Chapter 4

LIE-ADMISSIBLE BRANCH OF HADRONIC MECHANICS AND ITS ISODUAL

4.1 INTRODUCTION

4.1.1 The Scientific Imbalance Caused by Irreversibility

As recalled in Chapter 1, physical, chemical or biological systems are called *irreversible* when their images under time reversal $t \rightarrow -t$ are prohibited by causality and/or other laws, as it is generally the case for nuclear transmutations, chemical reactions and organism growth.

Systems are called *reversible* when their time reversal images are as causal as the original ones, as it is the case for planetary and atomic structures when considered isolated from the rest of the universe (see reprint volume [1] on irreversibility and vast literature quoted therein).

Another large scientific imbalance of the 20-th century studied in this monograph is the treatment of irreversible systems via the mathematical and physical formulations developed for reversible systems, since these formulations are themselves reversible, resulting in serious limitations in virtually all branches of science.

The problem is compounded by the fact that all used formulations are of Hamiltonian type, under the awareness that all known Hamiltonians are reversible (since all known potentials, such as the Coulomb potential $V(r)$, etc., are reversible).

This scientific imbalance was generally dismissed in the 20-th century with unsubstantiated statements, such as “irreversibility is a macroscopic occurrence that disappears when all bodies are reduced to their elementary constituents”.

These academic beliefs have been disproved by Theorem 1.3.3 according to which a classical irreversible system cannot be consistently decomposed into a finite number of elementary constituents all in reversible conditions and, vice-versa, a finite collection of elementary constituents all in reversible conditions cannot yield an irreversible macroscopic ensemble.

The implications of the above theorem are quite profound because it establishes that, contrary to academic beliefs, *irreversibility originates at the most primitive levels of nature, that of elementary particles, and then propagates all the way to our macroscopic environment.*

4.1.2 The Forgotten Legacy of Newton, Lagrange and Hamilton

The scientific imbalance on irreversibility was created in the early part of the 20-th century when, to achieve compatibility with quantum mechanics and special relativity, the entire universe was reduced to potential forces and the analytic equations were "truncated" with the removal of the external terms.

In reality, Newton [2] *did not* propose his celebrated equations to be restricted to forces derivable from a potential $F = \partial V/\partial r$, but proposed them for the most general possible forces,

$$m_a \times \frac{dv_{ka}}{dt} = F_{ka}(t, r, v), \quad k = 1, 2, 3; \quad a = 1, 2, \ldots, N, \tag{4.1.1}$$

where the conventional associative product of numbers, matrices, operators, etc. is continued to be denoted hereon with the symbol $\times$ so as to distinguish it from numerous other products needed later on.

Similarly, to be compatible with Newton's equations, Lagrange [3] and Hamilton [4] decomposed Newton's force into a potential and a nonpotential component, represented all potential forces with functions today known as the Lagrangian and the Hamiltonian, and proposed their celebrated equations with external terms,

$$\frac{d}{dt}\frac{\partial L(t, r, v)}{\partial v_a^k} - \frac{\partial L(t, r, v)}{\partial r_a^k} = F_{ak}(t, r, v), \tag{4.1.2a}$$

$$\frac{dr_a^k}{dt} = \frac{\partial H(t, r, p)}{\partial p_{ak}}, \quad \frac{dp_{ak}}{dt} = -\frac{\partial H(t, r, p)}{\partial r_a^k} + F_{ak}(t, r, p), \tag{4.1.2b}$$

$$L = \Sigma_a \frac{1}{2} \times m_a \times v_a^2 - V(t, r, v), \quad H = \Sigma_a \frac{\mathbf{p}_a^2}{2 \times m_a} + V(t, r, p), \tag{4.1.2c}$$

$$V = U(t, r)_{ak} \times v_a^k + U_o(t, r), \quad F(t, r, v) = F(t, r, p/m). \tag{4.1.2d}$$

More recently, Santilli [5] conducted comprehensive studies on the integrability conditions for the existence of a potential or a Lagrangian or a hamiltonian, called *conditions of variational selfadjointness.* These study permit the rigorous decomposition of Newtonian forces into a component that is variationally selfadjoint (SA) and a component that is not (NSA),

$$m_a \times \frac{dv_{ka}}{dt} = F_{ka}^{SA}(t, r, v) + F_{ka}^{NSA}(t, r, v). \quad (4.1.3)$$

Consequently, the true Lagrange and Hamilton equations can be more technically written

$$\Big[\frac{d}{dt}\frac{\partial L(t, r, v)}{\partial v_a^k} - \frac{\partial L(t, r, v)}{\partial r_a^k}\Big]^{SA} = F_{ak}^{NSA}(t, r, v), \quad (4.1.4a)$$

$$\Big[\frac{dr_a^k}{dt} - \frac{\partial H(t, r, p)}{\partial p_{ak}}\Big]^{SA} = 0, \quad \Big[\frac{dp_{ak}}{dt} + \frac{\partial H(t, r, p)}{\partial r_a^k}\Big]^{SA} = F_{ak}^{NSA}(t, r, p), \quad (4.1.4b)$$

The *forgotten legacy of Newton, Lagrange and Hamilton is that irreversibility originates precisely in the truncated NSA terms,* because all known potential-SA forces are reversible. The scientific imbalance of Section 1.3 is then due to the fact that no serious scientific study on irreversibility can be done with the truncated analytic equations and their operator counterpart, since these equations can only represent reversible systems.

Being born and educated in Italy, during his graduate studies at the University of Torino, the author had the opportunity of studying in the late 1960s the original works by Lagrange that were written precisely in Torino and most of them in Italian.

In this way, the author had the opportunity of verifying *Lagrange's analytic vision of representing irreversibility precisely via the external terms,* due to the impossibility of representing all possible physical events via the sole use of the Lagrangian, since the latter was solely conceived for the representation of reversible and potential events. As the reader can verify, Hamilton had, independently, the same vision.

Consequently, the truncation of the basic analytic equations caused the impossibility of a credible treatment of irreversibility at the purely classical level. The lack of a credible treatment of irreversibility then propagated at the subsequent operator level.

It then follows that *quantum mechanics cannot possibly be used for serious studies on irreversibility* because the discipline was constructed for the description of reversible quantized atomic orbits and not for irreversible systems.

In plain terms, while the validity of quantum mechanics for the arena of its original conception and construction is beyond scientific doubt, the assumption of quantum mechanics as the final operator theory for all conditions existing in the universe is outside the boundaries of serious science.

This establishes the need for the construction of a broadening (generalization) of quantum mechanics specifically conceived for quantitative studies of irreversibility. Since reversible systems are a *particular case* of irreversible ones, the broader mechanics must be a *covering* of quantum mechanics, that is, admitting the latter under a unique and unambiguous limit.

It is easy to see that the needed broader mechanics must also be a covering of the Lie-isotopic formulations, thus being a new branch of hadronic mechanics.

4.1.3 Early Representations of Irreversible Systems

As reviewed in Section 1.5.2, the brackets of the time evolution of an observable $A(r,p)$ in phase space according to the analytic equations with external terms,

$$\frac{dA}{dt} = (A, H, F) = \frac{\partial A}{\partial r_a^k} \times \frac{\partial H}{\partial p_{ka}} - \frac{\partial H}{\partial r_a^k} \times \frac{\partial A}{\partial p_{ka}} + \frac{\partial A}{\partial r_a^k} \times F_{ka}, \tag{4.1.5}$$

violate the right associative and scalar laws.

Therefore, the presence of external terms in the analytic equations causes not only the loss of *all* Lie algebras in the study of irreversibility, but actually the loss of all possible algebras as commonly understood in mathematics.

To resolve this problem, the author initiated a long scientific journey beginning with his graduate studies at the University of Torino, Italy, following the reading of Lagrange's papers.

