-isotopic branch of nonrelativistic hadronic mechanics* for matter and its isodual for antimatter, more simply referred to as *operator isomechanics,* and its isodual for antimatter referred to as *isodual operator isomechanics.* The new mechanics will then be used in subsequent sections for various developments, experimental verifications and industrial applications.

The extension of the results of this section to *relativistic operator isomechanics* is elementary and will be done in the following sections whenever needed for specific applications. the case of it operator genomechanics with a Lie-admissible, rather than the Lie-isotopic structure, will be studied in the next chapter.

A knowledge of Section 3.2 is necessary for a technical understanding of operator isomechanics. For the mathematically non-inclined readers, we present in Section 3.4.8 as very elementary construction of operator isomechanics via nonunitary transforms.

Unless otherwise specified, all quantities and operations represented with conventional symbols A, H, $\times$, *etc.*, denote quantities and operations on conventional Hilbert spaces over conventional fields. All quantities and symbols of the type $\hat{A}$, $\hat{H}$, $\hat{\times}$, *etc.*, are instead defined on isohilbert spaces over isofields.

Note the use of the terms "operator" isomechanics, rather than "quantum" isomechanics, because, as indicated in Chapter 1, the notion of quantum is fully established within the arena of its conception, the transition of electrons between different stable orbits of atomic structure (exterior problem), while the assumption of the same quantum structure for the same electrons when in the core of a star (interior problems) is a scientific religion at this writing deprived of solid experimental evidence.

3.4.2 Naive Isoquantization and its Isodual

An effective way to derive the basic dynamical equations of operator isomechanics is that via the isotopies of the conventional map of the classical Hamilton-Jacobi equations into their operator counterpart, known as *naive quantization.*. More rigorous methods, such as the isotopies of symplectic quantization, essentially yields the same operator equations and will not be treated in this section for brevity (see monograph [7] for a presentation).

Recall that the *naive quantization* can be expressed via the following map of the canonical action functional

$$A^o = \int_{t_1}^{t_2} (p_k \times dr^k - H \times dt) \rightarrow -i \times \hbar \times \ln |\psi\rangle, \tag{3.4.1}$$

under which the conventional Hamilton-Jacobi equations are mapped into the Schrödinger equations,

$$-\partial_t A^o = H \rightarrow i \times \hbar \times \partial_t |\psi\rangle = H \times |\psi >, \tag{3.4.2a}$$

$$p_k = \partial_k A^o \rightarrow p_k \times |\psi >= -i \times \hbar \times \partial_k |\psi >, \tag{3.4.2b}$$

where $|\psi >$ is the *wavefunction,* or, more technically, a state in a Hilbert space $\mathcal{H}$.

Isocanonical action (3.3.38) is evidently different than the conventional canonical action, e.g., because it is of higher order derivatives. As such, the above naive quantization does not apply.

In its place we have the following *naive isoquantization* first introduced by Animalu and Santilli [44] of 1990, and here extended to the use of the isodifferential calculus

$$\hat{A}^o = \int_{t_1}^{t_2} (\hat{p}_k \hat{\times} \hat{d}\hat{x}^k - \hat{H} \hat{\times} \hat{d}\hat{t}) \rightarrow -i \times \hat{I} \times \ln |\hat{\psi} >, \tag{3.4.3}$$

where $\hat{i} = i \times \hat{I}$, $|\hat{\psi} >$ is the *ospwavefunction,* or, more precisely, a state of the iso-Hilbert space $\hat{\mathcal{H}}$ outlined in the next section, and we should note that $\hat{i} \hat{\times} \hat{I} \times \ln |\hat{\psi}\rangle = i \times \text{isoln} |\hat{\psi} >$.

The use of Hamilton-Jacobi-Santilli isoequations (3.3.60) yields the following operator equations (here written for the simpler case in which $\hat{T}$ has no dependence on r, but admits a dependence on velocities and higher derivatives)

$$-\hat{\partial}_t \hat{A}^o = \hat{H} \rightarrow i \times \hat{\partial}_t |\hat{\psi}> = \hat{H} \times \hat{T} \times |\hat{\psi}> = \hat{H} \hat{\times} |\hat{\psi}>, \tag{3.4.4a}$$

$$\hat{p}_k = \hat{\partial}_k \hat{A}^o \rightarrow \hat{p}_k \times \hat{T} \times |\hat{\psi}> = \hat{p}_k \hat{\times} |\hat{\psi}> = -\hat{i} \hat{\times} \hat{\partial}_k |\hat{\psi}>, \tag{3.4.4b}$$

that constitutes the fundamental equations of operator isomechanics, as we shall see in the next section.

As it is well known, *Planck's constant $\hbar$ is the basic unit of quantum mechanics.* By comparing Eqs. (3.4.xx) and (3.4.xx) it is easy to see that $\hat{I}$ is the basic unit of operator isomechanics. Recall also that the isounits are defined at short distances as in Eqs. (3.1.xxx). We therefore have the following important

POSTULATE 3.4.1 [5]: In the transition from quantum mechanics to operator isomechanics Planck's unit $\hbar$ is replaced by the integrodifferential unit $\hat{I}$ under the condition of recovering the former at sufficiently large mutual distances,

$$\lim_{r \rightarrow \infty} \hat{I} = \hbar = 1. \tag{3.4.5}$$

Consequently, in the conditions of deep mutual penetration of the wavepackets and/or charge distributions of particles of particles as studied by operator isomechanics there is the superposition of quantized and continuous exchanges of energy.

3.4.3 Isohilbert Spaces and Their Isoduals

As it is well known, the Hilbert space $\mathcal{H}$ used in quantum mechanics is expressed in terms of states $|\psi\rangle,\ |\phi\rangle, \ldots,$ with normalization

$$\langle\psi| \times |\psi\rangle = 1, \tag{3.4.6}$$

and inner product

$$\langle\phi| \times |\psi\rangle = \int dr^3 \phi^\dagger(r) \times \psi(r), \tag{3.4.7}$$

defined over the field of complex numbers $\mathcal{C} = C(c, +, \times)$.

The lifting $C(c, +, \times) \rightarrow \hat{C}(\hat{c}, \hat{+}, \hat{\times})$, requires a compatible lifting of $\mathcal{H}$ into the *isohilbert space* $\hat{\mathcal{H}}$ with *isostates* $|\hat{\psi}\rangle, |\hat{\phi}\rangle, \ldots,$ *isoinner product* and *isonormalization*

$$\langle\hat{\psi}| \hat{\times} |\hat{\psi}\rangle \times \hat{I} = \left[\hat{\int}^{\wedge} \hat{d}\hat{r}^3 \ \hat{\psi}^\dagger(\hat{r}) \times \hat{T} \times \hat{\psi}(\hat{r}) \right] \times \hat{I} \in \hat{\mathcal{C}}, \tag{3.4.8a}$$

$$\langle\hat{\psi}|\hat{\times}|\hat{\psi}\rangle = 1, \tag{3.4.8b}$$

first introduced by Myung and Santilli in 1982 [45] (see also monographs [6,7] for a comprehensive study).

It is easy to see that the isoinner product is still inner (because $\hat{T} > 0$). Thus, $\hat{\mathcal{H}}$ is still Hilbert and the lifting $\mathcal{H} \to \hat{\mathcal{H}}$ is an isotopy. Also, it is possible to prove that *iso-Hermiticity coincides with conventional Hermiticity*,

$$< \hat{\psi}|\hat{\times}(\hat{H}\hat{\times}|\hat{\psi} >) \equiv (< \hat{\psi}|\hat{\times}\hat{H}^{\hat{\dagger}})\hat{\times}|\hat{\psi} >, \tag{3.4.9a}$$

$$\hat{H}^{\hat{\dagger}} \equiv \hat{H}^{\dagger} = \hat{H}. \tag{3.4.9b}$$

As a result, *all quantities that are observable for quantum mechanics remain so for hadronic mechanics.*

For consistency, the conventional eigenvalue equation $H \times |\psi\rangle = E \times |\psi\rangle$ must also be lifted into the *isoeigenvalue form* [7]

$$\hat{H}\hat{\times}|\hat{\psi}\rangle = \hat{H} \times \hat{T} \times |\hat{\psi}\rangle = \hat{E}\hat{\times}|\hat{\psi}\rangle =$$

$$= (E \times \hat{I}) \times \hat{T} \times |\hat{\psi}\rangle = E \times |\hat{\psi}\rangle, \tag{3.4.10}$$

where, as one can see, the final results are ordinary numbers.

Note the *necessity* of the isotopic action $\hat{H}\hat{\times}|\hat{\psi}\rangle$, rather than $\hat{H} \times |\hat{\psi}\rangle$. In fact, only the former admits $\hat{I}$ as the correct unit,

$$\hat{I}\hat{\times}|\hat{\psi}\rangle = \hat{T}^{-1} \times \hat{T} \times |\hat{\psi}\rangle \equiv |\hat{\psi}\rangle. \tag{3.4.11}$$

It is possible to prove that *the isoeigenvalues of isohermitean operators are isoreal*, i.e., they have the structure $\hat{E} = E \times \hat{I},\ E \in R(n, +, \times)$. As a result all real eigenvalues of quantum mechanics remain real for hadronic mechanics.

We also recall the notion of *isounitary operators* as the isooperators $\hat{U}$ on $\hat{\mathcal{H}}$ over $\hat{C}$ satisfying the isolaws

$$\hat{U}\hat{\times}\hat{U}^{\hat{\dagger}} = \hat{U}^{\hat{\dagger}}\hat{\times}\hat{U} = \hat{I}, \tag{3.4.12}$$

where we have used the identity $\hat{U}^{\hat{\dagger}} \equiv \hat{U}^{\dagger}$.

We finally indicate the notion of *isoexpectation value* of an isooperators $\hat{H}$ on $\hat{\mathcal{H}}$ over $\hat{C}$

$$\langle\hat{H}\rangle = \frac{\langle\hat{\psi}|\hat{\times}\hat{H}\hat{\times}|\hat{\psi}\rangle}{\langle\hat{\psi}|\hat{\times}|\hat{\psi}\rangle}. \tag{3.4.131}$$

It is easy to see that *the isoexpectation values of isohermitean operators coincide with the isoeigenvalues*, as in the conventional case.

Note also that *the isoexpectation value of the isounit is the isounit*,

$$\langle\hat{I}\rangle = \hat{I}, \tag{3.4.14}$$

provided, of course, that one uses the isoquotient (otherwise $\langle \hat{I} \rangle = I$).

The isotopies of quantum mechanics studied in the next sections are based on the following novel invariance property of the conventional Hilbert space [xxx], here expressed in term of a non-null scalar n independent from the integration variables,

$$\langle \hat{\phi} | \times | \hat{\psi} \rangle \times I \equiv$$
$$\equiv \langle \hat{\phi} | \times n^{-2} \times | \hat{\psi} \rangle \times (n^2 \times I) = \langle \phi | \hat{\times} | \psi \rangle \times \hat{I}. \tag{3.4.15}$$

Note that new invariances (3.4.15) remained undetected throughout the 20-th century because they required the prior discovery of *new numbers*, those with arbitrary units.

3.4.4 Structure of Operator Isomechanics and its Isodual

The structure of operator isomechanics is essentially given by the following main steps [47]:

1) The description of closed-isolated systems is done via *two* quantities, the Hamiltonian representing all action-at-a-distance potential interactions, plus the isounit representing all nonlinear, nonlocal and non-Hamiltonian effects,

$$H(t, r, p) = p^2/2m + V(r), \tag{3.4.16a}$$

$$\hat{I} = \hat{I}(t, r, p, \psi, \nabla\psi, \dots), \tag{3.4.16b}$$

The explicit form of the Hamiltonian is that conventionally used in quantum mechanics although written on isospaces over isofields,

$$\hat{H} = \hat{p} \hat{\times} \hat{p} \hat{/} \hat{2} \hat{\times} \hat{m} + \hat{V}(\hat{r}). \tag{3.4.17}$$

A generic expression of the isounit for the representation of two spinning particles with point-like change (such as the electrons) in conditions of deep penetration of their wavepackets (as occurring in chemical valence bonds and many other cases) is given by

$$\hat{I} = \exp \left[\Gamma(\psi, \psi^\dagger) \times \int dv \, \psi_\downarrow^\dagger(r) \psi_\uparrow(r) \right], \tag{3.4.18}$$

where the nonlinearity is expressed by $\Gamma(\psi, \psi^\dagger)$ and the nonlocality is expressed by the volume integral of the deep wave-overlappings $\int dv \, \psi_\downarrow^\dagger(r) \psi_\uparrow(r)$. All isounits will be restricted by the conditions of being positive-definite (thus everywhere invertible) as well as of recovering the trivial unit of quantum mechanics for sufficiently big mutual distances r,

$$\lim_{r \to \infty} \int dv \, \psi_\downarrow^\dagger(r) \psi_\uparrow(r) = 0. \tag{3.4.19}$$

2) The lifting of the multiplicative unit $I > 0 \rightarrow \hat{I} = 1/\hat{T} > 0$ requires the reconstruction of the entire formalism of quantum mechanics into such a form to admit $\hat{I}$ as the correct left and right unit at all levels of study, including numbers and angles, conventional and special functions, differential and integral calculus, metric and Hilbert spaces, algebras and groups, etc., without any exception known to the authors. This reconstruction is "isotopic" in the sense of being axiom-preserving. Particularly important is the preservation of all conventional quantum laws as shown below.

3) The mathematical structure of nonrelativistic hadronic mechanics is characterized by [6]:

3a) The isofield $\hat{C} = \hat{C}(\hat{c}, +, \hat{\times})$ with *isounit* $\hat{I} = 1/\hat{T} > 0$, *isocomplex numbers* and related *isoproduct*

$$\begin{gathered} \hat{c} = c \times \hat{I} = (n_1 + i \times n_2) \times \hat{I}, \quad \hat{c} \hat{\times} \hat{d} = (c \times d) \times \hat{I}, \\ \hat{c}, \hat{d} \in \hat{\mathcal{C}}, \quad c, d \in C, \end{gathered} \tag{3.4.20}$$

the isofield $\hat{R}(\hat{n}, +, \hat{\times})$ of *isoreal numbers* $\hat{n} = n \times \hat{I},\ n \in R$, being a particular case;

3b) The iso-Hilbert space $\hat{\mathcal{H}}$ with isostates $|\hat{\psi}\rangle, |\hat{\phi}\rangle, \dots$, *isoinner product* and *isonormalization*

$$\langle \hat{\phi} | \hat{\times} | \hat{\psi} \rangle \times \hat{I} \ \in \hat{S}, \quad \langle \hat{\psi} | \hat{\times} | \hat{\psi} \rangle = 1, \tag{3.4.21}$$

and related theory of isounitary operators;

3c) The Euclid-Santilli isospace $\hat{E}(\hat{r}, \hat{\delta}, \hat{\mathrm{R}})$ with *isocoordinates, isometric and isoinvariant* respectively given by

$$\hat{r} = \{r^k\} \times \hat{I}, \tag{3.4.22a}$$

$$\hat{\delta} = \hat{T}(t, r, p, \psi, \nabla\psi, \dots) \times \delta, \tag{3.4.22b}$$

$$\delta = Diag.(1, 1, 1), \tag{3.4.22c}$$

$$\hat{r}^{\hat{2}} = (r^i \times \hat{\delta}_{ij} \times r^j) \times \hat{I} \in \hat{\mathrm{R}}; \tag{3.4.22d}$$

3d) The isodifferential calculus and the isofunctional analysis (see Section 3.2);

3e) The Lie-Santilli isotheory with enveloping isoassociative algebra $\hat{\xi}$ of operators $\hat{A}$, $\hat{B}$, $\dots$, with isounit $\hat{I}$, isoassociative product $\hat{A} \hat{\times} \hat{B} = \hat{A} \times \hat{T} \times \hat{B}$, *Lie-Santilli isoalgebra with brackets and isoexponentiation*

$$[\hat{A} \hat{,} \hat{B}] = \hat{A} \hat{\times} \hat{B} - \hat{B} \hat{\times} \hat{A}, \tag{3.4.23a}$$

$$\hat{U} = \hat{e}^X = (e^{X \times \hat{T}}) \times \hat{I} = \hat{I} \times (e^{\hat{T} \times X}), \quad X = X^\dagger, \tag{3.4.13b}$$

and related isosymmetries characterizing groups of isounitary transforms on $\hat{\mathcal{H}}$ over $\hat{C}$,

$$\hat{U}\hat{\times}\hat{U}^\dagger = \hat{U}^\dagger\hat{\times}\hat{U} = \hat{I}. \tag{3.4.24}$$

As we shall see in Sections 3.4.8 and 3.4.9, the above entire mathematical structure can be achieved in a truly elementary way via nonunitary transforms of quantum formalisms. Their isotopic reformulations then proves the invariance of hadronic mechanics, namely, its capability of predicting the same numbers for the same conditions at different times.