The original argument [7-9], still valid today, is to select analytic equations characterizing brackets in the time evolution verifying the following conditions:

(1) The brackets of the time evolution must verify the right and left associative and scalar laws to characterize an algebra;

(2) Said brackets must not be invariant under time reversal as a necessary condition to represent irreversibility *ab initio;*

(3) Said algebra must be a covering of Lie algebras as a necessary condition to have a covering of the truncated analytic equations, namely, as a condition for the selected representation of irreversibility to admit reversibility as a particular case.

Condition (1) requires that said brackets must be bilinear, e.g., of the form (A, B) with properties

$$(n \times A, B) = n \times (A, B), \quad (A, m \times B) = m \times (A, B); \quad n, m \in C, \tag{4.1.6a}$$

$$(A \times B, C) = A \times (B, C), \quad (A, B \times C) = (A, B) \times C. \tag{4.1.6b}$$

Condition (2) requires that brackets (A, B) should not be totally antisymmetric as the conventional Poisson brackets,

$$(A, B) \neq -(B, A), \tag{4.1.7}$$

because time reversal is realized via the use of Hermitean conjugation.

Condition (3) then implies that brackets (A, B) characterize *Lie-admissible algebras* in the sense of Albert [10], namely, the brackets are such that the attached antisymmetric algebra is Lie.[1]

$$[A, B]^* = (A, B) - (B, A) = Lie. \tag{4.1.8}$$

In particular, the latter condition implies that the new brackets are formed by the superposition of totally antisymmetric and totally symmetric brackets,

$$(A, B) = [A, B]^* + \{A, B\}^*. \tag{4.1.9}$$

It should be noted that the operator realization of brackets (A, B) is also *Jordan-admissible* in the sense of Albert [7], namely, the attached symmetric brackets $\{A, B\}^*$ characterize a *Jordan algebra.* Consequently, *hadronic mechanics provides a realization of Jordan's dream, that of seeing his algebra applied to physics.*

However, the reader should be aware that, for certain technical reasons beyond the scope of this monograph, the classical realizations of brackets (A, B) are Lie-admissible but not Jordan-admissible. Therefore, Jordan-admissibility appears to emerge exclusively for operator theories.[2]

After identifying the above lines, Santilli [9] proposed in 1967 the following *generalized analytic equations*

$$\frac{dr_a^k}{dt} = \alpha \times \frac{\partial H(t, r, p)}{\partial p_{ak}}, \quad \frac{dp_{ak}}{dt} = -\beta \times \frac{\partial H(t, r, p)}{\partial r_a^k}, \tag{4.1.10}$$

(where α and β are real non-null parameters) that are manifestly irreversible. The brackets of the time evolution are then given by

$$\frac{dA}{dt} = (A, H) =$$

[1] More technically, a generally nonassociative algebra U with elements $a, b, c, \ldots$ and abstract product ab is said to be Lie-admissible when the attached algebra U^- characterized by the product $[a, b] = ab - ba$ verifies the*Lie axioms*

$$[a, b] = -[b, a],$$
$$[[a, b], c] + [[b, c], a] + [[c, b], a] = 0.$$

[2] More technically, a generally nonassociative algebra U with elements a, b, c, $\ldots$ and abstract product ab is said to be Jordan-admissible when the attached algebra U^+ characterized by the product $\{a, b\} = ab + bA$ verifies the *Jordan axioms*

$$\{a, b\} = \{b, a\},$$
$$\{\{a, b\}, a^2\} = \{a, \{b, a^2\}\}.$$

In classical realizations of the algebra U the first axiom of Jordan-admissibility is generally verified but the second is generally violated, while in operator realizations both axioms are generally verified.

$$= \alpha \times \frac{\partial A}{\partial r_a^k} \times \frac{\partial H}{\partial p_{ka}} - \beta \times \frac{\partial H}{\partial r_a^k} \times \frac{\partial A}{\partial p_{ka}}, \tag{4.1.11}$$

whose brackets are manifestly Lie-admissible, but *not* Jordan-admissible as the interested reader is encouraged to verify.

The above analytic equations characterize the time-rate of variation of the energy

$$\frac{dH}{dt} = (\alpha - \beta) \times \frac{\partial H}{\partial r_a^k} \times \frac{\partial H}{\partial p_{ka}}. \tag{4.1.12}$$

Also in 1967, Santilli [7,8] proposed an operator counterpart of the preceding classical setting consisting in the first known *Lie-admissible parametric generalization of Heisenberg's equation* in the following infinitesimal form

$$i \times \frac{dA}{dt} = (A, B) = p \times A \times H - q \times H \times A =$$

$$= m \times (A \times B - B \times A) + n \times (A \times B + B \times A), \tag{4.1.13a}$$

$$m = p + q, \quad n = q - p, \tag{4.1.13b}$$

where $p, q, p \pm q$ are non-null parameters, with finite counterpart

$$A(t) = e^{i \times H \times q} \times A(0) \times e^{-i \times p \times H}. \tag{4.1.14}$$

Brackets (A, B) are manifestly Lie-admissible with attached antisymmetric part

$$[A, B]^* = (A, B) - (B, A) = (p - q) \times [A, B]. \tag{4.1.15}$$

The same brackets are also Jordan-admissible as interested readers are encouraged to verify.

The resulting time evolution is then manifestly irreversible (for $p \neq q$) with nonconservation of the energy

$$i \times \frac{dH}{dt} = (H, H) = (p - q) \times H \times H \neq 0, \tag{4.1.16}$$

as necessary for an open system.

Subsequently, Santilli realized that the above formulations are not invariant under their own time evolution because Eq. (4.1.11) is manifestly *nonunitary.*

The application of nonunitary transforms to brackets (4.1.12) then led to the proposal in memoir [11,12] of 1978 of the following *Lie-admissible operator generalization of Heisenberg equations* in their infinitesimal form

$$\frac{dA}{dt} = A \times P \times H - H \times Q \times A = (A, H)^*, \tag{4.1.17}$$

with finite counterpart

$$A(t) = e^{i \times H \times Q} \times A(0) \times e^{-i \times P \times H}, \tag{4.1.18}$$

where

$$P = Q^{\dagger}. \tag{4.1.19}$$

where P, Q and $P \pm Q$ are now nonsingular operators (or matrices), and Eq. (4.1.16b) is a basic consistency condition explained later on in this section.

Eqs. (4.1.15)–(4.1.16) are the *fundamental equations of hadronic mechanics.* Their basic brackets are manifestly Lie-admissible and Jordan admissible with structure

$$(A, B)^* = A \times P \times B - B \times Q \times A =$$

$$= (A \times T \times B - B \times T \times A) + (A \times R \times B + B \times R \times A), \tag{4.1.20a}$$

$$T = P + Q, \quad R = Q - P. \tag{4.1.20b}$$

As indicated in Section 1.5.2, it is easy to see that the application of a nonunitary transform to the parametric brackets (4.1.11) leads precisely to the operator brackets (4.1.18) and that the application of additional nonunitary transforms preserves their Lie-admissible and Jordan-admissible characters. Consequently, fundamental equations (4.1.15)–(4.1.18) are "directly universal" in the sense of Lemma 1.5.2.

However, the above equations *are not invariant* and, consequently, are afflicted by the catastrophic inconsistencies of Theorem 1.5.2.

Moreover, in the form presented above, the dynamical equations are not derivable from a variational principle. Consequently, they admit no known unique map into their operator counterpart.

In view of these insufficiencies, said equations cannot be assumed in the above given form as the basic equations of any consistent physical theory.

4.2 ELEMENTS OF SANTILLI GENOMATHEMATICS AND ITS ISODUAL

4.2.1 Genounits, Genoproducts and their Isoduals

The "direct universality" of Eq. (4.1.15) voids any attempt at seeking further generalization in the hope of achieving invariance, since any nontrivial generalization would imply the loss of any algebra in the brackets of the time evolution with consequential inability to achieve any physically meaningful theory.

The preceding occurrences leave no alternative other than that of seeking a fundamentally new mathematics permitting Eq. (4.1.15) to achieve the needed invariance.