Under the above outlined structure we have the following main features:

I) Hadronic mechanics is a covering of quantum mechanics, because the latter theory is admitted uniquely and unambiguously at the limit when the isounit recovers the conventional unit, $\hat{I} \rightarrow I$;

II) Said covering is further characterized by the fact that hadronic mechanics coincides with quantum mechanics everywhere except for (as we shall see, generally small) non-Hamiltonian corrections at short mutual distances of particles caused by deep mutual overlapping of the wavepackets and/or charge distributions of particles;

III) Said covering is finally characterized by the fact that the indicated non-Hamiltonian corrections are restricted to verify all abstract axioms of quantum mechanics, with consequential preservation of is basic laws for closed non-Hamiltonian systems as a whole, as we shall see shortly.

Note that *composite hadronic systems*, such as *hadrons, nuclei, isomolecules, etc.*, are represented via the tensorial product of the above structures. This can be best done via the identification first of the *total isounit, total isofields, total isohilbert spaces*, *etc.*,

$$\hat{I}_{\text{tot}} = \hat{I}_1 \times \hat{I}_2 \times \ldots\ ,\ \hat{C}_{\text{tot}} = \hat{C}_1 \times \hat{C}_2 \times \ldots\ ,\ \hat{\mathcal{H}}_{\text{tot}} = \hat{\mathcal{H}}_1 \times \hat{\mathcal{H}}_2 \times \ldots\ . \tag{3.4.25}$$

Note also that some of the units, fields and Hilbert spaces in the above tensorial products can be *conventional*, namely, the composite structure may imply *local-potential long range interactions* (e.g., those of Coulomb type), which require the necessary treatment via *conventional* quantum mechanics, and *nonlocal-nonpotential short range interactions* (e.g., those in deep wave-overlappings), which require the use of operator isomechanics.

3.4.5 Dynamical Equations of Operator Isomechanics and Their Isoduals

The formulations of the preceding sections permit the identification of the following fundamental dynamical equations of the Lie-isotopic branch of hadronic mechanics, known under the name of *iso-Heisenberg equations,* that were identified in the original proposal of 1978 to build hadronic mechanics [5], are

can be presented in their finite and infinitesimal forms,

$$\hat{A}(\hat{t}) = \hat{U}\hat{\times}\hat{A}(\hat{0})\hat{\times}\hat{U}^{\hat{\dagger}} = \{\hat{e}^{\hat{i}\hat{\times}\hat{H}\hat{\times}\hat{t}}\}\hat{\times}\hat{A}(\hat{0})\hat{\times}\{\hat{e}^{-\hat{i}\hat{\times}\hat{t}\hat{\times}\hat{H}}\}, \tag{3.4.26a}$$

$$\hat{i}\hat{\times}\hat{d}\hat{A}\hat{/}\hat{d}\hat{t} = [\hat{A}\hat{,}\hat{H}] = A\hat{\times}\hat{H} - \hat{H}\hat{\times}\hat{A} = \hat{A}\times\hat{T}\times\hat{H} - \hat{H}\times\hat{T}\times\hat{A}, \tag{3.4.26b}$$

with the corresponding *fundamental hadronic isocommutation rules*

$$[\hat{b}^{\mu}\hat{,}\hat{b}^{\nu}] = \hat{i}\hat{\times}\hat{\omega}^{\mu\nu} = i\times\omega^{\mu\nu}\times\hat{I}_6, \quad \hat{b} = (\hat{r}^k, \hat{p}_k), \tag{3.4.27}$$

with corresponding *i-Soschrödinger equations* for the energy identified by Myung and Santilli [45] and Mignani [48] in 1982 over conventional fields and first formulated in an invariant way by Santilli in monograph [7] of 1995

$$\hat{i}\hat{\times}\hat{\partial}_{\hat{t}}|\hat{\psi}\rangle = \hat{H}\hat{\times}|\hat{\psi}\rangle = \hat{H}\times\hat{T}\times|\hat{\psi}\rangle = \hat{E}\hat{\times}|\hat{\psi}\rangle = E\times|\hat{\psi}\rangle, \tag{3.4.28a}$$

$$|\hat{\psi}(\hat{t})\rangle = \hat{U}\hat{\times}|\hat{\psi}(\hat{0})\rangle = \{\hat{e}^{i\hat{H}\hat{\times}\hat{t}}\}\hat{\times}|\hat{\psi}(\hat{0})\rangle, \tag{3.4.28b}$$

and *isolinear momentum* first identified by Santilli in Ref. [7] of 1995 thanks to the discovery of the isodifferential calculus

$$\hat{p}_k\hat{\times}|\hat{\psi}\rangle = \hat{p}_k\times\hat{T}\times|\hat{\psi}\rangle - \hat{i}\hat{\times}\hat{\partial}_k|\hat{\psi}\rangle = -i\times\hat{I}^i_k\times\partial_i|\hat{\psi}\rangle, \tag{3.4.29}$$

It is evident that the iso-Heisenberg equations in their infinitesimal and exponentiated forms are a realization of the Lie-Santilli isotheory of Section 3.2, which is therefore the algebraic and group theoretical structure of the isotopic branch of hadronic mechanics.

Note that Eqs. (3.4.26) and (3.4.28) automatically bring into focus the general need for a *time isounit* and related characterization of the time isodifferential and isoderivative

$$\hat{I}_t(t, r, \psi, \dots) = \hat{T}_t > 0, \tag{3.4.30a}$$

$$\hat{d}\hat{t} = \hat{I}_t\times d\hat{t}, \quad \hat{\partial}_{\hat{t}} = \hat{I}_t\times\partial_t. \tag{3.4.30b}$$

Note also that $\omega^{\mu\nu}$ in Eqs. (3.4.xxx) is the *conventional* Lie tensor, namely, the same tensor appearing in the conventional canonical commutation rules, thus confirming the axiom-preserving character of isomechanics.

The limited descriptive capabilities of quantum models should be kept in mind, purely Hamiltonian and, as such, they can only represent systems which are linear, local and potential. By comparison, we can write Eq. (3.4.xxx) in its explicit form

$$\begin{aligned}\hat{i}\hat{\times}\hat{\partial}_{\hat{t}}\hat{\psi} = i\times\hat{I}_t\times\partial_{\hat{t}}|\hat{\psi}\rangle = \hat{H}\hat{\times}|\hat{\psi}\rangle = \hat{H}\times\hat{T}\times|\hat{\psi}\rangle = \\ = \{\hat{p}_k\times\hat{p}_k/\hat{2}\hat{\times}\hat{m} + \hat{U}_k(\hat{t},\hat{r})\hat{\times}\hat{v}^k + \\ +\hat{U}_0(\hat{t},\hat{r})\}\times\hat{T}(\hat{t},\hat{r},\hat{p},\hat{\psi},\nabla\psi,\dots)\times|\hat{\psi}(\hat{t},\hat{r})\rangle = \\ = \hat{E}\hat{\times}|\hat{\psi}(t,\hat{x})\rangle = E\times|\hat{\psi}(\hat{t},\hat{x})\rangle,\end{aligned} \tag{3.4.31}$$

thus proving the following

THEOREM 3.4.1 [7]: Hadronic mechanics id "directly universal" for all infinitely possible, sufficiently smooth and regular, closed non-Hamiltonian systems, namely, it can represent in the fixed coordinates of the experimenter all infinitely possible closed-isolated systems with linear and nonlinear, local and nonlocal, and potential as well as nonpotential internal forces verifying the conservation of the total energy.

A consistent formulation of the isolinear momentum (3.4.29) escaped identification for two decades, thus delaying the completion of the construction of hadronic mechanics, as well as its practical applications. The consistent and invariant form (3.4.29) with consequential isocanonical commutation rules were first identified by Santilli in the second edition of Vol. II of this series, Ref. [7] of 1995 and memoir [10], following the discovery of the isodifferential calculus.

3.4.6 Preservation of Quantum Physical Laws

As one can see, the fundamental assumption of isoquantization is the lifting of the basic unit of quantum mechanics, Planck's constant $\hbar$, into a matrix $\hat{I}$ with nonlinear, integro-differential elements which also depend on the wavefunction and its derivatives

$$\hbar = I > 0 \rightarrow \hat{I} = \hat{I}(t, r, p, \psi, \hat{\psi}, \dots) = \hat{I}^{\dagger} > 0. \tag{3.4.32}$$

It should be indicated that the above generalization is only *internal* in closed non-Hamiltonian because, when measured from the outside, *the isoexpectation values and isoeigenvalues of the isounit recover Planck's constant identically* [46],

$$\langle \hat{I} \rangle = \frac{\langle \hat{\psi} | \hat{\times} \hat{I} \hat{\times} | \hat{\psi} \rangle}{\langle \hat{\psi} | \hat{\times} | \hat{\psi} \rangle} = 1 = \hbar, \tag{3.4.33a}$$

$$\hat{I} \hat{\times} | \hat{\psi} \rangle = \hat{T}^{-1} \times \hat{T} \times | \hat{\psi} \rangle = 1 \times | \hat{\psi} \rangle = | \hat{\psi} \rangle. \tag{3.4.33b}$$

Moreover, the isounit is the fundamental invariant of isomechanics, thus preserving all axioms of the conventional unit $I = \hbar$, e.g.,

$$\hat{I}^{\hat{n}} = \hat{I} \hat{\times} \hat{I} \hat{\times} \dots \hat{\times} \hat{I} \equiv \hat{I}, \tag{3.4.34a}$$

$$\hat{I}^{\hat{\frac{1}{2}}} \equiv \hat{I}, \tag{3.4.34b}$$

$$\hat{i} \hat{\times} \hat{d} \hat{I} \hat{/} \hat{d} t = [\hat{I} \hat{,} \hat{H}] = \hat{I} \hat{\times} \hat{H} - \hat{H} \hat{\times} \hat{I} \equiv 0. \tag{3.4.34c}$$

Despite their generalized structure, *Eqs. (3.4.26) and (3.4.28) preserve conventional quantum mechanical laws under nonlinear, nonlocal and nonpotential interactions* [7].

To begin an outline, the preservation of Heisenberg's uncertainties can be easily derived from isocommutation rules (3.4.27):

$$\Delta x^k \times \Delta p_k \geq \frac{1}{2} \times \langle [\hat{x}^k \hat{,} \hat{p}_k] \rangle = \frac{1}{2}. \tag{3.4.35}$$

To see the preservation of Pauli's exclusion principle, recall that the regular (two-dimensional) representation of $SU(2)$ is characterized by the conventional *Pauli matrices* σ_k with familiar commutation rules and eigenvalues on $\mathcal{H}$ over C,

$$[\sigma_i, \sigma_j] = \sigma_i \times \sigma_j - \sigma_j \times \sigma_i = 2 \times i\varepsilon_{ijk} \times \sigma_k, \tag{3.4.36a}$$

$$\sigma^2 \times |\psi\rangle = \sigma_k \times \sigma^k \times |\psi\rangle = 3 \times |\psi\rangle, \tag{3.4.36b}$$

$$\sigma_3 \times |\psi\rangle = \pm 1 \times |\psi\rangle. \tag{3.4.36c}$$

The isotopic branch of hadronic mechanics requires the construction of *nonunitary images of Pauli's matrices* first constructed in Ref. [49] that, for diagonal nonunitary transforms and isounits, can be written (see also Section 3.3.6)

$$\hat{\sigma}_k = U \times \sigma_k \times U^\dagger, \quad U \times U^\dagger = \hat{I} \neq I, \tag{3.4.37a}$$

$$\begin{gathered} U = \begin{pmatrix} i \times n_1 & 0 \\ 0 & i \times n_2 \end{pmatrix}, \quad U^\dagger = \begin{pmatrix} -i \times n_1 & 0 \\ 0 & -i \times n_2 \end{pmatrix}, \\ \hat{I} = \begin{pmatrix} m_1^2 & 0 \\ 0 & n_2^2 \end{pmatrix}, \quad \hat{T} = \begin{pmatrix} n_1^{-2} & 0 \\ 0 & n_2^{-2} \end{pmatrix}, \end{gathered} \tag{3.4.37b}$$

where the n's are well behaved nowhere null functions, resulting in the *regular Pauli-Santilli isomatrices* [49]

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & n_1^2 \\ n_2^2 & 0 \end{pmatrix}, \ \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \times n_1^2 \\ i \times n_2^2 & 0 \end{pmatrix}, \ \hat{\sigma}_3 = \begin{pmatrix} n_1^2 & 0 \\ 0 & n_2^2 \end{pmatrix}. \tag{3.4.38}$$

Another realization is given by *nondiagonal unitary transforms* [*loc. cit.*],

$$\begin{gathered} U = \begin{pmatrix} 0 & n_1 \\ n_2 & 0 \end{pmatrix}, \quad U^\dagger = \begin{pmatrix} 0 & n_2 \\ m_1 & 0 \end{pmatrix}, \\ \hat{I} = \begin{pmatrix} n_1^2 & 0 \\ 0 & n_2^2 \end{pmatrix}, \quad \hat{T} = \begin{pmatrix} n_1^{-2} & 0 \\ 0 & n_2^{-2} \end{pmatrix}, \end{gathered} \tag{3.4.39}$$

with corresponding *regular Pauli-Santilli isomatrices*,

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & n_1 \times n_2 \\ n_1 \times n_2 & 0 \end{pmatrix}, \ \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \times n_1 \times n_2 \\ i \times m_1 \times n_2 & 0 \end{pmatrix},$$

$$\hat{\sigma}_3 = \begin{pmatrix} n_1^2 & 0 \\ 0 & n_2^2 \end{pmatrix}, \tag{3.4.40}$$

or by more general realizations with Hermitean nondiagonal isounits $\hat{I}$ [15].

All Pauli-Santilli isomatrices of the above regular class verify the following isocommutation rules and isoeigenvalue equations on $\hat{\mathcal{H}}$ over $\hat{\mathcal{C}}$

$$[\hat{\sigma}_i \hat{,} \hat{\sigma}_j] = \hat{\sigma}_i \times \hat{T} \times \hat{\sigma}_j - \hat{\sigma}_j \times \hat{T} \times \hat{\sigma}_i = 2 \times i \times \varepsilon_{ijk} \times \hat{\sigma}_k, \tag{3.4.41a}$$

$$\hat{\sigma}^{\hat{2}} \hat{\times} |\hat{\psi}\rangle = (\hat{\sigma}_1 \hat{\times} \hat{\sigma}_1 + \hat{\sigma}_2 \hat{\times} \hat{\sigma}_2 + \hat{\sigma}_3 \hat{\times} \hat{\sigma}_3) \hat{\times} |\hat{\psi}\rangle = 3 \times |\hat{\psi}\rangle, \tag{3.4.41b}$$

$$\hat{\sigma}_3 \hat{\times} |\hat{\psi}\rangle = \pm 1 \times |\hat{\psi}\rangle, \tag{3.4.41c}$$

thus preserving conventional spin 1/2, and establishing the preservation in isochemistry of the Fermi-Dirac statistics and Pauli's exclusion principle.

It should be indicated for completeness that the representation of the isotopic $S\hat{U}(2)$ also admit *irregular isorepresentations*, that no longer preserve conventional values of spin [49]. The latter structures are under study for the characterization of spin under the most extreme conditions, such as for protons and electrons in the core of collapsing stars and, as such, they have no known relevance for isomechanics.

The preservation of the superposition principle under nonlinear interactions occurs because of the reconstruction of linearity on isospace over isofields, thus regaining the applicability of the theory to composite systems.

Recall in this latter respect that conventionally nonlinear models,

$$H(t, x, p, \psi, \dots) \times |\psi\rangle = E \times |\psi\rangle, \tag{3.4.42}$$

violate the superposition principle and have other shortcomings (see Section 1.5). As such, they cannot be applied to the study of composite systems such as molecules. All these models can be *identically* reformulated in terms of the isotopic techniques via the embedding of all nonlinear terms in the isotopic element,

$$H(t, x, p, \psi, \dots) \times |\psi\rangle \equiv H_0(t, x, p) \times \hat{T}(\psi, \dots) \times |\psi\rangle = E \times |\psi\rangle, \tag{3.4.43}$$

by regaining the full validity of the superposition principle in isospaces over isofields with consequential applicability to composite systems.

The preservation of causality follows from the one-dimensional isounitary group structure of the time evolution (3.4.28) (which is isomorphic to the conventional one); the preservation of probability laws follows from the preservation of the axioms of the unit and its invariant decomposition as indicated earlier; the preservation of other quantum laws then follows.

The same results can be also seen from the fact that operator isomechanics coincides at the abstract level with quantum mechanics by conception and

construction. As a result, hadronic and quantum versions are *different realizations of the same abstract axioms and physical laws.*

Note that the preservation of conventional quantum laws under nonlinear, nonlocal and nonpotential interactions is crucially dependent on the capability of isomathematics to reconstruct linearity, locality and canonicity-unitarity on isospaces over isofields.

The preservation of conventional physical laws by the isotopic branch of hadronic mechanics was first identified by Santilli in report [47]. It should be indicated that the same quantum laws *are not* generally preserved by the broader genomechanics, evidently because the latter must represent by assumption *non*-conservation laws and other *departures* from conventional quantum settings.