After numerous attempts and a futile search in the mathematical literature, Santilli proposed in Refs. [11,12] of 1978 the construction of a *new mathematics* specifically conceived for the indicated task, that eventually reached mathematical maturity for numbers only in paper [13] of 1993, mathematical maturity for the new differential calculus only in memoir [14] of 1996, and, finally, an invariant formulation of lie-admissible formulations only in paper [15] of 1997.

The new Lie-admissible mathematics is today known as *Santilli genomathematics*, where the prefix "geno" suggested in the original proposal [11,12] is used in the Greek meaning of "inducting" new Axioms (as compared to the prefix "iso" of the preceding chapter denoting the preservation of the axioms).

The basic idea is to lift the isounits of the preceding chapter into a form that is still nowhere singular, but *non-Hermitean*, thus implying the existence of *two* different generalized units, today called *Santilli genounits* for the description of matter, that are generally written [13]

$$\hat{I}^{>} = 1/\hat{T}^{>}, \quad {}^{<}\hat{I} = 1/{}^{<}\hat{T}, \tag{4.2.1a}$$

$$\hat{I}^{>} \neq^{<} \hat{I}, \quad \hat{I}^{>} = ({}^{<}\hat{I})^{\dagger}, \tag{4.2.1b}$$

with two additional *isodual genounits* for the description of antimatter [14]

$$(\hat{I}^{>})^{d} = -(\hat{I}^{>})^{\dagger} = -{}^{<}\hat{I} = -1/{}^{<}\hat{T}, \quad ({}^{<}\hat{I})^{d} = -\hat{I}^{>} = -1/\hat{T}^{>}. \tag{4.2.2}$$

Jointly, all conventional and/or isotopic products $A\hat{\times}B$ among generic quantities (numbers, vector fields, operators, etc.) are lifted in such a form to admit the genounits as the correct left and right units at all levels, i.e.,

$$A > B = A \times \hat{T}^{>} \times B, \quad A > \hat{I}^{>} = \hat{I}^{>} > A = A, \tag{4.2.3a}$$

$$A < B = A \times^{<} \hat{T} \times B, \quad A <^{<} \hat{I} =^{<} \hat{I} < A = A, \tag{4.2.3b}$$

$$A >^{d} B = A \times \hat{T}^{>d} \times B, \quad A >^{d} \hat{I}^{>d} = \hat{I}^{>d} >^{d} A = A, \tag{4.2.3c}$$

$$A <^{d} B = A \times^{<} \hat{T}^{d} \times B, \quad A <^{d} {}^{<}\hat{I}^{d} = {}^{<}\hat{I}^{d} <^{d} A = A, \tag{4.2.3d}$$

for all elements A, B of the set considered.

As we shall see in Section 4.3, the above basic assumptions permit the representation of irreversibility with the most primitive possible quantities, the basic units and related products.

In particular, as we shall see in Section 4.3 and 4.4, genounits permit an invariant representation of the external forces in Lagrange's and Hamilton's equations (4.2.2). As such, they are generally dependent on time, coordinates, momenta., wavefunctions and any other needed variable, e.g., $\hat{I}^{>} = \hat{I}^{>}(t^{>}, r^{>}, p^{>}, \psi^{>}, \ldots)$.

In fact, the assumption of all *ordered product to the right* $>$ represents matter systems moving forward in time, the assumption of all *ordered products to the left* $<$ represents matter systems moving backward in time, with the irreversibility being represented *ab initio* by the inequality $A > B \neq A < B$. Similar representation of irreversible antimatter systems occurs via isodualities.

4.2.2 Genonumbers, Genofunctional Analysis and their Isoduals

The author has repeatedly indicated in his writings that "there cannot be a really new physical theory without a new mathematics, and there cannot be a really new mathematics without new numbers".

Genomathematics began to reach maturity with the discovery made, apparently for the first time in paper [13] of 1993, that *the axioms of a field still hold under the ordering of all products to the right or, independently, to the left.*

This unexpected property permitted the formulation of *new numbers,* more general than the isodual numbers of Chapter 2 and the isonumbers of Chapter 3, that can be expressed via the following:

DEFINITION 4.2.1 [13]: Let $F = F(a, +, \times)$ be a field of characteristic zero as per Definitions 2.2.1 and 3.2.1. Santilli's forward genofields are rings $\hat{F}^> = \hat{F}(\hat{a}^>, \hat{+}^>, \hat{\times}^>)$ with: elements

$$\hat{a}^> = a \times \hat{I}^>, \tag{4.2.4}$$

where $a \in F$, $\hat{I}^> = 1/\hat{T}^>$ is a non singular non-Hermitean quantity (number, matrix or operator) generally outside F and $\times$ is the ordinary product of F; the genosum $\hat{+}^>$ coincides with the ordinary sum $+$,

$$\hat{a}^> \hat{+}^> \hat{b}^> \equiv \hat{a}^> + \hat{b}^>, \ \ \forall \hat{a}^>, \ \hat{b}^> \in \hat{F}^>, \tag{4.2.5}$$

consequently, the additive forward genounit $\hat{0}^> \in \hat{F}$ coincides with the ordinary $0 \in F$; and the forward genoproduct $>$ is such that $\hat{I}^>$ is the right and left isounit of $\hat{F}^>$,

$$\hat{I}^> \hat{\times} \hat{a}^> = \hat{a}^> > \hat{I}^> \equiv \hat{a}^>, \ \ \forall \hat{a}^> \in \hat{F}^>. \tag{4.2.6}$$

Santilli's forward genofields verify the following properties:

1) For each element $\hat{a}^> \in \hat{F}^>$ there is an element $\hat{a}^{>-\hat{1}^>}$, called forward genoinverse, for which

$$\hat{a}^> > \hat{a}^{>-\hat{I}^>} = \hat{I}^>, \ \ \forall \hat{a}^> \in \hat{F}^>; \tag{4.2.7}$$

2) The genosum is commutative

$$\hat{a}^> \hat{+}^> \hat{b}^> = \hat{b}^> \hat{+}^> \hat{a}^>, \tag{4.2.8}$$

and associative

$$(\hat{a}^> \hat{+}^> \hat{b}^>) +^> \hat{c}^> = \hat{a}^> \hat{+}^> (\hat{b}^> \hat{+}^> \hat{c}^>), \ \ \forall \hat{a}, \hat{b}, \hat{c} \in \hat{F}; \tag{4.2.9}$$

3) The forward genoproduct is associative

$$\hat{a}^> > (\hat{b}^> > \hat{c}^>) = (\hat{a}^> > \hat{b}^>) > \hat{c}^>, \quad \forall \hat{a}^>, \hat{b}^>, \hat{c}^> \in \hat{F}^>; \tag{4.2.10}$$

but not necessarily commutative

$$\hat{a}^> > \hat{b}^> \neq \hat{b}^> > \hat{a}^>, \tag{4.2.11}$$

4) The set $\hat{F}^>$ is closed under the genosum,

$$\hat{a}^> \hat{+}^> \hat{b}^> = \hat{c}^> \in \hat{F}^>, \tag{4.2.12}$$

the forward genoproduct,

$$\hat{a}^> > \hat{b}^> = \hat{c}^> \in \hat{F}^>, \tag{4.2.13}$$

and right and left genodistributive compositions,

$$\hat{a}^> > (\hat{b}^> \hat{+}^> \hat{c}^>) = \hat{d}^> \in \hat{F}^>, \tag{4.2.14a}$$

$$(\hat{a}^> \hat{+}^> \hat{b}^>) > \hat{c}^> = \hat{d}^> \in \hat{F}^> \quad \forall \hat{a}^>, \hat{b}^>, \hat{c}^>, \hat{d}^> \in \hat{F}^>; \tag{4.2.14b}$$

5) The set $\hat{F}^>$ verifies the right and left genodistributive law

$$\hat{a}^> > (\hat{b}^> \hat{+}^> \hat{c}^>) = (\hat{a}^> \hat{+}^> \hat{b}^>) > \hat{c}^> = \hat{d}^>, \ \forall \hat{a}^>, \hat{b}^>, \hat{c}^>, \in \hat{F}^>. \tag{4.2.15}$$

In this way we have the forward genoreal numbers $\hat{R}^>$, the forward genocomplex numbers $\hat{C}^>$ and the forward genoquaternionic numbers $\hat{Q}C^>$ while the forward genooctonions $\hat{O}^>$ can indeed be formulated but they do not constitute genofields [14].