With the understanding that the theory does not receive the classical determinism, it is evident that isomechanics provides a variety of "completions" of quantum mechanics according to the celebrated E-P-R argument [50], such as:

1) Isomechanics "completes" quantum mechanics via the addition of nonpotential-nonhamiltonian interactions represented by nonunitary transforms.

2) Isomechanics "completes" quantum mechanics via the broadest possible (non-oriented) realization of the associative product into the isoassociative form.

3) Isomechanics "completes" quantum mechanics in its classical image.

In fact, as proved by well known procedures based on *Bell's inequalities*, quantum mechanics does not admit direct classical images on a number of counts. On the contrary, as studied in details in Refs. [51], the nonunitary images of Bell's inequalities permit indeed direct and meaningful classical limits which do not exist for the conventional formulations.

Similarly, it is evident that isomechanics constitutes a specific and concrete realization of "hidden variables" [52] λ which are explicitly realized by the isotopic element, $\lambda = \hat{T}$, and actually turned into an operator hidden variables. The "hidden" character of the realization is expressed by the fact that hidden variables are embedded in the unit and product of the theory.

In fact, we can write the iso-Schrödinger equation $\hat{H}\hat{\times}|\hat{\psi}\rangle = \hat{H} \times \lambda \times |\hat{\psi}\rangle = E \times |\hat{\psi}\rangle$, $\lambda = \hat{T}$. As a result, the "variable" λ (now generalized into the operator $\hat{T}$) is "hidden" in the modular associative product of the Hamiltonian $\hat{H}$ and the state $|\hat{\psi}\rangle$.

Alternatively, we can say that hadronic mechanics provides an explicit and concrete realization of hydden variables because all distinctions between $\hat{H}\hat{\times}|\hat{\psi} >$ and $H \times |\psi >$ cease to exist at the abstract realization-free level.

For studies on the above and related issues, we refer the interested reader to Refs. [51] and quoted literature.

3.4.7 Isoperturbation Theory and its Isodual

We are now sufficiently equipped to illustrate the computational advantages in the use of isotopies.

THEOREM 3.4.2 [7]: Under sufficient continuity conditions, all perturbative and other series that are conventionally divergent (weakly convergent) can be turned into convergent (strongly convergent) forms via the use of isotopies with sufficiently small isotopic element (sufficiently large isounit),

$$|\hat{T}| \ll 1, \quad |\hat{I}| \gg 1. \tag{3.4.44}$$

The emerging perturbation theory was first studied by Jannussis and Mignani [53], and then studied in more detail in monograph [7] under the name of *isoperturbation theory.*

Consider a Hermitean operator on $\mathcal{H}$ over C of the type

$$H(k) = H_0 + k \times V, \quad H_0 \times |\psi\rangle = E_0 \times |\psi\rangle, \tag{3.4.45a}$$

$$H(k) \times |\psi(k)\rangle = E(k) \times |\psi(k)\rangle, \quad k \gg 1. \tag{3.4.45b}$$

Assume that H_0 has a nondegenerate discrete spectrum. Then, conventional perturbative series are *divergent*, as well known. In fact, the eigenvalue $E(k)$ of $H(k)$ up to second order is given by

$$\begin{aligned} E(k) &= E_0 + k \times E_1 + k^2 \times E_2 = \\ &= E_0 + k \times \langle\psi| \times V \times |\psi\rangle + k^2 \times \sum_{p \neq n} \frac{|\langle\psi_p| \times V \times |\psi_n\rangle|^2}{E_{0n} - E_{0p}}. \end{aligned} \tag{3.4.46}$$

But under isotopies we have

$$H(k) = H_0 + k \times V, \quad H_0 \times \hat{T} \times |\tilde{\psi}\rangle = \tilde{E}_0 \times |\tilde{\psi}\rangle, \quad \tilde{E}_0 \neq E_0, \tag{3.4.47a}$$

$$H(k) \times \hat{T} \times |\hat{\psi}(k)\rangle = \tilde{E}(k) \times |\hat{\psi}(k)\rangle, \quad \tilde{E} \neq E, \quad k > 1. \tag{3.4.47b}$$

A simple lifting of the conventional perturbation expansion then yields

$$\tilde{E}(k) = \tilde{E}_0 + k \times \tilde{E}_1 + k^2 \times \tilde{E}_2 + \hat{O}(k^2) =$$

$$= \tilde{E}_0 + k \times \langle\tilde{\psi}| \times \hat{T} \times V \times \hat{T} \times |\tilde{\psi}\rangle + \tag{3.4.48a}$$

$$+k^2 \times \sum_{p \neq n} \frac{|\langle\hat{\psi}_p| \times \hat{T} \times V \times \hat{T} \times |\hat{\psi}_n\rangle|^{\hat{2}}}{\tilde{E}_{0n} - \tilde{E}_{0p}}, \tag{3.4.48b}$$

whose convergence can be evidently reached via a suitable selection of the isotopic element, e.g., such that $|\hat{T}| \ll k$.

As an example, for a positive-definite constant $\hat{T} \ll k^{-1}$, expression (3.4.46) becomes

$$\tilde{E}(k) = \tilde{E}_0 + k \times \hat{T}^2 \times \langle \hat{\psi}| \times V \times |\psi*\rangle + k^2 \times T^5 \times \\ \times \sum_{p \neq n} \frac{|\langle \psi_p| \times V \times |\psi_n\rangle|^2}{\tilde{E}_{0n} - \tilde{E}_{0p}}. \tag{3.4.49}$$

This shows that the original divergent coefficients $1, k, k^2, \ldots$ are now turned into the manifestly convergent coefficients $1, k \times T^2, k^2 \times T^5, \ldots$, with $k > 1$ and $\hat{T} \ll 1/k$, thus ensuring isoconvergence for a suitable selection of $\hat{T}$ for each given k and V.

A more effective reconstruction of convergence can be seen in the algebraic approach. At this introductory stage, we consider a divergent canonical series,

$$A(k) = A(0) + k \times [A, H]/1! + \\ + k^2 \times [[A, H], H]/2! + \ldots \to \infty, \quad k > 1, \tag{3.4.50}$$

where $[A, H] = A \times H - H \times A$ is the familiar Lie product, and the operators A and H are Hermitean and sufficiently bounded. Then, under the isotopic lifting the preceding series becomes [7]

$$\hat{A}(k) = \hat{A}(0) + k \times [A\hat{,}H]/1! + k^2 \times [[A\hat{,}H]\hat{,}H]/2! + \cdots \leq |N| < \infty, \tag{3.4.51a}$$

$$[A\hat{,}H] = A \times \hat{T} \times H - H \times \hat{T} \times A, \tag{3.4.51b}$$

which holds, e.g., for the case $T = \varepsilon \times k^{-1}$, where ε is a sufficiently small positive-definite constant.

In summary, the studies on the construction of hadronic mechanics have indicated that the apparent origin of divergences (or slow convergence) in quantum mechanics and chemistry is their lack of representation of nonlinear, nonlocal, and nonpotential effects because when the latter are represented via the isounit, full convergence (much faster convergence) can be obtained.

As we shall see, all known applications of hadronic mechanics verify the crucial condition $|\hat{I}| >> 1$, $|\hat{T}| << 1$, by permitting convergence of perturbative series. For instance, in the case of chemical bonds, hadronic chemistry allows computations at least one thousand times faster than those of quantum chemistry, with evident advantages, e.g., a drastic reduction of computer time (see Chapter 9). Essentially the same results are expected for hadronic mechanics and hadronic superconductivity.

The reader should meditate a moment on the evident possibility that *hadronic mechanics offers realistic possibilities of constructing a convergent perturbative theory for strong interactions.* As a matter of fact, the divergencies that have afflicted strong interactions through the 20-th century originates precisely

from the excessive approximation of hadrons as point, with the consequential sole potential interactions and related divergencies.

In fact, whenever hadrons are represented as they actually are in reality, extended and hyperdense particles, with consequential potential as well as nonpotential interactions, all divergencies are removed by the isounit.

3.4.8 Simple Construction of Operator Isomechanics

Despite their *mathematical equivalence*, it should be indicated that quantum and hadronic mechanics are *physically inequivalent*, or, alternatively, hadronic mechanics is outside the classes of equivalence of quantum mechanics because the former is a *nonunitary image* of the latter.

As we shall see in the next chapters, the above property provides means for the explicit construction of the new model of isomechanics bonds from the conventional model. The main requirement is that of identifying the *non-hamiltonian* effects one desires to represent, which as such, are necessarily *nonunitary*. The resulting nonunitary transform is then assumed as the fundamental space isounit of the new isaomechanics [46]

$$U \times U^\dagger = \hat{I} \neq I, \tag{3.4.52}$$

under which transform we have the liftings of: the quantum unit into the isounit,

$$I \to \hat{I} = U \times I \times U^\dagger; \tag{3.4.53}$$

numbers into isonumbers,

$$a \to \hat{a} = U \times a \times U^\dagger = a \times (U \times U^\dagger) = a \times \hat{I}; \quad a = n, c, \tag{3.4.54}$$

associative products $A \times B$ into the isoassociative form with the correct isotopic element,

$$A \times B \to \hat{A} \hat{\times} \hat{B} = \hat{A} \times \hat{T} \times \hat{B}, \tag{3.4.55a}$$

$$\hat{A} = U \times A \times U^\dagger, \ \hat{B} = U \times B \times U^\dagger, \ \hat{T} = (U \times U^\dagger)^{-1} = T^\dagger; \tag{3.4.55b}$$

Schrödinger's equation into the isoschrödinger's equations

$$\begin{aligned} H \times |\psi\rangle = E \times |\psi\rangle \to U(H \times |\psi\rangle) = \\ = (U \times H \times U^\dagger) \times (U \times U^\dagger)^{-1} \times (U \times |\psi\rangle) = \\ = \hat{H} \times \hat{T} \times |\hat{\psi}\rangle = \hat{H} \hat{\times} |\hat{\psi}\rangle; \end{aligned} \tag{3.4.56}$$

Heisenberg's equations into their isoheisenberg generalization

$$\begin{aligned} i \times dA/dt - A \times H - H \times A = 0 \to \\ \to U \times (i \times dA/dt) \times U^\dagger - U(A \times H - H \times A) \times U^\dagger = \\ = \hat{i} \hat{\times} d\hat{A}/dt - \hat{A} \hat{\times} \hat{H} - \hat{H} \hat{\times} \hat{A} = 0; \end{aligned} \tag{3.4.57}$$

the Hilbert product into its isoinner form

$$\begin{gathered}\langle\psi|\times|\psi\rangle \to U\times\langle\psi|\times|\psi\rangle\times U^\dagger = \\ = (\langle\psi|\times U^\dagger)\times(U\times U)^{-1}\times(U\times|\psi\rangle)\times(U\times U)^{-1} = \langle\hat{\psi}|\hat{\times}|\hat{\psi}\rangle\times\hat{I};\end{gathered} \tag{3.4.58}$$

canonical power series expansions into their isotopic form

$$\begin{gathered}A(k) = A(0) + k\times[A,H] + k^2\times[[A,H,],H] + \ldots \to U\times A(k)\times U^\dagger = \\ = U\times\Big[A(0) + k\times[A,H] + k^2\times[[A,H],H] + \ldots\Big]\times U^\dagger = \\ = \hat{A}(\hat{k}) = \hat{A}(0) + \hat{k}\hat{\times}[\hat{A}\hat{,}\hat{H}] + \hat{k}^{\hat{2}}\hat{\times}[[\hat{A}\hat{,}\hat{H}]\hat{,}\hat{H}] + \ldots, \\ k > 1, \quad |\hat{T}| \ll 1;\end{gathered} \tag{3.4.59}$$

Schrödinger's perturbation expansion into its isotopic covering (where the usual summation over states $p \neq n$ is assumed)

$$\begin{gathered}E(k) = E(0) + k\times\langle\psi|\times V\times|\psi\rangle + k^2\frac{|\langle\psi|\times V\times|\psi\rangle|^2}{E_{0n}-E_{0p}} + \ldots \to \\ \to U\times E(k)\times U^\dagger = U\times\Big[E(0) + k\times\langle\psi|\times V\times|\psi\rangle + \ldots\Big]\times U^\dagger = \\ = \hat{E}(\hat{k}) = \hat{E}(0) + \hat{k}\hat{\times}\langle\hat{\psi}|\times\hat{T}\times\hat{V}\times\hat{T}\times|\hat{\psi}\rangle + \ldots, \\ k > 1, \quad |\hat{T}| \ll 1;\end{gathered} \tag{3.4.60}$$

etc. All remaining aspects of operator isochemistry can then be derived accordingly, including the isoexsponent, isologarithm, isodeterminant, isotrace, isospecial functions and transforms, *etc.*

Note that the above construction via a nonunitary transform is the correct operator image of the derivability of the classical isohamiltonian mechanics from the conventional form via noncanonical transforms (Section 3.2.12).

The construction of hadronic mechanics via nonunitary transforms of quantum mechanics was first identified by Santilli in the original proposal [5e], and then worked out in subsequent contributions (see [12] for the latest presentation).

3.4.9 Invariance of Operator Isomechanics

It is important to see that, in a way fully parallel to the classical case (Section 3.3.7), operator isomechanics is indeed invariant under the most general possible nonlinear, nonlocal and nonhamiltonian-nonunitary transforms, provided that, again, the invariance is treated via the isomathematics. In fact, any given nonunitary transform $U\times U^\dagger \neq I$ can always be decomposed into the form [12]

$$U = \hat{U}\times\hat{T}^{1/2},$$

under which nonunitary transforms on $\mathcal{H}$ over C are identically reformulated as isounitary transforms on the isohilbert space $\hat{\mathcal{H}}$ over the isofield $\hat{}$,

$$U \times U^\dagger \equiv \hat{U}\hat{\times}\hat{U}^\dagger = \hat{U}^\dagger\hat{\times}\hat{U} = \hat{I}. \tag{3.4.61}$$

The form-invariance of operator isomechanics under isounitary transforms then follows,

$$\hat{I} \rightarrow \hat{I}' = \hat{U}\hat{\times}\hat{I}\hat{\times}\hat{U}^\dagger \equiv \hat{I},\ \hat{A}\hat{\times}\hat{B} \rightarrow \hat{U}\hat{\times}(\hat{A}\hat{\times}\hat{B})\hat{\times}\hat{U}^\dagger = \hat{A}'\hat{\times}\hat{B}',\ \textit{etc.}, \tag{3.4.62a}$$

$$\begin{aligned} \hat{H}\hat{\times}|\hat{\psi}\rangle = \hat{E}\hat{\times}|\hat{\psi}\rangle \rightarrow \hat{U} \times \hat{H}\hat{\times}|\hat{\psi}\rangle = \\ = (\hat{U} \times \hat{H} \times \hat{U}^\dagger)\hat{\times}(\hat{U}\hat{\times}|\hat{\psi}\rangle) = \hat{H}'\hat{\times}|\hat{\psi}'\rangle = \\ = \hat{U}\hat{\times}\hat{E}\hat{\times}|\hat{\psi}\rangle = \hat{E}\hat{\times}\hat{U}\hat{\times}|\hat{\psi}\rangle = \hat{E}\hat{\times}|\hat{\psi}'\rangle, \end{aligned} \tag{3.4.62b}$$

where one should note the preservation of the *numerical values* of the isounit, isoproducts and isoeigenvalues, as necessary for consistent applications.

Note that the invariance in quantum mechanics holds only for transformations $U \times U^\dagger = I$ *with fixed* I. Similarly, the invariance of isomechanics holds only for all nonunitary transforms such $\hat{U}\hat{\times}\hat{U}^{\hat{\dagger}} = \hat{I}$ *with fixed* $\hat{I}$, and *not* for a transform $\hat{W}\hat{\times}\hat{W}^{\hat{\dagger}} = \hat{I}' \neq \hat{I}$ because the change of the isounit $\hat{I}$ implies the transition to a *different physical system*.

The form-invariance of hadronic mechanics under isounitary transforms was first studied by Santilli in memoir [46].

3.5 ISORELATIVITIES AND THEIR ISODUAL

3.5.1 Space, the Final Frontier

As it is well known, we would not be able to hear each other voices without our atmosphere, because sound is a longitudinal wave requiring a compressible medium for its existence and propagation.

Similarly, we would not be able to see each other faces without the *aether* conceived as a universal medium with characteristics similar to very high rigidity, because light is a transversal wave that, as such, requires a rigid medium for its existence and propagation.

The reduction of light to photons in an attempt to avoid a universal medium has the sole value of political posturing in support of Einsteinian doctrines, since radio waves with one meter in wavelength do indeed propagate through empty interplanetary spaces, while their reduction to photons has no scientific credibility due to the excessive size of the wavelength.

The first paper written by the author back in 1956 [54] when a high school student was intended to stress the need for the universal medium with features similar to high rigidity due to the transversal character of light, and to indicate

that the so-called *aethereal wind* under discussion in the 1950s "cannot" exist (out of the vast literature on the aether, see Ref. [55] for an early account, and Refs. [56,57] for more recent treatments).