The backward genofields and the isodual forward and backward genofields are defined accordingly. Santilli's genofields are called of the first (second) kind when the genounit is (is not) an element of F.

The basic axiom-preserving character of genofields is illustrated by the following:

LEMMA 4.2.1 [13]: Genofields of first and second kind are fields (namely, they verify all axioms of a field).

In Section 2.1 we pointed out that the conventional product "2 multiplied by 3" is not necessarily equal to 6 because, depending on the assumed unit and related product, it can be -6.

In Section 3.2 we pointed out that the same product "2 multiplied by 3" is not necessarily equal to $+6$ or -6, because it can also be equal to an arbitrary number, or a matrix or an integrodifferential operator.

In this section we point out that "2 multiplied by 3" can be ordered to the right or to the left, and the result is not only arbitrary, but yielding different numerical results for different orderings, $2 > 3 \neq 2 < 3$, all this by continuing to verify the axioms of a field per each order [13].

Once the forward and backward genofields have been identified, the various branches of genomathematics can be constructed via simple compatibility arguments.

For specific applications to irreversible processes there is first the need to construct the *genofunctional analysis,* studied in Refs. [6,18] that we cannot review here for brevity. the reader is however warned that any elaboration of irreversible processes via Lie-admissible formulations based on conventional or isotopic functional analysis leads to catastrophic inconsistencies because it would be the same as elaborating quantum mechanical calculations with genomathematics.

As an illustration, Theorems 1.5.1 and 1.5.2 of catastrophic inconsistencies are activated unless one uses the ordinary differential calculus is lifted, for ordinary motion in time of matter, into the following *forward genodifferentials and genoderivatives*

$$\hat{d}^{>}x = \hat{T}^{>}_{x} \times dx, \quad \frac{\hat{\partial}^{>}}{\hat{\partial}^{>}x} = \hat{I}^{>}_{x} \times \frac{\partial}{\partial x}, \text{ etc.} \tag{4.2.16}$$

with corresponding backward and isodual expressions here ignored,

Similarly, all conventional functions and isofunctions, such as isosinus, isocosinus, isolog, etc., have to be lifted in the genoform

$$\hat{f}^{>}(x^{>}) = f(\hat{x}^{>}) \times \hat{I}^{>}, \tag{4.2.17}$$

where one should note the necessity of the multiplication by the genounit as a condition for the result to be in $\hat{R}^{>}$, $\hat{C}^{>}$, or $\hat{O}^{>}$.

4.2.3 Genogeometries and Their Isoduals

Particularly intriguing are the *genogeometries* [16] (see also monographs [18,19] for detailed treatments) characterized by a simple genotopy of the isogeometries of the preceding chapter.

As an illustration, the *Minkowski-Santilli forward genospace* $\hat{M}^{>}(\hat{x}^{>}, \hat{\eta}^{>}, \hat{R}^{>})$ over the genoreal $\hat{R}^{>}$ is characterized by the following spacetime, *genocoordinates, genometric and genoinvariant*

$$\hat{x}^{>} = x\hat{I}^{>} = \{x^{\mu}\} \times \hat{I}^{>}, \quad \hat{\eta}^{>} = \hat{T}^{>} \times \eta, \quad \eta = Diag.(1,1,1,-1), \tag{4.2.18a}$$

$$\hat{x}^{>2^{>}} = \hat{x}^{>\mu}\hat{\times}^{>}\hat{\eta}^{>}_{\mu\nu}\hat{\times}^{>}\hat{x}^{>\nu} = (x^{\mu} \times \hat{\eta}^{>}_{\mu\nu} \times x^{\nu}) \times \hat{I}^{>}, \tag{4.2.18b}$$

where the first expression of the genoinvariant is on genospaces while the second is its projection in our spacetime.

Note that the minkowski-Santilli genospace has, in general, an explicit dependence on spacetime coordinates. Consequently, it is equipped with the entire formalism of the conventional Riemannian spaces covariant derivative, Christoffel's symbols, Bianchi identity, etc. only lifted from the isotopic form of the preceding chapter into the genotopic form.

A most important feature is that *genospaces permit, apparently for the first time in scientific history, the representation of irreversibility directly via the basic genometric*. This is due to the fact that genometrics are nonsymmetric by conception, e.g.,

$$\hat{\eta}^{>}_{\mu\nu} \quad \neq \quad \hat{\eta}^{>}_{\nu\mu}. \tag{4.2.19}$$

Consequently, *genotopies permit the lifting of conventional symmetric metrics into nonsymmetric forms,*

$$\eta^{Minkow.}_{Symm} \quad \rightarrow \quad \hat{\eta}^{>}{}^{Minkow.-Sant.}_{NonSymm} \tag{4.2.20}$$

Remarkably, *nonsymmetric metrics bare indeed permitted by the axioms of conventional spaces* as illustrated by the invariance

$$(x^{\mu} \times \eta_{\mu\nu} \times x^{\nu}) \times I \equiv [x^{\mu} \times (\hat{T}^{>} \times \eta_{\mu\nu}) \times x^{\nu}] \times T^{>-1} \equiv$$
$$\equiv (x^{\mu} \times \hat{\eta}^{>}_{\mu\nu} \times x^{\nu}) \times \hat{I}^{>}, \tag{4.2.21}$$

where $\hat{T}^{>}$ is assumed in this simple illustration to be a complex number.

Interested readers can then work out backward genogeometries and the isodual forward and backward genogeometries with their underlying genofunctional analysis.

This basic geometric feature was not discovered until recently because hidden where nobody looked for, in the basic unit. However, this basic geometric advance in the representation of irreversibility required the prior discovery of basically new numbers, Santilli's genonumbers with nonsymmetric unit and ordered multiplication [14].

4.2.4 Lie-Santilli Genotheory and its Isodual

Particularly important for irreversibility is the lifting of Lie's and Lie-Santilli's theories permitted by genomathematics, first identified by Santilli in Ref. [12] of 1978 (and then studied in various works, e.g., [7,18,19]) via the following genotopies:

(1) The *forward and backward universal enveloping genoassociative algebra* $\hat{\xi}^{>}$, ${}^{<}\hat{\xi}$, with infinite-dimensional basis characterizing the *Poincaré-Birkhoff-Witt-Santilli genotheorem*

$$\hat{\xi}^{>} : \hat{I}^{>},\ \hat{X}_i,\ \hat{X}_i > \hat{X}_j,\ \hat{X}_i > \hat{X}_j > \hat{X}_k,\ \ldots,\ i \leq j \leq k, \tag{4.2.22a}$$

$${}^{<}\hat{\xi} : \hat{I},\ {}^{<}\hat{X}_i,\ \hat{X}_i < \hat{X}_j,\ \hat{X}_i < \hat{X}_j < \hat{X}_k,\ \ldots,\ i \leq j \leq k; \tag{4.2.22b}$$

where the "hat" on the generators denotes their formulation on genospaces over genofields and their Hermiticity implies that $\hat{X}^{>} =^{<} \hat{X} = \hat{X}$;

(2) The *Lie-Santilli genoalgebras* characterized by the universal, jointly Lie- and Jordan-admissible brackets,

$$^{<}\hat{L}^{>} : (\hat{X}_i \hat{,} \hat{X}_j) = \hat{X}_i < \hat{X}_j - \hat{X}_j > \hat{X}_i = C_{ij}^k \times \hat{X}_k, \tag{4.2.23}$$

here formulated formulated in an invariant form (see below);

(3) The *Lie-Santilli genotransformation groups*

$$^{<}\hat{G}^{>} : \hat{A}(\hat{w}) = (\hat{e}^{\hat{i} \hat{\times} \hat{X} \hat{\times} \hat{w}})^{>} > \hat{A}(\hat{0}) <^{<} (\hat{e}^{-\hat{i} \hat{\times} \hat{w} \hat{\times} \hat{X}}) =$$

$$= (e^{i \times \hat{X} \times \hat{T}^{>} \times w}) \times A(0) \times (e^{-i \times w \times ^{<}\hat{T} \times \hat{X}}), \tag{4.2.24}$$

where $\hat{w}^{>} \in \hat{R}^{>}$ are the *genoparameters;* the *genorepresentation theory,* etc.