As it is well known, the electron is a pure oscillation with the well known frequency of 1.236×10^{20} Hz, thus requiring a universal medium for its very existence, since no ordinary mass or other material entity oscillates. We merely have a dimensionless point of the universal medium that oscillates at the given frequency.

Consequently, when the electron moves, it "cannot" experience any aethereal wind as claimed in earlier references [55] because we merely move its characteristic oscillation from one point of the aether to another.

The main argument of paper [54] is that exactly the same occurrence holds for all other elementary and composite particles. Consequently, the aether as a universal medium is necessary not only for the existence and propagation of electromagnetic waves, but also for the very existence of all elementary and composite particles and, therefore, of matter as perceived by our senses.

The main conclusion of paper [54] is that, contrary to our sensory perception, *matter is entirely empty and space is entirely full,* because mass can be entirely reduced to oscillations of the aether, and the inertia (from which we compute the mass) is in actuality a tendency of the aether to oppose variation of speed motion of said oscillations.

When initiating his academic life in the late 1960s, the author soon discovered that any mention of the aether as a universal medium would imply instant disqualification by the organized academic interests on Einsteinian doctrines in control of the world wide physics community, and, consequently, the author had to dedicate himself to other studies.

Nevertheless, *physical veritas* is not established by academic power, but by evidence. The need of a universal medium for the existence and propagation of electromagnetic waves, particles and all matter perceived by our senses is beyond any possible, credible or otherwise plausible doubt.

It then follows that, no matter how beloved and supported a theory is at a given time, no relativity will resist the test of time without the achievement of full compatibility with the aether as a universal medium and, of course, the unavoidable *privileged reference frame* at rest with respect to said universal medium.

At any rate, space is the true ultima frontier of the physics of the third millennium, in the sense that no other physical problem can compared to studies on the universal medium in view of potential discoveries simply beyond our imagination.

As an indication, if the aether has such a high rigidity to propagate a *transversal* wave at the speed of 2.997×10^{10} cm/sec, it is expected to propa-

gate *longitudinal* waves at such a speed to render interstellar communications virtually instantaneous.[24]

The fact that Maxwell's equations do not admit longitudinal solutions is, on scientific grounds outside academic politics, a confirmation of their limitations due to their known incompatibility with the privileged reference frame of a universal medium.

Alternatively, the dismissal of possible longitudinal waves in space because not predicted by Maxwell's equations is academic politics because in Chapter 11 we shall study new clean fuels in industrial sale whose production is based precisely on longitudinal forces, in this case within a material medium, not predicted by Maxwell's equations.

As we shall see in Chapter 13, the aether as a universal medium can well permit, at a time in the far future, the *isogeometric locomotion,* that is, a locomotion based on the alteration of the local geometry without any Newtonian action and reaction, which locomotion is predicted to permit arbitrary speeds without any conflict with Einsteinian doctrines.

Similarly, recent studies have indicated that the aether as a universal medium is composed of a superposition of positive and negative energies with a very high density of the order of 10^{25} erg/cm^3, whose coexistence has been rendered possible by the isodual theory because positive and negative energies exist in *different spacetimes.*

Consequently, the aether has recently emerged as being, by far, the biggest reservoir of energy available to mankind. The suppression of its study just to support any beloved theory would clearly be anti-scientific and anti-social.

In reality, when systematic studies are initiated on the aether as a universal medium, hadronic mechanics will emerge as *the first known mechanics for interplay between matter and aether.*

More specifically, studies presented in subsequent chapters show that *the isotopic realization of hidden variables can characterize the interplay between matter and space* with far reaching possibilities simply beyond our imagination at this time.

In view of all the above, our studies on relativities will be based on the fundamental assumption of the privileged reference frame of the universal medium necessary for the existence and propagation of electromagnetic waves, particles

[24]It is generally believed that there exist no intelligent life within a radius of 70 light years from Earth (the period of time since the initiation of our radio waves activity) because SETI has failed to detect any intelligent radio signal. in reality, it is possible that a civilization with millions of years of scientific evolution (as expected from the life of near-by star systems) looks at our communications with electromagnetic waves in the same way we look at the ancient communications via smoke signals in view of the fact that, despite its large value, the speed of light is dramatically insufficient for interstellar communication. Longitudinal waves propagating though the aether are then the only conceivable alternative on ground of our limited knowledge since they will propagate at a speed predicted to be a large multiple of that for the transversal electromagnetic waves.

and matter. All admitted relativities will be restricted to those compatible in one way or another with this fundamental assumption.

3.5.2 Lorentzian versus Galilean Relativities

Special relativity is generally presented in contemporary academia as providing a descriptions of all infinitely possible conditions existing in the universe.

The scientific reality is somewhat different than this academic posture. In fact, in Section 1.1 we have shown that special relativity *cannot* provide a consistent classical description of point antiparticles moving in vacuum because leading to inconsistent quantum images consisting of particles (rather than charge conjugated antiparticles) with the wrong sign of the charge, besides having no distinction for bodies with null total charge.

In Section 1.3 we have established that special relativity *cannot* be exactly valid for extended particles and antiparticles moving within physical media, because of an axiomatic inability to represent extended and nonspherical shapes, an incompatibility with the deformation theory, the lack of existence of a consistent reduction of classical nonconservative forces to potential abstractions at the particle level, and other reasons.

In Section 1.3 we have also established that special relativity *cannot* describe irreversible processes for both matter and antimatter due to unavoidable inconsistencies caused by its notoriously reversible axioms and other reasons.

In Section 1.3 we have also established that the complexity of biological systems is immensely beyond the rather limited descriptive capacity of special relativity.

When restricting the possible validity to the arena of its original conception (propagation of point particles and electromagnetic waves in vacuum), special relativity results to be incompatible with the privileged frame of the universal medium.

Consequently, **contrary to popular beliefs, special relativity has no uncontested arena at all in which it can be claimed on strict scientific grounds to be exactly valid.**

Nevertheless, special relativity is known to work well in particle accelerators up to very high speeds. Consequently, we shall indeed study special relativity in this section but only on grounds of its *pragmatic validity.*

We shall then seek conditions under which special relativity can be made compatible with a privileged reference frame. A logical line of study is *the assumption that the two-way speed of light used in all historical and more recent experiments is indeed an invariant leading to Lorentz transforms and special relativity, while the one way speed of light is strictly Galilean, thus leading to Galileo's relativity* [31,32].

In different terms, the alternative as to whether the ultimate relativity is of Lorentzian or Galilean type is far from being resolved. It is an easy prediction that such an alternative will not be resolved in this volume. Consequently, at the current state of our knowledge we are left with no other alternative other than the study of both relativities and leave the final selection to future generations.

In the following we shall first study the isotopies of special relativity in order to broader the conditions of its pragmatic applicability as much as possible, including the achievement of an invariant representation of arbitrary speeds of light, its extension to noninertial frames, the geometric unification of special and general relativities with consequential isotopic formulation of gravitation (used for the grand unification of Chapter 14), and related topics.

Subsequently, we shall first review a relativistic extension of Galileo's relativity, and then formulate its isotopies so as to broaden the arena of applicability also to locally varying speed of light within physical media, noninertial frames and other advances.

We shall finally address the problem of a conceivable mutual compatibility of these seemingly discordant relativities.

3.5.3 Iso-Minkowskian Spaces and Their Isoduals

As studied in Section 1.2, the "universal constancy of the speed of light" is a philosophical abstraction because experimental evidence establishes that *the speed of light is a local variable depending on the medium in which it propagates,* with well known expression

$$c = c_\circ / n, \tag{3.5.1}$$

where the familiar *index of refraction* n is a function of a variety of characteristics of the medium, including time t, coordinates r, density μ, temperature τ, frequency ω, etc., $n = n(t, r, \mu, \tau, \omega, \dots)$.

In particular, the speed of light is generally smaller than that in vacuum when propagating within media of low density, such as atmospheres or liquids,

$$c \ll c_\circ, \quad n \gg 1, \tag{3.5.2}$$

while the speed of light is generally bigger than that in vacuum when propagating within special guides, or within media of very high density, such as the interior of stars and quasars,

$$c \gg c_\circ, \quad n \ll 1 \tag{3.5.3}$$

Academic claims of recovering the speed of light in water via photons scattering among the water molecules are afflicted by numerous inconsistencies inconsistencies studied in Section 1.2, and the same hoolds for otehr aspects.

Assuming that via some unknown manipulation special relativity is shown to represent consistently the propagation of light within physical media, such a representation would activate the catastrophic inconsistencies of Theorem 1.5.1.

This is due to the fact that *the transition from the speed of light in vacuum to that within physical media requires a noncanonical or nonunitary transform.*

This point can be best illustrated by using the metric originally proposed by Minkowski, which can be written

$$\eta = Diag.(1, 1, 1, -c_\circ^2). \tag{3.5.4}$$

Then, the transition from $c_\circ$ to $c = c_\circ/n$ in the metric can only be achieved via a noncanonical or nonunitary transform

$$\eta = Diag.(1, 1, 1, -c_\circ^2) \rightarrow \hat{\eta} =$$

$$= Diag.(1, 1, 1, -c_\circ/n^2) = U \times \eta \times U^\dagger, \tag{3.5.5a}$$

$$U \times U^\dagger = Diag.(1,\ 1,\ 1,\ 1/n^2) \neq I. \tag{3.5.5b}$$

An invariant resolution of the limitations of special relativity for closed and reversible systems of extended and deformable particules under Hamiltonian and non-Hamiltonian interactions has been provided by the lifting of special relativity into a new formulation today known as *Santilli isorelativity,* or *isorelativity* for short, where: the prefix "iso" stands to indicate that relativity principles apply on isospacetime over isofields; and the characterization of "special" or "general" is inapplicable because, as shown below, *isorelativity achieves a geometric unification of special and general relativities.*

Irreversible systems of extended and deformable particles with Hamiltonian and non-Hamiltonian interactions are studied in the next chapter. They are also characterized by isorelativity chen closed via the use of a time-dependent hermitean isounit, and require the broader genorelativity when open.

Isorelativity was first proposed by R. M. Santilli in Ref. [58] of 1983 via the first invariant formulation of *iso-Minkowskian spaces* and related *iso-Lorentz symmetry.*

The studies were then continued in: Ref. [59] of 1985 with the first isotopies of the rotational symmetry; Ref. [49] of 1993 with the first isotopies of the SU(2)-spin symmetry; Ref. [60] of 1993 with the first isotopies of the Poincaré symmetry; Ref. [51] of 1998 with the first isotopies of the SU(2)-isospin symmetries, Bell's inequalities and local realism; and Refs. [61,62] on the first isotopies of the spinorial covering of the Poincaré symmetry.

The studies were then completed with memoir [26] of 1998 presenting a comprehensive formulation of the iso-Minkowskian geometry and its capability to unify the Minkowskian and Riemannian geometries, including its formulation

via the mathematics of the Riemannian geometry (such iso-Christoffel's symbols, isocovariant derivatives, etc.).

Numerous independent studies on isorelativity are available in the literature, one can inspect in this respect Refs. [32–43] and papers quoted therein; Aringazin's proof [63] of the direct universality of the Lorentz-Poincaré-Santilli isosymmetry for all infinitely possible spacetimes with signature (+, +, +, −); Mignani's exact representation [64] of the large difference in cosmological redshifts between quasars and galaxies when physically connected; the exact representation of the anomalous behavior of the meanlives of unstable particles with speed by Cardone et al. [65–66]; the exact representation of the experimental data on the Bose-Einstein correlation by Santilli [67] and Cardone and Mignani [68]; the invariant and exact validity of the iso-Minkowskian geometry within the hyperdense medium in the interior of hadrons by Arestov et al. [69]; the first known exact representation of molecular features by Santilli and Shillady [70,71]; and numerous other contributions.

Evidently we cannot review isorelativity in the necessary details to avoid a prohibitive length. Nevertheless, to achieve minimal self-sufficiency of this presentation, it is important to outline at least its main structural lines (see monograph [55] for detailed studies).

The central notion of isorelativity is the lifting of the basic unit of the Minkowski space and of the Poincaré symmetry, $I = Diag.(1, 1, 1, 1)$, into a 4×4-dimensional, nowhere singular and positive-definite matrix $\hat{I} = \hat{I}_{4\times 4}$ with an unrestricted functional dependence on local spacetime coordinates x, speeds v, accelerations a, frequencies ω, wavefunctions ψ, their derivative $\partial\psi$, and/or any other needed variable,

$$I = Diag.(1, 1, 1) \rightarrow \hat{I}(x, v, a, \omega, \psi, \partial\psi, \dots) =$$

$$= 1/\hat{T}(x, v, \omega, \psi, \partial\psi, \dots) > 0. \tag{3.5.6}$$

Isorelativity can then be constructed via the method of Section 3.4.6, namely, by assuming that the basic noncanonical or nonunitary transform coincides with the above isounit

$$U \times U^{\dagger} = \hat{I} = Diag.(g_{11}, g_{22}, g_{33}, g_{44}),$$

$$g_{\mu\mu} = g_{\mu\mu}(x, v, \omega, \psi, \partial\psi, \dots) > 0, \quad \mu = 1, 2, 3, 4, \tag{3.5.7}$$

and then subjecting the *totality* of quantities and their operation of special relativity to the above transform.

This construction is, however, selected here only for simplicity in pragmatic applications, since the rigorous approach is the construction of isorelativity from its abstract axioms, a task we have to leave to interested readers for brevity (see the original derivations [7]).

This is due to the fact that the former approach evidently preserves the original eigenvalue spectra and does not allow the identification of anomalous eigenvalues emerging from the second approach, such as those of the $SU(2)$ and $SU(3)$ isosymmetries [51].

Let $M(x, \eta, R)$ be the Minkowski space with local coordinates $x = (x^\mu)$, metric $\eta = Diag.(1, 1, 1, -1)$ and invariant

$$x^2 = (x^\mu \times \eta_{\mu\nu} \times x^\nu) \times I \in R. \tag{3.5.8}$$

The fundamental space of isorelativity is the *Minkowski-Santilli isospace* [58] and related topology [10,22–25], $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$ characterized by the liftings

$$I = Diag.(1, 1, 1, 1) \to U \times I \times U^\dagger = \hat{I} = 1/\hat{T}, \tag{3.5.9a}$$

$$\eta = Diag.(1, 1, 1, -1) \times I \to (U^{\dagger -1} \times \eta \times U^{-1}) \times \hat{I} = \hat{\eta} =$$
$$= \hat{T} \times \eta = Diag.(g_{11}, g_{22}, g_{33}, -g_{44}) \times \hat{I}, \tag{3.5.9b}$$

with consequential isotopy of the basic invariant

$$x^2 = (x^\mu \times \eta_{\mu\nu} \times x^\nu) \times I \in R \to$$
$$\to U \times x^2 \times U^\dagger = \hat{x}^{\hat{2}} = (\hat{x}^\mu \hat{\times} \hat{m}_{\mu\nu} \times x^\nu) \times I \in R, \tag{3.5.10}$$

whose projection in conventional spacetime can be written

$$\hat{x}^{\hat{2}} = [x^\mu \times \hat{\eta}_{\mu\nu}(x, v, a, \omega, \psi, \partial\psi, \dots) \times x^\nu] \times \hat{I}, \tag{3.5.11}$$

The nontriviality of the above lifting is illustrated by the following:

THEOREM 3.5.1: The Minkowski-Santilli isospaces are directly universal, in the sense of admitting as particular cases all possible spaces with the same signature $(+, +, +, -)$*, such as the Minkowskian, Riemannian, Finslerian and other spaces (universality), directly in terms of the isometric within fixed local variables (direct universality).*

Therefore, the correct formulation of the *the Minkowski-Santilli isogeometry* requires the isotopy of all tools of the Riemannian geometry, such as the iso-Christoffel symbols, isocovariant derivative, etc. (see for brevity Ref. [15]).

Despite that, one should keep in mind that, in view of the positive-definiteness of the isounit [34,79], *the Minkowski-Santilli isogeometry coincides at the abstract level with the conventional Minkowski geometry, thus having a null isocurvature* (because of the basic mechanism of deforming the metric η by the amount $\hat{T}(x, \dots)$ while deforming the basic unit of the inverse amount $\hat{I} = 1/\hat{T}$).

The geometric unification of the Minkowskian and Riemannian geometries achieved by the Minkowski-Santilli isogeometry constitute the evident geometric foundation for the unification of special and general relativities studied below.

It should be also noted that, following the publication in 1983 of Ref. [58], numerous papers on "deformed Minkowski spaces" have appeared in the physical and mathematical literature (generally without a quotation of their origination in Ref. [58]).

These "deformations" are ignored in these studies because they are formulated via conventional mathematics and, consequently, they all suffer of the catastrophic inconsistencies of Theorem 1.4.1.

By comparison, isospaces are formulated via isomathematics and, therefore, they resolve the inconsistencies of Theorem 1.5.1, as shown in Section 3.5.9. This illustrates again the necessity of lifting the basic unit and related field jointly with all remaining conventional mathematical methods.

3.5.4 Poincaré-Santilli Isosymmetry and its Isodual

Let $P(3.1)$ be the conventional Poincaré symmetry with the well known ten generators $J_{\mu\nu}, P_\mu$ and related commutation rules hereon assumed to be known.

The second basic tool of isorelativity is the *Poincaré-Santilli isosymmetry* $\hat{P}(3.1)$ studied in detail in monograph [55] that can be constructed via the isotheory of Section 3.2, resulting in the isocommutation rules [58,60]

$$[J_{\mu\nu}\hat{,}J_{\alpha\beta}] = i \times (\hat{\eta}_{\nu\alpha} \times J_{\beta\mu} - \hat{\eta}_{\nu\alpha} \times J_{\beta\nu} - \hat{\eta}_{nu\beta} \times J_{\alpha\mu} + \hat{\eta}_{\mu\beta} \times J_{\alpha\nu}), \quad (3.5.12a)$$

$$[J_{\mu\nu}\hat{,}P_\alpha] = i \times (\hat{\eta}_{\mu\alpha} \times P_\nu - \hat{\eta}_{\nu\alpha} \times P_\mu), \quad (3.5.12b)$$

$$[P_\mu\hat{,}P\nu] = 0, \quad (3.5.12c)$$

where we have followed the general rule of the Lie-Santilli isotheory according to which isotopies leave observables unchanged (since Hermiticity coincides with iso-Hermiticity) and merely change the *operations* among them.

The *iso-Casimir invariants* of $\hat{P}(3.1)$ are given by

$$P^{\hat{2}} = P_\mu \hat{\times} P^\mu = P^\mu \times \hat{\eta}_{\mu\nu} \times P^\nu = P_k \times g_{kk} \times P_k - p_4 \times g_{44} \times P_4, \quad (3.5.13a)$$

$$W^{\hat{2}} = W_\mu \hat{\times} W^\mu, \;\; W_\mu = \hat{\epsilon}_{\mu\alpha\beta\rho} \hat{\times} J^{\alpha\beta} \hat{\times} P^\rho. \quad (3.5.13b)$$

and they are at the foundation of classical and operator *isorelativistic kinematics.*

Since $\hat{I} > 0$, it is easy to prove that *the Poincaré-Santilli isosymmetry is isomorphic to the conventional symmetry.* It then follows that *the isotopies increase dramatically the arena of applicability of the Poincaré symmetry, from the sole Minkowskian spacetime to all infinitely possible spacetimes.*

Next, the reader should be aware that *the Poincaré-Santilli isosymmetry characterizes "isoparticles" (and not particles) via its irreducible isorepresentations.*

A mere inspection of the isounit shows that the Poincaré-Santilli isosymmetry characterizes actual nonspherical and deformable shapes as well as internal densities and the most general possible nonlinear, nonlocal and nonpotential interactions.

Since any interaction implies a renormalization of physical characteristics, it is evident that *the transition from particles to isoparticles, that is, from motion in vacuum to motion within physical media, causes an alteration (called isorenormalization), in general, of all intrinsic characteristics, such as rest energy, magnetic moment, charge, etc.*

As we shall see later on, the said isorenormalization has permitted the first exact numerical representation of nuclear magnetic moments, molecular binding energies and other data whose exact representation resulted to be impossible for nonrelativistic and relativistic quantum mechanics despite all possible corrections conducted over 75 years of attempts.

The explicit form of the *Poincaré-Santilli isotransforms* leaving invariant line element (3.5.11) can be easily constructed via the Lie-Santilli isotheory and are given:

(1) The **isorotations** [11]

$$\hat{O}(3) : \hat{\mathbf{x}}' = \hat{\Re}(\hat{\theta}) \hat{\times} \hat{\mathbf{x}}, \quad \hat{\theta} = \theta \times \hat{I}_\theta \in \hat{R}_\theta, \tag{3.5.14}$$

that, for isotransforms in the (1, 2)-isoplane, are given by

$$x^{1'} = x^1 \times \cos[\theta \times (g_{11} \times g_{22})^{1/2}] - x^2 \times g_{22} \times g_{11}^{-1} \times \sin[\theta \times (g_{11} \times g_{22})^{1/2}], \tag{3.5.15a}$$

$$x^{2'} = x^1 \times g_{11} \times g_{22}^{-1} \times \sin[\theta \times (g_{11} \times g_{22})^{1/2}] + x^2 \times \cos[\theta \times (g_{11} \times g_{22})^{1/2}]. \tag{3.5.15b}$$

For the general expression in three dimensions interested reader can inspect Ref. [7] for brevity.

Note that, since $\hat{O}(3)$ is isomorphic to $O(3)$, Ref. [59] proved, contrary to a popular belief throughout the 20-th century, that

LEMMA 3.5.1: The rotational symmetry remains exact for all possible signature-preserving $(+,+,+)$ *deformations of the sphere.*

The rotational symmetry was believed to be "broken" for ellipsoidal and other deformations of the sphere merely due to insufficient mathematics for the case considered because, when the appropriate mathematics is used, the rotational symmetry returns to be exact, and the same holds for virtually all "broken" symmetries.

The above reconstruction of the exact rotational symmetry can be geometrically visualized by the fact that *all possible signature-preserving deformations of the sphere are perfect spheres in isospace called isosphere.*

This is due to the fact that ellipsoidal deformations of the semiaxes of the perfect sphere are compensated on isospaces over isofields by the *inverse* deformation of the related unit

$$Radius \quad 1_k \to 1/n_k^2, \tag{3.5.16a}$$

$$Unit \quad 1_k \to n_k^2. \tag{3.5.16b}$$

We recover in this way the perfect sphere on isospaces over isofields

$$\hat{r}^{\hat{2}} = \hat{r}_1^2 + \hat{r}_2^2 + \hat{r}_3^2 \tag{3.5.17}$$

with exact $\hat{O}(3)$ symmetry, while its projection on the conventional Euclidean space is the ellipsoid

$$r_1^2/n_1^2 + r_2^2/n_2^2 + r_3^2/n_3^2, \tag{3.5.18}$$

with broken $O(3)$ symmetry.

(2) The **Lorentz-Santilli isotransforms** [26,29]

$$\hat{O}(3.1) : \hat{x}' = \hat{\Lambda}(\hat{v}, \dots)\hat{\times}\hat{x}, \ \hat{v} = v \times \hat{I}_v \in \hat{R}_v, \tag{3.5.19}$$

that, for isotransforms in the (3,4)-isoplane, can be written

$$x^{1'} = x^1, \tag{3.5.20a}$$

$$x^{2'} = x^2, \tag{3.5.20b}$$

$$\begin{aligned} x^{3'} &= x^3 \times \cosh[v \times (g_{33} \times g_{44})^{1/2}] - \\ &-x^4 \times g_{44} \times (g_{33} \times g_{44})^{-1/2} \times \sinh[v \times (g_{33} \times g_{44})^{1/2}] = \\ &= \hat{\gamma} \times (x^3 - \beta \times x^4), \end{aligned} \tag{3.5.20c}$$

$$\begin{aligned} x^{4'} &= -x^3 \times g_{33} \times (g_{33} \times g_{44})^{-1/2} \times \sinh[v(g_{33} \times g_{44})^{1/2}] + \\ &+x^4 \times \cosh[v \times (g_{33} \times g_{44})^{1/2}] = \\ &= \hat{\gamma} \times (x^4 - \hat{\beta} \times x^3), \end{aligned} \tag{3.5.20b}$$

where

$$\hat{\beta} = \frac{v_k \times g_{kk} \times v_k}{c_o \times g_{44} \times c_o}, \ \ \hat{\gamma} = \frac{1}{(1 - \hat{\beta}^2)^{1/2}}. \tag{3.5.21}$$

For the general expression interested readers can inspect Ref. [7].

Contrary to another popular belief throughout the 20-th century, Ref. [58] proved that

LEMMA 3.5.2: The Lorentz symmetry remains exact for all possible signature preserving $(+,+,+,-)$ *deformations of the Minkowski space.*

Again, the symmetry remains exact under the use of the appropriate mathematics.

The above reconstruction of the exact Lorentz symmetry can be geometrically visualized by noting that the light cone

$$x_2^2 + x_3^2 - c_o^2 \times t^2 = 0, \tag{3.5.22}$$

can only be formulated in vacuum, while within physical media we have the *light isocone*

$$\frac{r_2^2}{n_2^2} + \frac{r_3^2}{n_3^2} - \frac{c_o^2 \times t^2}{n^2(\omega, \dots)} = 0. \tag{3.5.23}$$

that, when formulated on isospaces over isofield, it is also a perfect cone, as it is the case for the isosphere. This property then explains how the Lorentz symmetry is reconstructed as exact according to Lemma 3.5.2 or, equivalently, that $\hat{O}(3.1)$ is isomorphic to $O(3.1)$.

(3) The **isotranslations** [29]

$$\hat{\mathcal{T}}(4) : \hat{x}' = \hat{\mathcal{T}}(\hat{a}, \dots)\hat{]} \times x = \hat{x} + \hat{A}(\hat{a}, x, \dots), \;\; \hat{a} = a \times \hat{I}_a \in \hat{R}_a, \tag{3.5.24}$$

that can be written

$$x^{\mu'} = x^\mu + A^\mu(a, \dots), \tag{3.5.25a}$$

$$A^\mu = a^\mu(g_{\mu\mu} + a^\alpha \times [g_{i\mu}\hat{,}P_\alpha]/1! + \dots), \tag{3.5.25b}$$

and there is no summation on the μ indices.

We reach in this way the following important result:

LEMMA 3.5.3 [55]: Isorelativity permits an axiomatically correct extention of relativity laws to noninertial frames.

In fact, noninertial frames are transformed into a forms that are inertial on isospaces over isofields, called *isoinertial,* as established by the fact that isotranslations (3.5.25) are manifestly nonlinear and, therefore, noninertial on conventional spaces while they are isolinar on isospaces, according to a process similar to the reconstruction of locality, lineary and canonicity.

The isoinertial character of the frames can also be seem from the isocommutativity of the linear momenta, Eqs. (3.5.12c), while such a commutativity is generally lost in the projection of Eqs. (3.5.12c) on ordinary spaces over ordinary fields, thus confirming the lifting of conventional noninertial frames into an isoinertial form.

This property illustrates again the origin of the name "isorelativity" to indicate that conventional relativity axioms are solely applicable in isospacetime.

(4) The novel **isotopic transformations** [60]

$$\hat{\mathcal{I}}(1) : \hat{x}' = \hat{w}^{-\hat{1}} \hat{\times} \hat{x} = w^{-1} \times \hat{x}, \ \hat{I}' = w^{-2} \times \hat{I}, \tag{3.5.26}$$

where w is a constant,

$$\hat{I} \rightarrow \hat{I}' = \hat{w}^{-2} \hat{\times} \hat{I} = w^{-2} \times \hat{I} = 1/\hat{T}', \tag{3.5.27a}$$

$$\hat{x}^{\hat{2}} = (x^\mu \times \hat{\eta}_{\mu\nu} \times x^\nu) \times \hat{I} \equiv \hat{x}'^{\hat{2}} =$$
$$= [x^\mu \times (w^2 \times \hat{\eta}_{\mu\nu}) \times x^\nu] \times (w^2 \times \hat{I}). \tag{3.5.27b}$$

Contrary to another popular belief throughout the 20-th century, we therefore have the following

THEOREM 3.5.2: The Poincaré-Santilli isosymmetry, hereon denoted with

$$\hat{P}(3.1) = \hat{O}(3.1) \hat{\times} \hat{\mathcal{T}}(4) \hat{\times} \hat{\mathcal{I}}(1), \tag{3.5.28}$$

and, therefore, the conventional Poincaré symmetry, are eleven dimensional.

The increase of dimensionality of the fundamental spacetime symmetry as, predictably, far reaching implications, including a basically novel and axiomatically consistent grand unification of electroweak and gravitational interactions studied in Chapter 5.

The simplest possible realization of the above formalism for isorelativistic kinematics can be outlined as follows. The first application of isorelativity is that of providing *an invariant description of locally varying speeds of light propagating within physical media.* For this purpose a realization of isorelativity requires the known;edge of the *density* of the medium in which motion occurs.

The simplest possible realization of the fourth component of the isometric is then given by the function

$$g_{44} = n_4^2(x, \omega, \dots), \tag{3.5.29}$$

normalized to the value $n_4 = 1$ for the vacuum (note that the density of the medium in which motion occurs *cannot* be described by special relativity). The above representation then follows with invariance under $\hat{P}(3.1)$.

In this case the quantities n_k, $k = 1, 2, 3$, represent the *inhomogeneity and anisotropy of the medium considered.* For instance, if the medium is

homogeneous and isotropic (such as water), all metric elements coincide, in which case

$$\hat{I} = Diag.(g_{11}, g_{22}, g_{33}, g_{44}) = n_4^2 \times Diag.(1, 1, 1, 1), \tag{3.5.30a}$$

$$\hat{x}^{\hat{2}} = \frac{x^2}{n_4^2} \times n_4^2 \times I \equiv x^2, \tag{3.5.30b}$$

thus confirming that *isotopies are hidden in the Minkowskian axioms,* and this may be a reason why they were not been discovered until recently.

Next, isorelativity has been constricted for the invariant description of systems of extended, nonspherical and deformable particles under Hamiltonian and non-Hamiltonian interactions.

Practical applications then require the knowledge of the actual shape of the particles considered, here assumed for simplicity as being spheroidal ellipsoids with semiaxes n_1^2, n_2^2, n_3^2.

Note that the minimum number of constituents of a closed non-Hamiltonian system is two. In this case we have shapes represented with $n_{\alpha k}$, $\alpha = 1, 2, , \ldots , n$.

Specific applications finally require the identification of the nonlocal interactions, e.g., whether occurring on an extended *surface* or *volume.* As an illustration, two spinning particles denoted 1 and 2 in condition of deep mutual penetration and overlapping of their wavepackets (as it is the case for valence bonds), can be described by the following Hamiltonian and total isounit

$$H = \frac{p_1 \times p_1}{2 \times m_1} + \frac{p_2 \times p_2}{2 \times m_2} + V(r), \tag{3.5.31a}$$

$$\hat{I}_{Tot} = Diag.(n_{11}^2, n_{12}^2, n_{13}^2, n_{14}^2) \times Diag.(n_{21}^2, n_{22}^2, n_{23}^2, n_{24}^2) \times \\ \times e^{N \times (\hat{\psi}_1/\psi_1 + \hat{\psi}_2/\psi_2) \times \int \hat{\psi}_{1\uparrow}(r)^\dagger \times \hat{\psi}_{2\downarrow}(r) \times dr^3}, \tag{3.5.31b}$$

where N is a positive constant.

The above realization of the isounit has permitted the first known *invariant and numerically exact* representation of the binding energy and other features of the hydrogen, water and other molecules [71,72] (see Chapter 9) for which a historical 2% has been missing for about one century. The above isounit has also been instrumental for a number of additional data on two-body systems whose representation had been impossible with quantum mechanics, such as the origin of the spin 1 of the ground state of the deuteron that, according to quantum axioms, should be zero.

Note in isounit (3.5.31) the nonlinearity in the wave functions, the nonlocal-integral character and the impossibility of representing any of the above features with a Hamiltonian.

¿From the above examples interested readers can then represent any other closed non-Hamiltonian systems.

3.5.5 Isorelativity and its Isodual

The third important part of the new isorelativity is given by the following isotopies of conventional relativistic axioms that, for the case of motion along the third axis, can be written [29] as follows [60]:

ISOAXIOM I. The projection in our spacetime of the maximal causal invariant speed is given by:

$$V_{Max} = c_\circ \times \frac{g_{44}^{1/2}}{g_{33}^{1/2}} = c_\circ \frac{n_3}{n_4} = c \times n_3. \tag{3.5.32}$$

This isoaxiom resolves the inconsistencies of special relativity recalled earlier for particles and electromagnetic waves propagating within physical media such as water.

In fact, water is homogeneous and isotropic, thus requiring that

$$g_{11} = g_{22} = g_{33} = g_{44} = 1/n^2, \tag{3.5.33}$$

where n is the index of refraction.

In this case the maximal causal speed for a massive particle is $c_\circ$ as experimentally established, e.g., for electrons, while the local speed of electromagnetic waves is $c = c_\circ/n$, as also experimentally established.

Note that such a resolution requires *the abandonment of the speed of light as the maximal causal speed for motion within physical media, and its replacement with the maximal causal speed of particles.*

It happens that in vacuum these two maximal causal speeds coincide. However, even in vacuum the correct maximal causal speed remains that of particles and *not* that of light, as generally believed.

At any rate, physical media are generally opaque to light but not to particles. Therefore, the assumption of the speed of light as the maximal causal speed within media in which light cannot propagate would be evidently vacuous.

It is an instructive exercise for interested readers to prove that

LEMMA 3.5.4: The maximal causal speed of particles on isominkowski space over an isofield remains $c_\circ$.

In fact, on isospaces over isofields $c_\circ^2$ is deformed by the index of refraction into the form $c_\circ^2/n_4^2$, but the corresponding unit cm^2/sec^2 is deformed by the inverse amount, $n_4^2 \times \text{cm}^2/\text{sec}^2$, thus preserving the numerical value $c_\circ^2$ due to the structure of the isoinvariant studied earlier.