4.2.5 Genosymmetries and Nonconservation Laws

The implications of the Lie-Santilli genotheory are significant mathematically and physically. On mathematical grounds, the Lie-Santilli genoalgebras are "directly universal" and include as particular cases all known algebras, such as Lie, Jordan, Flexible algebras, power associative algebras, quantum, algebras, supersymmetric algebras, Kac-Moody algebras, etc. (Section 1.5).

Moreover, when computed on the *genobimodule*

$$^{<}\hat{B}^{>} =^{<} \hat{\xi} \times \hat{\xi}^{>}, \tag{4.2.25}$$

Lie-admissible algebras verify all Lie axioms, while deviations from Lie algebras emerge only in their *projection* on the conventional bimodule

$$^{<}B^{>} =^{<} \xi \times \xi^{>}, \tag{4.2.26}$$

of Lie's theory (see Ref. [17] for the initiation of the genorepresentation theory of Lie-admissible algebras on bimodules).

This is due to the fact that the computation of the left action $A < B = A \times^{<} \hat{T} \times B$ on $^{<}\hat{\xi}$ (that is, with respect to the genounit $^{<}\hat{I} = 1/^{<}\hat{T}$) yields the save value as the computation of the conventional product $A \times B$ on $^{<}\xi$ (that is, with respect to the trivial unit I), and the same occurs for the value of $A > B$ on $\hat{\xi}^{>}$.

The above occurrences explain the reason the structure constant and the product in the r.h.s. of Eq. (4.2.23) are those of a conventional Lie algebra.

In this way, thanks to genomathematics, *Lie algebras acquire a towering significance in view of the possibility of reducing all possible irreversible systems to primitive Lie axioms.*

The physical implications of the Lie-Santilli genotheory are equally far reaching. In fact, Noether's theorem on the reduction of reversible conservation laws to primitive Lie symmetries can be lifted to the *reduction, this time, of irreversible nonconservation laws to primitive Lie-Santilli genosymmetries.*

As a matter of fact, this reduction was the very first motivation for the construction of the genotheory in memoir [12] (see also monographs [7,18,19,20]). The reader can then foresee similar liftings of all remaining physical aspects treated via Lie algebras.

The construction of the isodual Lie-Santilli genotheory is an instructive exercise for readers interested in learning the new methods.

4.3 CLASSICAL LIE-ADMISSIBLE MECHANICS

4.3.1 Fundamental Ordering Assumption on Irreversibility

The discovery [12] of two complementary orderings of the product and related units while preserving the abstract axioms of a field has truly fundamental implications for irreversibility, since it permits the axiomatically consistent and invariant representation of irreversibility via the most ultimate and primitive axioms, those on the product and related unit. We, therefore, have the following:

FUNDAMENTAL ORDERING ASSUMPTION ON IRREVERSIBILITY [15,19,20]: *Dynamical equations for motion forward in time of matter (antimatter) are characterized by genoproducts to the right and related genounits (their isoduals), while dynamical equations for the motion backward in time of matter (antimatter) are characterized by genoproducts to the left and related genounits (their isoduals) under the condition that said genoproducts and genounits are interconnected by time reversal expressible for generic quantities A, B with the relation,*

$$(A > B)^\dagger = (A > \hat{T}^> \times B)^\dagger = B^\dagger \times (\hat{T}^>)^\dagger \times A^\dagger, \tag{4.3.1}$$

namely,

$$\hat{T}^> = ({}^<\hat{T})^\dagger \tag{4.3.2}$$

thus recovering the fundamental complementary conditions (4.1.17) or (4.2.2).

Unless otherwise specified, from now on physical and chemical expression for irreversible processes will have no meaning without the selection of one of the indicated two possible orderings.

4.3.2 Geno-Newtonian Equations and Their Isoduals

Recall that, for the case of isotopies, the basic Newtonian systems are given by those admitting nonconservative internal forces restricted by certain constraints to verify total conservation laws (these are the *closed non-Hamiltonian systems* of Chapter 1).

For the case of the genotopies under consideration here, the basic Newtonian systems are the conventional nonconservative systems without subsidiary constraints (*open non-Hamiltonian systems*) with generic expression (4.1.3), in which case irreversibility is characterized by nonselfadjoint forces, since all conservative forces are reversible.

As it is well known, the above equations are not derivable from any variational principle in the fixed frame of the observer [6], and this is the reason why all conventional attempts for consistently quantizing nonconservative forces have failed for about one century. In turn, the lack of achievement of a consistent operator counterpart of nonconservative forces lead to the academic belief that they are illusory (Section 1.2).

Hadronic mechanics has achieved the first and only physically consistent operator formulation of nonconservative forces known to the author.[3] This goal was achieved by rewriting Newton's equations (4.1.3) into an identical form derivable from a variational principle. Still in turn, the latter objective was solely permitted by the novel genomathematics.

It is appropriate to recall that Newton was forced to discover new mathematics, the differential calculus, prior to being able to formulated his celebrated equations. Therefore, readers should not be surprised at the need for the new genodifferential calculus as a condition to represent all nonconservative Newton's systems from a variational principle.

Recall also from Section 2.3 that, contrary to popular beliefs, there exist *four* inequivalent directions of time, namely, motion forward in future times, motion backward in past time, motion backward from future times and motion forward in past times, each direction having its own unit.

Consequently, time reversal alone cannot represent all these possible motions, and isoduality results to be the only known additional conjugation that, when combined with time reversal, can represent all possible time evolutions of both matter and antimatter.

The above setting implies the existence of four different new mechanics first formulated by Santilli in memoir [14] of 1996, and today known as *Newton-Santilli genomechanics,* namely:

[3] The author would appreciate any indication of operator formulations of nonconservative forces under the conditions verified by hadronic mechanics shown in the next section, namely, that *nonconserved* quantities, such as the Hamiltonian, are *Hermitean* as a necessary condition to be *observable.*

A) *Forward genomechanics* for the representation of forward motion of matter systems;

B) *Backward genomechanics* for the representation of the time reversal image of matter systems;

C) *Isodual backward genomechanics* for the representation of motion backward in time of antimatter systems, and

D) *Isodual forward genomechanics* for the representation of time reversal antimatter systems.

These new mechanics are characterized by:

1) Four different times, *forward and backward genotimes for matter systems and the backward and forward isodual genotimes for antimatter systems*

$$\hat{t}^{>} = t \times \hat{I}^{>}_{t}, \quad -\hat{t}^{>}, \quad \hat{t}^{>d}, \quad -\hat{t}^{>d}, \tag{4.3.3}$$

with (nowhere singular and non-Hermitean) *forward and backward time genounits and their isoduals*[4],

$$\hat{I}^{>}_{t} = 1/\hat{T}^{>}_{t}, \quad -\hat{I}^{>}_{t}, \quad \hat{I}^{>d}_{t}, \quad -\hat{I}^{>d}_{t}; \tag{4.3.4}$$

2) The *forward and backward genocoordinates and their isoduals*

$$\hat{x}^{>} = x \times \hat{I}^{>}_{x}, \quad -\hat{x}^{>}, \quad \hat{x}^{>d}, \quad -\hat{x}^{>d}, \tag{4.3.5}$$

with (nowhere singular non-Hermitean) *coordinate genounit*

$$\hat{I}^{>}_{x} = 1/\hat{T}^{>}_{x}, \quad -\hat{I}^{>}_{x}, \quad \hat{I}^{>d}_{x}, \quad -\hat{I}^{>d}_{x}, \tag{4.3.6}$$

with *forward and backward coordinate genospace and their isoduals* $\hat{S}^{>}_{x}$, etc., and related *forward coordinate genofield and their isoduals* $\hat{R}^{>}_{x}$, etc.;

3) The *forward and backward genospeeds and their isoduals*

$$\hat{v}^{>} = \hat{d}^{>}\hat{x}^{>}/\hat{d}^{>}\hat{t}^{>}, \quad -\hat{v}^{>}, \quad \hat{v}^{>d}, \quad -\hat{v}^{>d}, \tag{4.3.7}$$

with (nowhere singular and non-Hermitean) *speed genounit*

$$\hat{I}^{>}_{v} = 1/\hat{T}^{>}_{v}, \quad -\hat{I}^{>}_{v}, \quad \hat{I}^{>d}_{v}, \quad -\hat{I}^{>d}_{v}, \tag{4.3.8}$$

with related *forward speed backward genospaces and their isoduals* $\hat{S}^{>}_{v}$, etc., over *forward and backward speed genofields* $\hat{R}^{>}_{v}$, etc.;

The above formalism then leads to the *forward genospace for matter systems*

$$\hat{S}^{>}_{tot} = \hat{S}^{>}_{t} \times \hat{S}^{>}_{x} \times \hat{S}^{>}_{v}, \tag{4.3.9}$$

[4] Note that, to verify the condition of non-Hermiticity, the time genounits can be *complex valued.*

defined over the it forward genofield

$$\hat{R}^{>}_{tot} = \hat{R}^{>}_{t} \times \hat{R}^{>}_{x} \times \hat{R}^{>}_{v}, \tag{4.3.10}$$

with *total forward genounit*

$$\hat{I}^{>}_{tot} = \hat{I}^{>}_{t} \times \hat{I}^{>}_{x} \times \hat{I}^{>}_{v}, \tag{4.3.11}$$

and corresponding expressions for the remaining three spaces obtained via time reversal and isoduality.

The basic equations are given by:

I) The *forward Newton-Santilli genoequations for matter systems* [14], formulated via the genodifferential calculus,

$$\hat{m}^{>}_{a} > \frac{\hat{d}^{>}\hat{v}^{>}_{ka}}{\hat{d}^{>}\hat{t}^{>}} = -\frac{\hat{\partial}^{>}\hat{V}^{>}}{\hat{\partial}^{>}\hat{x}^{>k}_{a}}; \tag{4.3.12}$$

II) The *backward genoequations for matter systems* that are characterized by time reversal of the preceding ones;

III) the *backward isodual genoequations for antimatter systems* that are characterized by the isodual map of the backward genoequations,

$$^{<}\hat{m}^{d}_{a} < \frac{^{<}\hat{d}^{d<}\hat{v}^{d}_{ka}}{^{<}\hat{d}^{d<}\hat{t}^{d}} = -\frac{^{<}\hat{\partial}^{d<}\hat{V}^{d}}{^{<}\hat{\partial}^{d<}\hat{x}^{dk}_{a}}; \tag{4.3.13}$$

IV) the *forward isodual genoequations for antimatter systems* characterized by time reversal of the preceding isodual equations.

Newton-Santilli genoequations (4.3.12) are "directly universal" for the representation of all possible (well behaved) Eqs. (4.1.3) in the frame of the observer because they admit a multiple infinity of solution for any given nonselfadjoint force.

A simple representation occurs under the conditions assumed for simplicity,

$$N = \hat{I}^{>}_{t} = \hat{I}^{>}_{v} = 1, \tag{4.3.14}$$

for which Eqs. (4.3.12) can be explicitly written

$$\hat{m}^{>} > \frac{\hat{d}^{>}\hat{v}^{>}}{\hat{d}^{>}t} = m \times \frac{d\hat{v}^{>}}{dt} =$$

$$= m \times \frac{d}{dt}\frac{d(x \times \hat{I}^{>}_{x})}{dt} = m \times \frac{dv}{dt} \times \hat{I}^{>}_{x} + m \times x \times \frac{d\hat{I}^{>}}{dt} = \hat{I}^{>}_{x} \times \frac{\partial V}{\partial x}, \tag{4.3.15}$$

from which we opbtain the genorepresentation

$$F^{NSA} = -m \times x \times \frac{1}{\hat{I}^{>}_{x}} \times \frac{d\hat{I}^{>}_{x}}{dt}, \tag{4.3.16}$$

that always admit solutions here left to the interested reader since in the next section we shall show a much simpler, universal, *algebraic* solution.

As one can see, in Newton's equations the nonpotential forces are part of the applied force, while in the Newton-Santilli genoequations nonpotential forces are represented by the genounits, or, equivalently, by the genodifferential calculus, in a way essentially similar to the case of isotopies.

The main difference between iso- and geno-equations is that isounits are Hermitean, thus implying the equivalence of forward and backward motions, while genounits are non-Hermitean, thus implying irreversibility.

Note also that the topology underlying Newton's equations is the conventional, Euclidean, local-differential topology which, as such, can only represent point particles.

By contrast, the topology underlying the Newton-Santilli genoequations is given by a genotopy of the isotopology studied in the preceding chapter, thus permitting for the representation of extended, nonspherical and deformable particles via forward genounits, e.g., of the type

$$\hat{I}^{>} = Diag.(n_1^2, n_2^2, n_3^2, n_4^2) \times \Gamma^{>}(t, r, v, \dots), \tag{4.3.17}$$

where $n_k^2,\ k = 1, 2, 3$ represents the semiaxes of an ellipsoid, n_4^2 represents the density of the medium in which motion occurs (with more general nondiagonal realizations here omitted for simplicity), and $\Gamma^{>}$ constitutes a nonsymmetric matrix representing nonselfadjoint forces, namely, the contact interactions among extended constituents occurring for the motion forward in time.

4.3.3 Lie-Admissible Classical Genomechanics and its Isodual

In this section we show that, once rewritten in their identical genoform (4.3.12), Newton's equations for nonconservative systems are indeed derivable from a variational principle, with analytic equations possessing a Lie-admissible structure and Hamilton-Jacobi equations suitable for the first know consistent and unique operator map studied in the next section.

The most effective setting to introduce real-valued non-symmetric genounits is in the $6N$-dimensional *forward genospace (genocotangent bundle)* with local genocoordinates and their conjugates

$$\hat{a}^{>\mu} = a^{\rho} \times \hat{I}_{1\ \rho}^{>\mu},\ \ (\hat{a}^{>\mu}) = \begin{pmatrix} \hat{x}_{\alpha}^{>k} \\ \hat{p}_{k\alpha}^{>} \end{pmatrix} \tag{4.3.18}$$

and

$$\hat{R}_{\mu}^{>} = R_{\rho} \times \hat{I}_{2\ \mu}^{>\rho},\ (\hat{R}_{\mu}^{>}) = (\hat{p}_{k\alpha}, \hat{0}), \tag{4.3.19a}$$

$$\hat{I}_1^{>} = 1/\hat{T}_1^{>} = (\hat{I}_2^{>})^T = (1/\hat{T}_2^{>})^T, \tag{4.3.19b}$$

$$k = 1,2,3;\ \alpha = 1,2,\dots,N;\quad \mu,\ \rho = 1,2,\dots 6N,$$

where the superscript T stands for transposed, and nowhere singular, real-valued and non-symmetric genometric and related invariant

$$\hat{\delta}^{>} = \hat{T}^{>}_{1\,6N\times 6N}\,\delta_{6N\times 6N}\times\delta_{6N\times 6N}, \tag{4.3.20a}$$

$$\hat{a}^{>\mu} > \hat{R}^{>}_{\mu} = \hat{a}^{>\rho}\times\hat{T}^{>\beta}_{1\,\rho}\times\hat{R}^{>}_{\beta} = a^{\rho}\times\hat{I}^{>\beta}_{2\,\rho}\times R_{\beta}. \tag{4.3.20b}$$

In this case we have the following *genoaction principle* [14]

$$\hat{\delta}^{>}\hat{\mathcal{A}}^{>} = \hat{\delta}^{>}\hat{\int}^{>}[\hat{R}^{>}_{\mu} >_a \hat{d}^{>}\hat{a}^{>\mu} - \hat{H}^{>} >_t \hat{d}^{>}\hat{t}^{>}] =$$

$$= \delta\int[R_{\mu}\times\hat{T}^{>\mu}_{1\,\nu}(t,x,p,\dots)\times d(a^{\beta}\times\hat{I}^{>\nu}_{1\,\beta}) - H\times dt] = 0, \tag{4.3.21}$$

where the second expression is the projection on conventional spaces over conventional fields and we have assumed for simplicity that the time genounit is 1.