The understanding of isorelativity requires the knowledge that, when formulated on the Minkowski-Santilli isospace over the isoreals, Isoaxiom I coincides

with the conventional axiom that is, the maximal causal speed returns to be c. The same happens for all remaining isoaxioms.

ISOAXIOM II. The projection in our spacetime of the isorelativistic addition of speeds within physical media is given by:

$$v_{Tot} = \frac{v_1 + v_2}{1 + \frac{v_1 \times g_{33} \times v_2}{c_\circ \times g_{44} \times c_\circ}} = \frac{v_1 + v_2}{1 + \frac{v_1 \times n_4^2 \times v_2}{c_\circ \times n_3^2 \times c_\circ}} \tag{3.5.34}$$

We have again the correct result that *the sum of two maximal causal speeds in water,*

$$V_{max} = c_\circ \times (n_3/n_4), \tag{3.5.35}$$

yields the maximal causal speed in water, as the reader is encouraged to verify.

Note that such a result is impossible for special relativity. Note also that *the "relativistic" sum of two speeds of lights in water,* $c = c_\circ/n$, *does not* yield the speed of light in water, thus confirming that the speed of light within physical media, assuming that they are transparent to light, is not the fundamental maximal causal speed.

ISOAXIOM III. The projection in our spacetime of the isorelativistic laws of dilation of time $t_\circ$ *and contraction of length* $\ell_\circ$ *and the variation of mass* $m_\circ$ *with speed are given respectively by:*

$$t = \hat{\gamma} \times t_\circ, \tag{3.5.36a}$$

$$\ell = \hat{\gamma}^{-1} \times \ell_\circ, \tag{3.5.36b}$$

$$m = \hat{\gamma} \times m_\circ. \tag{3.5.36c}$$

Note that in water these values coincide with the relativistic ones as it should be since particles such as the electrons have in water the maximal causal speed $c_\circ$.

Note again the necessity of avoiding the interpretation of the local speed of light as the maximal local causal speed. Note also that the mass diverges at the maximal local causal speed, but *not* at the local speed of light.

ISOAXIOM IV. The projection in our spacetime of the iso-Doppler law is given by the isolaw (here formulated for simplicity for $90°$ *angle of aberration):*

$$\omega = \hat{\gamma} \times \omega_\circ. \tag{3.5.37}$$

This isorelativistic axioms permits an *exact, numerical and invariant representation* of the large differences in cosmological redshifts between quasars and galaxies when physically connected.

In this case light simply exits the huge quasar chromospheres already redshifted due to the decrease of the speed of light, while the speed of the quasars can remain the *same* as that of the associated galaxy. Note again as this result is impossible for special relativity.

Isoaxiom IV also permits a numerical interpretation of the internal blue- and red-shift of quasars due to the dependence of the local speed of light on its frequency.

Finally, Isoaxiom IV predicts that *a component* of the predominance toward the red of sunlight at sunset is of iso-Doppler nature. This prediction is based on the different travel within atmosphere of light at sunset as compared to the zenith (evidently because of the travel within a comparatively denser atmosphere).

By contrast, the popular representation of the apparent redshift of sunlight at sunset is that via the scattering of light among the molecules composing our atmosphere. Had this interpretation be correct, the sky at the zenith should be red, while it is blue.

At any rate, the claim of representation of the apparent redshift via the scattering of light is political because of the impossibility of reaching the needed numerical value of the redshift, as serious scholars are suggested to verify.

ISOAXIOM V. The projection in our spacetime of the isorelativistic law of equivalence of mass and energy is given by:

$$E = m \times c_\circ^2 \times g_{44} = m_\times \frac{c_\circ^2}{n_4^2}. \tag{3.5.38}$$

Among various applications, *Isoaxiom V removes any need for the "missing mass" in the universe.* This is due to the fact that all isotopic fits of experimental data agree on values $g_{44} \gg 1$ within the hyperdense media in the interior of hadrons, nuclei and stars [7].

As a result, Isoaxiom V yields a value of the total energy of the universe dramatically bigger than that believed until now under the assumption of the universal validity of the speed of light in vacuum.

For other intriguing applications of Isoaxioms V, e.g., for the rest energy of hadronic constituents, we refer the interested reader to monographs [55,61].

The *isodual isorelativity* for the characterization of antimatter can be easily constructed via the isodual map of Chapter 2, and its explicit study is left to the interested reader for brevity.

3.5.6 Isorelativistic Hadronic Mechanics and its Isoduals

The isorelativistic extension of relativistic hadronic mechanics is readily permitted by the Poincaré-Santilli isosymmetry. In fact, iso-invariant (3.5.13a)

characterizes the following *iso-Gordon equation* on $\hat{\mathcal{H}}$ over $\hat{C}$ [55]

$$\hat{p}_\mu \hat{\times} |\hat{\psi}> = -\hat{i} \hat{\times} \hat{\partial}_\mu |\hat{\psi}> = -i \times \hat{I}^\nu_\mu \times \partial_\nu |\hat{\psi}>, \tag{3.5.39a}$$

$$(\hat{p}_\mu \hat{\times} \hat{p}^\mu + \hat{m}^2_\circ \hat{\times} \hat{c}^4) \hat{\times} |\hat{\psi}> = (\hat{\eta}^{\alpha\beta} \times \partial_\alpha \times \partial_\beta + m^2_\circ \times c^4) \times |\hat{\psi}> = 0. \tag{3.5.39b}$$

The linearization of the above second-order equations into the *Dirac-Santilli isoequation* has been first studied in Refs. [60–62] and then by other authors (although generally without the use of isomathematics, thus losing the invariance).

By recalling the structure of Dirac's equation as the Kronecker product of a spin 1/2 massive particle and its antiparticle of Chapter 2, the Dirac-Santilli isoequation is formulated on the total isoselfadjoint isospace and related isosymmetry

$$\hat{M}^{tot} = [\hat{M}^{orb}(\hat{x}, \hat{\eta}, \hat{R}) \times \hat{S}^{spin}(2)] \times$$

$$\times [\hat{M}^{d\,orb}(\hat{x}^d, \hat{\eta}^d, \hat{R}^d) \times \hat{S}^{d\,spin}(2)] = \hat{M}^{d\,tot}, \tag{3.5.40a}$$

$$\hat{S}^{tot} = \hat{P}(3.1) \times \hat{P}^d(3.1) = \hat{S}^{d\,tot}, \tag{3.5.40b}$$

and can be written [29]

$$[\hat{\gamma}^\mu \hat{\times} (\hat{p}_\mu - \hat{e} \hat{\times} \hat{A}_\mu) + \hat{i} \hat{\times} \hat{m}] \hat{\times} |\phi(x)> = 0, \tag{3.5.41a}$$

$$\hat{\gamma}^\mu = g^{\mu\mu} \times \gamma^\mu \times \hat{I} \tag{3.5.41b}$$

where the γ's are the conventional Dirac matrices.

Note the appearance of the isometric elements directly in the structure of the gamma matrices and their presence also when the equation is projected in the conventional spacetime.

The following generators

$$J_{\mu\nu} = (S_k,\, L_{k4}), P_\mu, \tag{3.5.42a}$$

$$S_k = (\hat{\epsilon}_{kij} \hat{\times} \hat{\gamma}_i \hat{\times} \hat{\gamma}_j)/2, \;\; L_{k4} = \hat{\gamma}_k \hat{\times} \hat{\gamma}_4/2, \;\; P_\mu = \hat{p}_\mu, \tag{3.5.42b}$$

characterize the *isospinorial covering of the Poincaré-Santilli isosymmetry.*

The notion of "isoparticle" can be best illustrated with the above realization because it implies that, *in the transition from motion in vacuum (as particles have been solely detected and studied until now) to motion within physical media, particles generally experience the alteration, called "mutation", of all intrinsic characteristics,* as illustrated by the following isoeigenvalues,

$$\hat{S}^{\hat{2}} \hat{\times} |\hat{\psi}> = \frac{g_{11} \times g_{22} + g_{22} \times g_{33} + g_{33} \times g_{11}}{4} \times |\hat{\psi}>, \tag{3.5.43a}$$

$$\hat{S}_3 \hat{\times} |\hat{\psi} >= \frac{(g_{11} \times g_{22})^{1/2}}{2} \times |\hat{\psi} > . \tag{3.5.43b}$$

The mutation of spin then characterizes a necessary mutation of the intrinsic magnetic moment given by [29]

$$\tilde{\mu} = \left(\frac{g_{33}}{g_{44}}\right)^{1/2} \times \mu, \tag{3.5.44}$$

where μ is the conventional magnetic moment for the same particle when in vacuum. The mutation of the rest energy and of the remaining characteristics has been identified before via the isoaxioms.

Note that the invariance under isorotations allows the rescaling of the radius of an isosphere. Therefore, for the case of the perfect sphere we can always have $g_{11} = g_{22} = g_{33} = g_{44}$ in which case the magnetic moment is not mutated. These results recover conventional classical knowledge according to which *the alteration of the shape of a charged and spinning body implies the necessary alteration of its magnetic moment.*

The construction of the isodual isorelativistic hadronic mechanics is left to the interested reader by keeping in mind that the iso-Dirac equation is isoselfdual as the conventional equation.

To properly understand the above results, one should keep in mind that *the mutation of the intrinsic characteristics of particles is solely referred to the constituents of a hadronic bound state under conditions of mutual penetration of their wave packets (such as one hadronic constituent) under the condition of recovering conventional characteristics for the hadronic bound state as a whole (the hadron considered),* much along Newtonian subsidiary constrains on non-Hamiltonian forces, Eqs. (3.1.6).

It should be also stressed that *the above indicated mutations violate the unitary condition when formulated on conventional Hilbert spaces, with consequential catastrophic inconsistencies, Theorem 1.5.2.*

As an illustration, the violation of causality and probability law has been established for all eigenvalues of the angular momentum M different than the quantum spectrum

$$M^2 \times |\psi >= \ell(\ell+1) \times |\psi >, \quad \ell = 0, 1, 2, 3, \ldots . \tag{3.5.45}$$

As a matter of fact, these inconsistencies are the very reason why the mutations of internal characteristics of particles for bound states at short distances could not be admitted within the framework of quantum mechanics.

By comparison, hadronic mechanics has been constructed to recover unitarity on iso-Hilbert spaces over isofields, thus permitting an invariant description of internal mutations of the characteristics of the constituents of hadronic bound states, while recovering conventional features for states as a whole.

Far from being mere mathematical curiosities, the above mutations permit basically new structure models of hadrons, nuclei and stars, with consequential, new clean energies and fuels (see Chapters 11, 12).

These new advances are prohibited by quantum mechanics precisely because of the preservation of the *intrinsic* characteristics of the constituents in the transition from bound states at large mutual distance, for which no mutation is possible, to the bound state of the same constituents in condition of mutual penetration, in which case mutations have to be admitted in order to avoid the replacement of a scientific process with unsubstantiated personal beliefs one way or the other (see Chapter 12 for details).

3.5.7 Isogravitation and its Isodual

As indicated in Section 1.4, there is no doubt that the classical and operator formulations of gravitation on a curved space have been the most controversial theory of the 20-th century because of an ever increasing plethora of problematic aspects remained vastly ignored. By contrast, as also reviewed in Section 1.4, special relativity in vacuum has a majestic axiomatic consistence is its *invariance* under the Poincaré symmetry.

Recent studies have shown that the formulation of gravitation on a curved space or, equivalently, the formulation of gravitation based on "covariance", is necessarily noncanonical at the classical level and nonunitary at the operator level, thus suffering of all catastrophic inconsistencies of Theorems 1.4.1 and 1.4.2.

These catastrophic inconsistencies can only be resolved via a new conception of gravity based on a *universal invariance,* rather than covariance.

Additional studies have identified profound axiomatic incompatibilities between gravitation on a curved space and electroweak interactions. These incompatibilities have resulted to be responsible for the lack of achievement of an axiomatically consistent grand unification since Einstein's times (see Chapter 14).

No knowledge of isotopies can be claimed without a knowledge that isorelativity has been constructed to resolve at least some of the controversies on gravitation. The fundamental requirement is *the abandonment of the formulation of gravity via curvature on a Riemannian space and its formulation instead on an iso-Minkowskian space* via the following steps characterizing *exterior isogravitation in vacuum,* first presented in Refs. [73,74]:

I) Factorization of any given Riemannian metric representing exterior gravitation $g^{ext}(x)$ into a nowhere singular and positive-definite 4×4-matrix $\hat{T}(x)$ times the Minkowski metric η,

$$g^{ext}(x) = \hat{T}^{ext}_{grav}(x) \times \eta; \tag{3.5.47}$$

II) Assumption of the inverse of $\hat{T}_{grav}$ as the fundamental unit of the theory,

$$\hat{I}^{ext}_{grav}(x) = 1/\hat{T}^{ext}_{grav}(x); \tag{3.5.48}$$

III) Submission of the totality of the Minkowski space and relative symmetries to the noncanonical/nonunitary transform

$$U(x) \times I^{\dagger}(x) = \hat{I}^{ext}_{grav}. \tag{3.5.49}$$

The above procedure yields the isominkowskian spaces and related geometry $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$, resulting in a new conception of gravitation, exterior isogravity, with the following main features [26]:

i) Isogravity is characterized by a universal *symmetry* (and not a covariance), the Poincaré-Santilli isosymmetry $\hat{P}(3.1)$ for the gravity of matter with isounit $\hat{I}^{ext}_{grav}(x)$, the isodual isosymmetry $\hat{P}^d(3.1)$ for the gravity of antimatter, and the isoselfdual symmetry $\hat{P}(3.1) \times \hat{P}^d(3.1)$ for the gravity of matter-antimatter systems;

ii) All conventional field equations, such as the Einstein-Hilbert and other field equations, can be formulated via the Minkowski-Santilli isogeometry since the latter preserves all the tools of the conventional Riemannian geometry, such as the Christoffel's symbols, covariant derivative, etc. [15];

iii) Isogravitation is isocanonical at the classical level and isounitarity at the operator level, thus resolving the catastrophic inconsistencies of Theorems 1.5.1 and 1.5.2;

iv) An axiomatically consistent operator version of gravity always existed and merely creeped in un-noticed through the 20-th century because gravity is imbedded where nobody looked for, in the *unit* of relativistic quantum mechanics, and it is given by isorelativistic hadronic mechanics outlined in the next section.

v) The basic feature permitting the above advances is the abandonment of curvature for the characterization of gravity (namely, curvature characterized by metric $g^{ext}(x)$ referred to the unit I) and its replacement with *isoflatness,* namely, the verification of the axioms of flatness in isospacetime, while preserving conventional curvature in its projection on conventional spacetime (or, equivalently, curvature characterized by the $g(x) = \hat{T}^{ext}_{grav}(x) \times \eta$ referred to the isounit $\hat{I}_{grav}(x)$ in which case curvature becomes null due to the inter-relation $\hat{I}^{ext}_{grav}(x) = 1/\hat{T}^{ext}_{grav}(x)$) [26].

A resolution of numerous controversies on classical formulations of gravity then follow from the above main features, such as:

a) The resolution of the century old controversy on the lack of existence of consistent total conservation laws for gravitation on a Riemannian space, which controversy is resolved under the universal $\hat{P}(3.1)$ symmetry by mere visual verification that the generators of the conventional and isotopic Poincaré

symmetry are the same (since they represent conserved quantities in the absence and in the preserve of gravity);

b) The controversy on the fact that gravity on a Riemannian space admits a well defined "Euclidean", but not "Minkowskian" limit, which controversy is trivially resolved by isogravity via the limit

$$\hat{I}^{ext}_{grav}(x) \rightarrow I; \tag{3.5.50}$$

c) The resolution of the controversy that Einstein's gravitation predicts a value of the bending of light that is twice the experimental value, one for curvature and one for newtonian attraction, which controversy is evidently resolved by the elimination of curvature as the origin of the bending, as necessary in any case for the free fall of a body along a straight radial line in which no curvature of any type is conceivably possible or credible; and other controversies.

A resolution of the controversies on quantum gravity can be seen from the property that relativistic hadronic mechanics of the preceding section *is* a quantum formulation of gravity whenever $\hat{T} = \hat{T}_{grav}$.

Such a form of operator gravity is as axiomatically consistent as conventional relativistic quantum mechanics because the two formulations coincide, by construction, at the abstract, realization-free level.

As an illustration, whenever

$$\hat{T}^{ext}_{grav} = Diag.(g^{ext}_{11}, g^{ext}_{22}, g^{ext}_{33}, g^{ext}_{44}), \quad g_{\mu\mu} > 0, \tag{3.5.51}$$

the Dirac-Santilli isoequation (3.5.41) provides a direct representation of the conventional electromagnetic interactions experienced by an electron, represented by the vector potential A_μ, plus gravitational interactions represented by the isogamma matrices.

Once curvature is abandoned in favor of the broader isoflatness, the axiomatic incompatibilities existing between gravity and electroweak interactions are resolved because:

i) isogravity possesses, at the abstract level, the *same* Poincaré invariance of electroweak interactions;

ii) isogravity can be formulated on the *same* flat isospace of electroweak theories; and

iii) isogravity admits positive and negative energies in the *same* way as it occurs for electroweak theories.

An axiomatically consistent *iso-grand-unification* then follows, as studied in Chapter 14.

Note that the above grand-unification requires the prior *geometric unification of the special and general relativities,* that is achieved precisely by isorelativity and its underlying iso-Minkowskian geometry.

In fact, special and general relativities are merely differentiated in isospecial relativity by the explicit realization of the unit. In particular, *black holes are now characterized by the zeros of the isounit* [7]

$$\hat{I}^{ext}_{grav}(x) = 0. \tag{3.5.52}$$

The above formulation recovers all conventional results on gravitational singularities, such as the singularities of the Schwarzschild's metric, since they are all described by the gravitational content $\hat{T}_{grav}(x)$ of $g(x) = \hat{T}_{grav}(x) \times \eta$, since η is flat.

This illustrates again that *all conventional results of gravitation, including experimental verifications, can be reformulated in invariant form via isorelativity.*

Moreover, the problematic aspects of general relativity mentioned earlier refer to the *exterior gravitational problem.* Perhaps greater problematic aspects exist in gravitation on a Riemannian space for *interior gravitational problems,* e.g., because of the lack of characterization of basic features, such as the density of the interior problem, the locally varying speed of light, etc.

These additional problematic aspects are also resolved by isorelativity due to the unrestricted character of the functional dependence of the isometric that, therefore, permits a direct geometrization of the density, local variation of the speed of light, etc.

The above lines constitute only the initial aspects of isogravitation since its most important branch is *interior isogravitation* as characterized by isounit and isotopic elements of the illustrative type

$$\hat{I}^{int}_{grav} = 1/\hat{T}^{int}_{grav} > 0, \tag{3.5.53a}$$

$$\hat{T}^{int}_{grav} = Diag.(g^{int}_{11}/n_1^2, g^{int}_{22}/n_2^2, g^{int}_{33}/n_3^2, g^{int}_{44}/n_4^2), \tag{3.5.53b}$$

permitting a it geometric representation directly via the isometric of the actual shape of the body considered, in the above case an ellipsoid with semiaxes n_1^2, n_2^2, n_3^2, as well as the (average) interior density n_4^2 with consequential representation of the (average value of the) interior speed of light $C = c/n_4$.

A most important point is that the invariance of interior isogravitation under the Poincaré-Santilli isosymmetry persists in its totality since the latter symmetry is completely independent from the explicit value of the isounit or isotopic element, and solely depends on their positive-definite character.

Needless to say, isounit (3.4.53) is merely illustrative because a more accurate interior isounit has a much more complex functional dependence with a locally varying density, light speed and other characteristics as they occur in reality.

Explicit forms of these more adequate models depends on the astrophysical body considered, e.g., whether gaseous, solid or a mixture of both, and their study is left to the interested reader.

It should also be noted that *gravitational singularities should be solely referred to interior models* evidently because exterior descriptions of type (3.5.52) are a mere approximation or a geometric abstraction.

In fact, *a gravitational singularities existing for exterior models are not necessarily confirmed by the corresponding interior formulations.* Consequently, the current views on black holes could well result to be pseudo-scientific beliefs because the only scientific statement that can be proffered at this time without raising issue of scientific ethics is that *the gravitational features of large and hyperdense aggregations of matter, whether characterizing a "black" or "brown" hole, are basically unresolved at this time.*

Needless to say, exterior isogravitation is a particular case of the interior formulation. Consequently, from now on, unless otherwise specified isogravitation will be referred to the interior form.

The cosmological implications are also intriguing and will be studied in Chapter 5.

It is hoped that readers with young minds of any age admit the incontrovertible character of the limitations of special and general relativities and participate in the laborious efforts toward new vistas because any lack of participation in new frontiers of science, whether for personal academic interest or other reason, is a gift of scientific priorities to others.

Appendix 3.A
Universal Enveloping Isoassociative Algebras

The main structural component of Lie's theory is its *universal enveloping associative algebra* $\xi(L)$ of a Lie algebra L. In fact, Lie algebras can be obtained as the attached antisymmetric part $[\xi(L)]^- \approx L$; the infinite dimensional basis of $\xi(l)$ permit the exponentiation to a finite transformation group G; and the representation theory is crucially dependence on the right and/or left modular associative action originally defined on G.

In Section 3.2.9B we have reviewed the rudiments of the *universal enveloping isoassociative algebras* $\hat{\xi}(L)$ of a Lie-Santilli isoalgebra $\hat{L}$. It is easy to see that all features occurring for $\xi(L)$ carry over to the covering isoform $\hat{\xi}(L)$.[25]

In this appendix we would like to outline a more technical definition of universal enveloping isoassociative algebras since they are at the foundations of the unification of simple Lie algebras of dimension N into a single Lie-Santilli isoalgebra of the same dimension (Section 3.2.13).

With reference to Figure 3.A.1, the envelop $\xi(L)$ can be defined as the (ξ, τ) where ξ is an associative algebra and τ is a homomorphism of L into the antisymmetric algebra ξ^- attached to ξ such that: if ξ' is another associative algebra and τ' is another homomorphism of L into ξ'^- a unique isomorphism γ exists between ξ and ξ' exists in such a way that the diagram in the l.h.s of Figure 3.A.1 is commutative. The above definition evidently expresses the uniqueness of the Lie algebra L up to local isomorphism, and illustrates the origin of the name "universal" enveloping algebra of L.

With reference to the r.h.s. diagram of Figure 3.A.1, the universal enveloping isoassociative algebra $\hat{\xi}(L)$ of a Lie algebra L was introduced in Ref. [4] as the set $\{(\xi, \tau), i, \hat{\xi}, \hat{\tau}\}$ where: (ξ, τ) is a conventional envelope of L; i is an isotopic mapping $L \to i(L) = \hat{L} \sim L$; $\hat{\xi}$ is an associative algebra generally non-isomorphic to ξ; $\hat{\tau}$ is a homomorphism of $\hat{L}$ into $\hat{\xi}^-$; such that: if $\hat{\xi}'$ is another associative algebra and $\hat{\tau}'$ is another homomorphism of $\hat{L}$ into $\hat{\xi}'^-$, there exists

[25] We use the denomination $\hat{\xi}(L)$ rather than $\hat{\xi}(\hat{L})$ to stress the fact that the generators of ξ are those of L and not of $\hat{L}$, a requirement that is essential for consistent physical applications because the generators of L represent ordinary physical quantities (such as total energy, total linear momentum, etc.) that, as such, cannot be changed by isotopies.

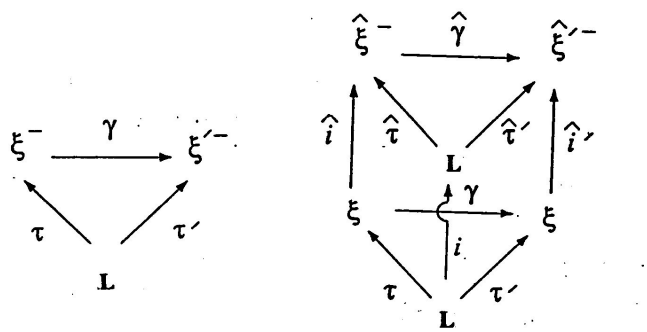

Figure 3.A.1. A schematic view of the universal enveloping associative algebra of a Lie algebra L and its lifting for the Lie-Santilli isoalgebra $\hat{L}$ according to the original proposal [4] of 1978.

a unique isomorphism $\hat{\gamma}$ of $\hat{\xi}$ into $\hat{\xi}'$ with $\hat{\tau}' = \gamma(\hat{\tau})$ and two unique isotopies $i(\xi) = \hat{\xi}$ and $i(\hat{\xi}) = \hat{\xi}'$.

A primary objective of the above definition of isoenvelope is the *lack of uniqueness of the Lie algebra characterized by the isoenvelope* or, equivalently, the *characterization of a family of generally nonisomorphic Lie algebras via the use of only one basis.* The above definition of isoenvelope also explains in more details the variety of realization of the simple 3-dimensional Lie-Santilli isoalgebra $\hat{L}_3$ provided in Eq. (3.2.236), and may be of assistance in extending the same classification to other isoalgebras.

The above notion of isoenvelope represents the essential mathematical structure of hadronic mechanics, namely, the preservation of the conventional basis, i.e., the set of observables of quantum mechanics, and the generalization of the operations on them via an infinite number of isotopies so as to admit a new class of interactions structurally beyond the possibilities of quantum mechanics.

Appendix 3.B
Recent Advances in Isotopology

In Section 3.2.7 we introduced the elements of the *Tsagas-Sourlas-Santilli-Falcón-Núñez isotopology* (or TSSFN Isotopology for short). In this appendix we ourline recent advances on the isotopology by the Spanish mathematicians R. M. Falcón Ganfornina and J. Núñez Valdés [24,25].

PROPOSITION 3.2.B1: Consider a mathematical structure

$$(E, +, \times, \circ, \bullet, \dots),$$

if we construct an isotopic lifting such that:

a) Both primaries $*$, $\widehat{I}$ *and secondaries* $\star$, $\widehat{S}$ *isotopic elements are used.*

b) $(E, \star, *, \dots)$ *is a structure of the same type as the initial, which is endowed with isounits* S, I, $\dots$, *with respect to* $\star$, $*$, $\dots$, *respectively.*

c) I *is an unit with respect to* $*$ *in the corresponding general set* V, *being* $T = \widehat{I}^{-I} \in V$ *the associated isotopic element.*

Then, by defining in the isotopic level the operations:

$$\widehat{a}\widehat{+}\widehat{b} = \widehat{a \star b}; \quad \widehat{a}\widehat{\times}\widehat{b} = \widehat{a * b}; \quad \dots \tag{3.B.1}$$

And being defined in the projection level:

$$\overline{\widehat{a}} = a * \widehat{I}; \quad \alpha\overline{\widehat{+}}\beta = ((\alpha * T) \star (\beta * T)) * \widehat{I}; \quad \alpha\overline{\widehat{\times}}\beta = \alpha * T * \beta; \ \dots \tag{3.B.2}$$

It is obtained that the isostructure $(\overline{\widehat{E}}, \overline{\widehat{+}}, \overline{\widehat{\times}}, \dots)$ *is of the same type as the initial one.*

The study in Refs. [24,25] is made by taking into consideration both isotopic and projection levels. Equivalent results related to injective isotopies are also obtained. In the first place, Proposition 3.2.A1 is verified for topological spaces and for their elements and basic properties: isotopologies, isoclosed sets, isoopen sets, T_2, etc:

A *topological isospace* is every isospace endowed with a topological space structure. If, besides, such an isospace is an isotopic projection of a topological space, it is called *isotopological isospace*.

Similarly, they are defined concepts of *(iso)boundary isopoint, closure of a set, closed set, isointerior isopoint, interior of a set, open set, (iso)Hausdorff isospace and second countable isospace, among others.*

PROPOSITION 3.2.B2: The space from which any topological isospace in the isotopic level is obtained can be endowed with the final topology relative to the mapping $\mathbf{I}$.

The isotopic projection of a topological space is an isotopological isospace in the projection level. If such a projection is injective, then every topological isospace in such a level is, in fact, isotopological.

Similar results are obtained for the concepts of (iso)boundary isopoint, isointerior isopoint and (iso)Hausdorff isospace.

Next, Refs. [24,25] generalize Kadeisvili's isocontinuity [19]. Particularly, the basic isofield can be endowed with an isoorder, according to the following procedure.

Let $\widehat{K}$ be an isofield associated with a field K, endowed with an order $\leq$, by using an isotopology which preserves the inverse element with respect to the addition. We define the *isoorder* $\widehat{\leq}$ as $\widehat{a}\widehat{\leq}\widehat{b}$ if and only if $a \leq b$. If the isotopy is injective, the isoorder $\overline{\widehat{\leq}}$ en $\overline{\widehat{K}}$ is defined in the same way.

PROPOSITION 3.2.B3: The isoorders $\widehat{\leq}$ *and* $\overline{\widehat{\leq}}$ *are orders over* $\widehat{K}$ *and* $\overline{\widehat{K}}$*, of the same type as* $\leq$.

Let $\widehat{U}$ be a $\widehat{R}$ isovectorspace with isonorm $\widehat{||}.\widehat{||} \equiv \widehat{||.||}$ and isoorder $\widehat{\leq}$, obtained from an isotopy compatible with respect to each one of the initial operations. It will be said that an isoreal isofunction $\widehat{f}$ of $\widehat{U}$ is *isocontinuous in* $\widehat{X} \in \widehat{U}$, if for all $\widehat{\epsilon}\widehat{>}\widehat{S}$, there exists $\widehat{\delta}\widehat{>}\widehat{S}$ such that for all $\widehat{Y} \in \widehat{U}$ with $\widehat{||}\widehat{X} - \widehat{Y}\widehat{||}\widehat{<}\widehat{\delta}$, it is verified that $\widehat{|}\widehat{f}(\widehat{X}) - \widehat{f}(\widehat{Y})\widehat{|}\widehat{<}\widehat{\epsilon}$. We will say that $\widehat{f}$ is *isocontinuous in* $\widehat{U}$ if it is isocontinuous in $\widehat{X}$, for all $\widehat{X} \in \widehat{U}$. Finally, when dealing with injective isotopies, the isocontinuity in the projection level is defined in a similar way.

PROPOSITION 3.2.B4: The isocontinuity in $\widehat{U}$ *is equivalent to the continuity in* U*. In the case of injective isotopies, both ones are equivalent to the one in* $\overline{\widehat{U}}$.

The isocontinuity on isotopological isospaces is also analyzed:

An *isocontinuous isomapping* in the isotopic level between two topological isospaces $\widehat{M}$ and $\widehat{N}$ is every isomapping $\widehat{f} : \widehat{M} \to \widehat{N}$ preserving closures. The definition in the projection level is given in a similar way.

PROPOSITION 3.2.B5: They are verified that:

a) $\widehat{f}$ is isocontinuous if and only if the mapping f from which comes from is continuous. That result is similar in the projection level by using injective isotopies.

b) Every isoconstant isomapping is isocontinuous.

c) Isocontinuity is preserved by both topological composition and product.

Finally, the analysis of (iso)(pseudo)metric isospaces is also concreted:

PROPOSITION 3.2.B6: Let $\widehat{M}$ be a $\widehat{K}$ isovectorspace, isotopic lifting of a vectorspace M, endowed with a (pseudo)metric d y defined on an ordered field K, by using an isotopy which preserves the inverse element and compatible with respect to the addition in K. Then, the isofunction $\widehat{d}$ is an iso(pseudo)metric.

Let $(\widehat{M}, d')$ be an (iso)(pseudo)metric $\widehat{K}$ isovectorspace, endowed with an isoorder $\widehat{\leq}$. $B_{d'}(\widehat{X_0}, \widehat{\epsilon}) = \{\widehat{X} \in \widehat{M} : d'(\widehat{X}, \widehat{X_0}) \widehat{<} \widehat{\epsilon}\}$ is called *metric ball* with center $\widehat{X_0} \in \widehat{M}$ and radius $\widehat{\epsilon} \widehat{>} \widehat{S}$. If M is endowed with a (pseudo)metric d, with $\widehat{d} = d'$, then every metric ball $B_{d'} = B_{\widehat{d}} = \widehat{B_d}$ in $\widehat{M}$, which is isotopic lifting of a metric ball B_d in M, is called *metric isoball* in $\widehat{M}$.

PROPOSITION 3.2.B7: Under conditions of Proposition XXX, if $B_d(X_0, \epsilon)$ is a metric ball in M, then $B_d(\widehat{X_0}, \epsilon) = B_{\widehat{d}}(\widehat{X_0}, \widehat{\epsilon})$ is a metric ball in $\widehat{M}$.

A *metric neighborhood of an isopoint* $\widehat{X} \in \widehat{M}$ is a subset $\widehat{A} \subseteq \widehat{M}$ containing a metric ball centered in $\widehat{X}$. The set of metric neighborhoods of $\widehat{X}$ is denoted by $\widehat{\aleph}^{d'}_{\widehat{X}}$. Finally, if d' is the isoEuclidean isodistance over $\widehat{R}^n$, the associated metric neighborhoods are called *isoEuclidean neighborhoods.*

PROPOSITION 3.2.B8: Let d' and d'' two (iso)(pseudo)metrics over an isovectorspace $\widehat{M}$. It is verified that $\widehat{\aleph}^{d'}_{\widehat{X}} = \widehat{\aleph}^{d''}_{\widehat{X}}$ if and only if every metric ball $B_{d'}(\widehat{X}, \widehat{\epsilon})$ contains a ball $B_{d''}(\widehat{X}, \widehat{\rho})$ and every ball $B_{d''}(\widehat{X}, \widehat{\delta})$ contains a ball $B_{d'}(\widehat{X}, \widehat{\mu})$.

PROPOSITION 3.2.B9: Every isospace endowed with an (iso)(pseudo)metric is an isotopological isospace.