It is easy to prove that the above genoprinciple characterizes the following *forward Hamilton-Santilli genoequations,* (originally proposed in Ref. [11] of 1978 with conventional mathematics and in Ref. [14] of 1996 with genomathematics (see also Refs. [18,19,20])

$$\hat{\omega}^{>}_{\mu\nu} > \frac{\hat{d}^{>}\hat{a}^{\nu>}}{\hat{d}^{>}\hat{t}^{>}} - \frac{\hat{\partial}^{>}\hat{H}^{>}(\hat{a}^{>})}{\hat{\partial}^{>}\hat{a}^{\mu>}} =$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\times\begin{pmatrix} dr/dt \\ dp/dt \end{pmatrix} - \begin{pmatrix} 1 & K \\ 0 & 1 \end{pmatrix}\times\begin{pmatrix} \partial H/\partial r \\ \partial H/\partial p \end{pmatrix} = 0, \tag{4.3.22a}$$

$$\hat{\omega}^{>} = \Big(\frac{\hat{\partial}^{>}R^{>}_{\nu}}{\hat{\partial}^{>}\hat{a}^{\mu>}} - \frac{\hat{\partial}^{>}\hat{R}^{>}_{\mu}}{\hat{\partial}^{>}\hat{a}^{\nu>}}\Big)\times\hat{I}^{>} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\times\hat{I}^{>}, \tag{4.3.22b}$$

$$K = F^{NSA}/(\partial H/\partial p), \tag{4.3.22c}$$

where one should note the "direct universality" of the simple algebraic solution (4.3.22c).

The time evolution of a quantity $\hat{A}^{>}(\hat{a}^{>})$ on the forward geno-phase-space can be written in terms of the following brackets

$$\frac{\hat{d}^{>}\hat{A}^{>}}{\hat{d}^{>}t^{>}} = (\hat{A}^{>},\hat{H}^{>}) = \frac{\hat{\partial}^{>}\hat{A}^{>}}{\hat{\partial}^{>}\hat{a}^{>\mu}} > \hat{\omega}^{\mu\nu>} > \frac{\hat{\partial}^{>}\hat{H}^{>}}{\hat{\partial}\hat{a}^{>\nu}} =$$

$$= \frac{\partial\hat{A}^{>}}{\partial\hat{a}^{>\mu}}\times S^{\mu\nu}\times\frac{\partial\hat{H}^{>}}{\partial\hat{a}^{>\nu}} =$$

$$= \Big(\frac{\partial \hat{A}^>}{\partial \hat{r}^{>k}_\alpha} \times \frac{\partial \hat{H}^>}{\partial \hat{p}^>_{ka}} - \frac{\partial \hat{A}^>}{\partial \hat{p}^>_{ka}} \times \frac{\partial \hat{H}^>}{\partial \hat{r}^{>k}_a}\Big) + \frac{\partial \hat{A}^>}{\partial \hat{p}^>_{ka}} \times F^{NSA}_{ka}, \tag{4.3.23a}$$

$$S^{>\mu\nu} = \omega^{\mu\rho} \times \hat{I}^{2\,\mu}_\rho, \omega^{\mu\nu} = (||\omega_{\alpha\beta}||^{-1})^{\mu\nu}, \tag{4.3.23b}$$

where $\omega^{\mu\nu}$ is the conventional Lie tensor and, consequently, $S^{\mu\nu}$ is Lie-admissible in the sense of Albert [7].

As one can see, the important consequence of genomathematics and its genodifferential calculus is that of turning the triple system (A, H, F^{NSA}) of Eqs. (4.1.5) in the bilinear form $(A\hat{,}B)$, thus characterizing a consistent algebra in the brackets of the time evolution.

This is the central purpose for which genomathematics was built (note that the multiplicative factors represented by K are fixed for each given system). The invariance of such a formulation will be proved shortly.

It is an instructive exercise for interested readers to prove that the brackets $(A\hat{,}B)$ are Lie-admissible, although not Jordan-admissible.

It is easy to verify that the above identical reformulation of Hamilton's historical time evolution correctly recovers the *time rate of variations of physical quantities* in general, and that of the energy in particular,

$$\frac{dA^>}{dt} = (A^>, H^>) = [\hat{A}^>, \hat{H}^>] + \frac{\partial \hat{A}^>}{\partial \hat{p}^>_{k\alpha}} \times F^{NSA}_{k\alpha}, \tag{4.3.24a}$$

$$\frac{dH}{dt} = [\hat{H}^>, \hat{H}^>] + \frac{\partial \hat{H}^>}{\partial \hat{p}^>_{k\alpha}} \times F^{NSA}_{ka} = v^k_\alpha \times F^{NSA}_{ka}. \tag{4.3.24b}$$

It is easy to show that genoaction principle (4.3.21) characterizes the following *Hamilton-Jacobi-Santilli genoequations* [14]

$$\frac{\hat{\partial}^> \mathcal{A}^>}{\hat{\partial}^> \hat{t}^>} + \hat{H}^> = 0, \tag{4.3.25a}$$

$$\Big(\frac{\hat{\partial}^> \mathcal{A}^>}{\hat{\partial}^> \hat{a}^{>\mu}}\Big) = \Big(\frac{\hat{\partial}^> \mathcal{A}^>}{\hat{\partial}^> x^{>k}_a}, \frac{\hat{\partial}^> \mathcal{A}^>}{\hat{\partial}^> p^>_{ka}}\Big) = (\hat{R}^>_\mu) = (\hat{p}^>_{ka}, \hat{0}), \tag{4.3.25b}$$

which confirm the property (crucial for genoquantization as shown below) that the genoaction is indeed independent of the linear momentum.

Note the *direct universality* of the Lie-admissible equations for the representation of all infinitely possible Newton equations (4.1.3) (universality) directly in the fixed frame of the experimenter (direct universality).

Note also that, *at the abstract, realization-free level, Hamilton-Santilli genoequations coincide* with Hamilton's equations without external terms, yet represent those with external terms.

The latter are reformulated via genomathematics as the only known way to achieve invariance and derivability from a variational principle while admitting a consistent algebra in the brackets of the time evolution [38].

Therefore, Hamilton-Santilli genoequations (3.6.66) are indeed irreversible for all possible reversible Hamiltonians, as desired. The origin of irreversibility rests in the contact nonpotential forces F^{NSA} according to Lagrange's and Hamilton's teaching that is merely reformulated in an invariant way.

The above Lie-admissible mechanics requires, for completeness, *three* additional formulations, the *backward genomechanics* for the description of *matter moving backward in time,* and the isoduals of both the forward and backward mechanics for the description of *antimatter.*

The construction of these additional mechanics is lefty to the interested reader for brevity.