PROPOSITION 3.2.B10: Let $\widehat{f} : (\widehat{M}, d') \to (\widehat{N}, d'')$ *be an isomapping between* $\widehat{K}$*-isospaces endowed with (iso)(pseudo)metric and let us consider* $\widehat{X} \in \widehat{M}$*. Then,* $\widehat{f}$ *is isocontinuous in* $\widehat{X}$ *if and only if for all* $\widehat{\epsilon} \widehat{>} \widehat{S}$ *there exists* $\widehat{\delta} \in \widehat{K}$ *such that* $\widehat{\delta} \widehat{>} \widehat{S}$*, and if* $\widehat{Y} \in B_{d'}(\widehat{X}, \widehat{\delta})$*, then it is verified that* $\widehat{f}(\widehat{Y}) \in B_{d''}(\widehat{f}(\widehat{X}), \widehat{\epsilon})$*.*

PROPOSITION 3.2.B11: Let $\widehat{f} : \widehat{M} \to \widehat{N}$ *be an isomapping between two isotopological isospaces* $\widehat{M}$ *and* $\widehat{N}$*. If conditions of the definition of isocontinuity are satisfied, then* $\widehat{f}$ *is isocontinuous if and only if* $\widehat{f}^{-1}(\widehat{U})$ *is an isoopen of* $\widehat{M}$*, for all isoopen* $\widehat{U}$ *of* $\widehat{N}$*.*

Appendix 3.C
Recent Advances on the Lie-Santilli Isotheory

In Section 3.2.9 we have outlined the rudiments of the *Lie-Santilli isotheory.* It may be useful for the mathematically oriented reader to outline recent developments achieved by by the Spanish mathematicians R. M. Falcón Ganfornina and J. Núñez Valdés [24,25,43] in the field beyond those presented in monographs [2,6,36,37].

Falcón and Núñez introduced in 2001 [37] a new construction model of isotopies which was similar to the one proposed by Santilli in 1978 although in its multivalued version presented by the same author later on [6] (see Chapter 4) because based on the use of several isolaws and isounits as operations existing in the initial mathematical structure. Such a model, which from now on will be called MCIM (*isoproduct construction model based on the multiplication*), was later generalized in Refs. [24,25,43]. In a schematic way, Santilli's isotopies can be described with the following diagram:

$$\begin{array}{ccc}
 & & \text{General Level} \\
\text{Conventional Level} & \longrightarrow & (V, *, \star, \dots) \\
 & & \cup \\
(E, +, \times, \dots) & & (E, \star, *, \dots) \\
\downarrow & \sharp & \downarrow \mathbf{I} \\
\text{Projection Level} & \xleftarrow{\quad\pi\quad} & \text{Isotopic level} \\
(\overline{\widehat{E}}, \overline{\widehat{+}}, \overline{\widehat{\times}}, \dots) & & (\widehat{E}, \widehat{+}, \widehat{\times}, \dots)
\end{array}$$

where, by construction:

a) The mapping $\mathbf{I} : (E, \star, *, \dots) \to (\widehat{E}, \widehat{+}, \widehat{\times}, \dots) : X \to \widehat{X}$ is an isomorphism.

b) The *isotopic projection* is onto:
$\pi : (\widehat{E}, \widehat{+}, \widehat{\times}, \dots) \to (\overline{\widehat{E}}, \overline{\widehat{+}}, \overline{\widehat{\times}}, \dots) : \widehat{a} \to \pi(\widehat{a}) = \overline{\widehat{a}} = a * \widehat{I}.$

c) $\widehat{a}\widehat{+}\widehat{b} = \widehat{a \star b}; \quad \widehat{a}\widehat{\times}\widehat{b} = \widehat{a * b}; \dots.$

d) $\overline{\widehat{a}} = a * \widehat{I}; \quad \alpha\overline{\widehat{+}}\beta = ((\alpha * T) \star (\beta * T)) * \widehat{I}; \quad \alpha\overline{\widehat{\times}}\beta = \alpha * T * \beta; \dots$

PROPOSITION 3.2.C1: The following properties are verified:

a) The isotopic projection associated with each injective isotopic lifting is an isomorphism.

b) If the isotopic lifting used is compatible with respect to all of initial operations, then the isostructure $\widehat{E}$ is isomorphic to the initial E.

c) The relation of being isotopically equivalents is of equivalence.

d) Every isotopy $\pi \circ \mathbf{I} : (E, +, \times, \circ, \bullet, \dots) \to (\overline{\widehat{E}}, \overline{\widehat{+}}, \overline{\widehat{\times}}, \overline{\widehat{\circ}}, \overline{\widehat{\bullet}}, \dots)$ can be considered as an isotopic lifting which follows the MCIM, that is, every mathematical isostructure is an isostructure with respect to the multiplication.

Then, it has a perfect sense to considerer each one of the isostructures which result of applying the MCIM to conventional structures. Particularly, we can consider the construction of *Santilli's isoalgebras* (as the isotopic lifting of each algebra, which is endowed with a structure of algebra).

PROPOSITION 3.2.C2: Let U be a K-algebra and let $\widehat{U}$ be a $\widehat{K}$-isovectorspace. If a $K(a, \star, *)$-algebra $(U, \diamond, \Box, \cdot)$ is used in the general level, then the isotopic lifting $\widehat{U}$ corresponding to the isotopy of primary elements $\widehat{I}$ and $\Box$ and secondary ones $\widehat{S}$ and $\diamond$, when MCIM is used, has a structure of isoalgebra on $\widehat{K}$, and it preserves the initial type of the algebra.

A particular type of isoalgebra is the *Lie-Santilli isoalgebra* [4]. Particularly, if $\overline{\widehat{U}}$ is the isotopic projection of a Lie-Santilli isoalgebra,

$$\widehat{I} = \widehat{I}(x, dx, d^2x, t, T, \mu, \tau, \dots)$$

is an isounit and a basis $\overline{\widehat{U}}$, $\{\overline{\widehat{e_1}}, \dots, \overline{\widehat{e_n}}\}$ is fixed, where $\overline{\widehat{e_i}} \,\widehat{\cdot}\, \overline{\widehat{e_j}} = \sum \overline{\widehat{c_{ij}^h}} \,\overline{\widehat{\bullet}}\, \overline{\widehat{e_h}}$, $\forall\ 1 \leq i, j \leq n$, then coefficients $\overline{\widehat{c_{ij}^h}} \in \overline{\widehat{K}}$ are the *Maurer-Cartan coefficient* of the isoalgebra, which constitute a generalization of the conventional case, since they are not constants in general, but functions dependent of the factors of $\widehat{I}$.

Another interesting isoalgebra is the *Santilli's Lie-admissible algebra* [4], that is, the isoalgebra $\widehat{U}$ such that with the commutator bracket $[.,.]_{\widehat{U}}$: $[\widehat{X}, \widehat{Y}]_{\widehat{U}} = (\widehat{X} \widehat{\cdot} \widehat{Y}) - (\widehat{Y} \widehat{\cdot} \widehat{X})$ is an isotopic Lie isoalgebra. The following result is satisfied:

PROPOSITION 3.2.C3: Under conditions of Proposition XXX, let us suppose that the law $\widehat{\circ}$ of the isoalgebra $\widehat{U}$ is defined according $\widehat{X} \widehat{\circ} \widehat{Y} = (X \circ Y) \Box \widehat{I}$,

for all $X, Y \in U$. If U is a Lie (admissible) algebra, then $\widehat{U}$ is a Lie isoadmissible isoalgebra.

In this way, Santilli's Lie-admissible isoalgebras inherit the usual properties of conventional (admissible) Lie algebras. In the same way, usual structures related with such algebras have also their analogue ones when isotopies are used.

For instance, an *isoideal* of a Lie isoalgebra $\widehat{U}$ is every isotopic lifting of an ideal $\Im$ of U, which is by itself an ideal. In particular, the *center* of an Lie isoalgebra $\widehat{U}$, $\left\{\widehat{X} \in \widehat{U} \text{ such that } \widehat{X}\widehat{\cdot}\widehat{Y} = \widehat{S},\ \forall \widehat{Y} \in \widehat{U}\right\}$, is an isoideal of $\widehat{U}$. In fact, it is verified the following result:

PROPOSITION 3.2.C4: Let $\widehat{U}$ be an Lie isoalgebra associated with a Lie algebra U and let $\Im$ be an ideal of U. Then, the corresponding isotopic lifting $\widehat{\Im}$ is an isoideal of $\widehat{U}$.

An isoideal $\widehat{\Im}$ of a Lie isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$, is called *isocommutative* if $\widehat{X}\widehat{\cdot}\widehat{Y} = \widehat{S}$, for all $\widehat{X} \in \widehat{\Im}$ and for all $\widehat{Y} \in \widehat{U}$, being $\widehat{U}$ *isocommutative* if it is so as an isoideal.

PROPOSITION 3.2.C5: $\widehat{U}$ is isocommutative if and only if U is commutative.

Lie-Santilli isoalgebras can also be introduced as fopllows. Given an $\widehat{K}$-isoassociative isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$, the commutator in $\widehat{U}$ associated with $\widehat{\cdot}$: $[\widehat{X}, \widehat{Y}]_S = (\widehat{X}\widehat{\cdot}\widehat{Y}) - (\widehat{Y}\widehat{\cdot}\widehat{X})$, for all $\widehat{X}, \widehat{Y} \in \widehat{U}$ is denominated *Lie-Santilli bracket product* $[.,.]_S$ with respect to $\widehat{\cdot}$. The isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, [.,.]_S)$ is then denominated *Lie-Santilli algebra*.

DEFINITION 3.2.C6: Let $\widehat{U}$ be an $\widehat{K}$-isoassociative isoalgebra associated with a K-algebra U, under conditions of Proposition XXX. Then, the Lie-Santilli algebra associated with $\widehat{U}$ is an Lie isoalgebra if the algebra U is either associative or Lie admissible.

Apart from that, a Lie-Santilli isoalgebra $\widehat{U}$ is said to be *isosimple* if, being an isotopy of a simple Lie algebra, it is not isocommutative and the only isoideals which contains are trivial. In an analogous way, $\widehat{U}$ is called *isosemisimple* if, being an isotopy of a semisimple Lie algebra, it does not contain non trivial isocommutative isoideals. Note that, this definition involves that every isosemisimple Lie isoalgebra is also isosimple. Moreover, it is verified:

PROPOSITION 3.2.C7: Under conditions of Proposition XXX, the isotopic lifting of a (semi)simple Lie algebra is an iso(semi)simple Lie isoalgebra. Particularly, every isosemisimple Lie isoalgebra is a direct sum of isosimple Lie isoalgebras.

A lie-Santilli isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ is said to be *isosolvable* if, being an isotopy of a solvable Lie algebra, in the *isosolvability series*

$$\widehat{U}_1 = \widehat{U}, \quad \widehat{U}_2 = \widehat{U}\widehat{\cdot}\widehat{U}, \quad \widehat{U}_3 = \widehat{U}_2\widehat{\cdot}\widehat{U}_2, \ \ldots \ , \widehat{U}_i = \widehat{U}_{i-1}\widehat{\cdot}\widehat{U}_{i-1}, \ldots$$

there exists a natural integer n such that $\widehat{U}_n = \{\widehat{S}\}$. The minor of such integers is called *isosolvability index* of the isoalgebra.

PROPOSITION 3.2.C8: Under conditions of Proposition XXX, the isotopic lifting of a solvable Lie algebra is an isosolvable Lie isoalgebra.

An easy example of isosolvable Lie isolgebras are the isocommutative isotopic Lie isoalgebras, since they verify, by definition, that $\widehat{U}\widehat{\cdot}\widehat{U} = \widehat{U}_2 = \{\widehat{S}\}$. It implies that every nonzero isocommutative Lie isoalgebra has an isosolvability index equals 2, being 1 the corresponding to the trivial isoalgebra $\{\widehat{S}\}$.

PROPOSITION 3.2.C9: Let $\widehat{U}$ be a Lie isoalgebra associated with a Lie algebra U. Under conditions of Proposition XXX, they are verified:

1) $\widehat{U}_i$ is an isoideal of $\widehat{U}$ and of $\widehat{U}_{i-1}$, for all $i \in N$.

2) If $\widehat{U}$ is isosolvable and U is solvable, then every isosubalgebra of $\widehat{U}$ is isosolvable.

3) La intersection and the product of a finite number of isosolvable isoideals of $\widehat{U}$ are isosolvable isoideals. Moreover, under conditions of Proposition XXX, the sum of a finite number of isosolvable isoideals is also an isosolvable isoideal.

By using this the last result it is deduced hat the sum of all isosolvable isoideals of $\widehat{U}$ is another isosolvable isoideal, which is called *isoradical* of $\widehat{U}$. Note that it is different from the *radical* of $\widehat{U}$, which would be the sum of all solvable ideals of $\widehat{U}$. The isoradical is denoted by $isorad\ \widehat{U}$, to not be confused with $rad\ \widehat{U}$, and it will always contain $\{\widehat{S}\}$, because this last one is a trivial isosolvable isoideal of every isoalgebra. Note also that as every isosolvable isoideal of $\widehat{U}$ is a solvable ideal of $\widehat{U}$, then $isorad\ \widehat{U} \subset rad\ \widehat{U}$. So, if $\widehat{U}$ is isosolvable, then $\widehat{U} = isorad\ \widehat{U} = rad\ \widehat{U}$, due to $\widehat{U}$ is solvable in particular.

PROPOSITION 3.2.C10: If $\widehat{U}$ is a semisimple Lie isoalgebra over a field of zero characteristic, then isorad $\widehat{U} = \{\widehat{S}\}$.

A Lie-Santilli isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ is called *isonilpotent* if, being an isotopy of a nilpotent Lie algebra, in the series

$$\widehat{U}^1 = \widehat{U}, \quad \widehat{U}^2 = \widehat{U} \widehat{\cdot} \widehat{U}, \quad \widehat{U}^3 = \widehat{U}^2 \widehat{\cdot} \widehat{U}, \ldots, \widehat{U}^i = \widehat{U}^{i-1} \widehat{\cdot} \widehat{U}, \ldots$$

(which is called *isonilpotency series*), there exists a natural integer n such that $\widehat{U}^n = \{\widehat{S}\}$. The minor of such integers is denominated *nilpotency index* of the isoalgebra.

As a consequence immediate of this definition it is deduced that every isonilpotent Lie isoalgebra is isosolvable and that every nonzero isocommutative Lie isoalgebra has an isonilpotency index equals 2, being 1 the corresponding of the isoalgebra $\{\widehat{S}\}$. Moreover, they are verified:

PROPOSITION 3.2.C11: Under conditions of Proposition XXX, the isotopic lifting of a nilpotent Lie algebra is an isonilpotent isotopic Lie isoalgebra.

PROPOSITION 3.2.C12: Let $\widehat{U}$ be a Lie isoalgebra associated with a Lie algebra U. They are verified:

1) *Under conditions of Proposition XXX, the sum of a finite number of isonilpotent isoideals of $\widehat{U}$ is another isonilpotent isoideal.*
2) *If $\widehat{U}$ is also isonilpotent and U is nilpotent, then*
   (a) *Every isosubalgebra of $\widehat{U}$ is isonilpotent.*
   (b) *Under conditions of Proposition XXX, if $\widehat{U}$ is nonzero isonilpotent, then its center is non null.*

In a similar way as the case isosolvable, the result (1) involves that the sum of all isonilpotent isoideals of $\widehat{U}$ is another isonilpotent isoideal, which is denoted by *isonihil-radical* of $\widehat{U}$, to be distinguished from the nihil-radical of $\widehat{U}$, which is the sum of the radicals ideals. It will be represented by *isonil-rad* $\widehat{U}$, which allows to distinguish it from the *nil-rad* $\widehat{U}$. It is immediate that *isonil-rad* $\widehat{U} \subset$ *nil-rad* $\widehat{U} \cap$ *isorad* $\widehat{U} \subset$ *nil-rad* $\widehat{U} \subset$ *rad* $\widehat{U}$.

Apart from that, it is possible to relate an isosolvable isotopic Lie isoalgebra with its derived Lie isoalgebra, by using the following:

PROPOSITION 3.2.C13: Under conditions of Proposition XXX, an Lie isotopic isoalgebra is isosolvable if and only if its derived Lie isoalgebra is isonilpotent.

Finally, an isonilpotent Lie isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ is called *isofiliform* if, being an isotopy of a filiform Lie algebra, it is verified that

$$\dim \widehat{U}^2 = n-2, \ \ \ldots, \ \ \dim \widehat{U}^i = n-i, \ \ \ldots, \ \ \dim \widehat{U}^n = 0,$$

where $\dim \widehat{U} = n$.

Note that the theory related with a filiform Lie algebra U is based on the use of a basis of such an algebra. So, starting from a basis $\{e_1, \ldots, e_n\}$ de U, which is preferably an *adapted basis*, we can deal with lots of concepts of it, such that dimensions of U and of elements of the nilpotency series, invariants i and j of U and, in general, the resting properties, starting from its structure coefficients, which are, in fact, responsible for the complete study of filiform Lie algebras.

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