4.4 LIE-ADMISSIBLE OPERATOR MECHANICS AND ITS ISODUAL

4.4.1 Basic Dynamical Equations

A simple genotopy of the naive or symplectic quantization applied to Eqs. (4.3.24) yields the *Lie-admissible branch of hadronic mechanics* comprising four different formulations, the *forward and backward genomechanics for matter and their isoduals for antimatter.* The forward genomechanics for matter is characterized by the following main topics:

1) The nowhere singular (thus everywhere invertible) non-Hermitean *forward genounit* for the representation of all effects causing irreversibility, such as contact nonpotential interactions among extended particles, etc. (see the subsequent chapters for various realizations)

$$\hat{I}^{>} = 1/\hat{T}^{>} \neq (\hat{I}^{>})^{\dagger}, \tag{4.4.1}$$

with corresponding ordered product and genoreal $\hat{R}^{>}$ and genocomplex $\hat{C}^{>}$ genofields;

2) The *forward genotopic Hilbert space* $\hat{\mathcal{H}}^{>}$ with *forward genostates* $|\hat{\psi}^{>}>$ and *forward genoinner product*

$$<^{<}\hat{\psi}| > |\hat{\psi}^{>}> \times \hat{I}^{>} = <^{<}\hat{\psi}| \times \hat{T}^{>} \times |\hat{\psi}^{>}> \times \hat{I}^{>} \in \hat{C}^{>}, \tag{4.4.2}$$

and fundamental property

$$\hat{I}^{>} > |\hat{\psi}^{>}> = |\hat{\psi}^{>}>, \tag{4.4.3}$$

holding under the condition that $\hat{I}^{>}$ is indeed the correct unit for motion forward in time, and *forward genounitary transforms*

$$\hat{U}^{>} > (^{<}\hat{U})^{\dagger>} = (^{<}\hat{U})^{\dagger>} > \hat{U}^{>} = \hat{I}^{>}; \tag{4.4.4}$$

3) The fundamental Lie-admissible equations, first proposed in Ref. [12] of 1974 (p. 783, Eqs. (4.18.16)) as the foundations of hadronic mechanics, formulated on conventional spaces over conventional fields, and first formulated in Refs. [14,19] of 1996 on genospaces and genodifferential calculus on genofields, today's known as *Heisenberg-Santilli genoequations*, that can be written in the finite form

$$\hat{A}(\hat{t}) = \hat{U}^{>} > \hat{A}(0) <^{<} \hat{U} = (\hat{e}_{>}^{\hat{i}\hat{\times}\hat{H}\hat{\times}\hat{t}}) > \hat{A}(\hat{0}) < ({}_{<}\hat{e}^{-\hat{i}\hat{\times}\hat{t}\hat{\times}\hat{H}}) =$$

$$= (e^{i\times\hat{H}\times\hat{T}^{>}\times t}) \times A(0) \times (e^{-i\times t\times{}^{<}\hat{T}\times\hat{H}}), \qquad (4.4.5)$$

with corresponding infinitesimal version

$$\hat{i}\hat{\times}\frac{\hat{d}\hat{A}}{\hat{d}\hat{t}} = (\hat{A}\hat{,}\hat{H}) = \hat{A} < \hat{H} - \hat{H} > \hat{A} =$$

$$= \hat{A} \times^{<} \hat{T}(\hat{t}, \hat{r}, \hat{p}, \hat{\psi}, \dots .) \times \hat{H} - \hat{H} \times \hat{T}^{>}(\hat{t}, \hat{r}, \hat{p}, \hat{\psi}, \dots) \times \hat{A}, \qquad (4.4.6)$$

where there is no time arrow, since Heisenberg's equations are computed at a fixed time.

4) The equivalent *Schrödinger-Santilli genoequations*, first suggested in the original proposal [12] to build hadroniuc mechanics (see also Refs. [17,23,24]), formulated via conventional mathematics and in Refs. [14,19] via genomathematics, that can be written

$$\hat{i}^{>} > \frac{\hat{\partial}^{>}}{\hat{\partial}^{>}\hat{t}^{>}}|\hat{\psi}^{>} >= \hat{H}^{>} > |\hat{\psi}^{>} >=$$

$$= \hat{H}(\hat{r}, \hat{v}) \times \hat{T}^{>}(\hat{t}, \hat{r}, \hat{p}, \hat{\psi}, \hat{\partial}\hat{\psi} \dots) \times |\hat{\psi}^{>} >= E^{>} > |\psi^{>} >, \qquad (4.4.7)$$

where the time orderings in the second term are ignored for simplicity of notation;

5) The *forward genomomentum* that escaped identification for two decades and was finally identified thanks to the genodifferential calculus in Ref. [14] of 1996

$$\hat{p}_k^{>} > |\hat{\psi}^{>} >= -\hat{i}^{>} > \hat{\partial}_k^{>}|\hat{\psi}^{>} >= -i \times \hat{I}_k^{>i} \times \partial_i|\hat{\psi}^{>} >, \qquad (4.4.8)$$

6) The *fundamental genocommutation rules* also first identified in Ref. [14],

$$(\hat{r}^i\hat{,}\hat{p}_j) = i \times \delta^i_j \times \hat{I}^{>}, \ \ (\hat{r}^i\hat{,}\hat{r}^j) = (\hat{p}_i\hat{,}\hat{p}_j) = 0, \qquad (4.4.9)$$

7) The *genoexpectation values* of an observable for the forward motion $\hat{A}^{>}$ [14,19]

$$\frac{<^{<} \hat{\psi}| > \hat{A}^{>} > |\hat{\psi}^{>} >}{<^{<} \hat{\psi}| > |\hat{\psi}^{>} >} \times \hat{I}^{>} \in \hat{C}^{>}, \qquad (4.4.10)$$

under which the genoexpectation values of the genounit recovers the conventional Planck's unit as in the isotopic case,

$$\frac{<\hat{\psi}| > \hat{I}^{>} > |\hat{\psi}>}{<\hat{\psi}| > |\hat{\psi}>} = I. \tag{4.4.11}$$

The following comments are now in order. Note first in the genoaction principle the crucial independence of isoaction $\hat{\mathcal{A}}^{>}$ in form the linear momentum, as expressed by the Hamilton-Jacobi-Santilli genoequations (4.3.25). Such independence assures that genoquantization yields a genowavefunction solely dependent on time and coordinates, $\hat{\psi}^{>} = \hat{\psi}^{>}(t, r)$.

Other geno-Hamiltonian mechanics studied previously [7] do not verify such a condition, thus implying genowavefunctions with an explicit dependence also on linear momenta, $\hat{\psi}^{>} = \hat{\psi}^{>}(t, r, p)$ that violate the abstract identity of quantum and hadronic mechanics whose treatment in any case is beyond our operator knowledge at this writing.

Note that *forward geno-Hermiticity coincides with conventional Hermiticity.* As a result, *all quantities that are observables for quantum mechanics remain observables for the above genomechanics.*

However, unlike quantum mechanics, physical quantities are generally *nonconserved,* as it must be the case for the energy,

$$\hat{i}^{>} > \frac{\hat{d}^{>}\hat{H}^{>}}{\hat{d}^{>}\hat{t}^{>}} = \hat{H} \times ({}^{<}\hat{T} - \hat{T}^{>}) \times \hat{H} \neq 0. \tag{4.4.12}$$

Therefore, *the genotopic branch of hadronic mechanics is the only known operator formulation permitting nonconserved quantities to be Hermitean as a necessary condition to be observability.*

Other formulation attempt to represent nonconservation, e.g., by adding an "imaginary potential" to the Hamiltonian, as it is often done in nuclear physics [25]. In this case the Hamiltonian is non-Hermitean and, consequently, the nonconservation of the energy cannot be an observable.

Besides, said "nonconservative models" with non-Hermitean Hamiltonians are nonunitary and are formulated on conventional spaces over conventional fields, thus suffering all the catastrophic inconsistencies of Theorem 1.4.2. For additional aspects of genomechanics interested readers may consult Ref. [61].

We should stress the representation of irreversibility and nonconservation beginning with the most primitive quantity, the unit and related product. *Closed irreversible systems* are characterized by the Lie-isotopic subcase in which

$$\hat{i}\hat{\times}\frac{\hat{d}\hat{A}}{\hat{d}\hat{t}} = [\hat{A}\hat{,}\hat{H}] = \hat{A} \times \hat{T}(t, \dots) \times \hat{H} - \hat{H} \times \hat{T}(t, \dots) \times \hat{A}, \tag{4.4.13a}$$

$${}^{<}\hat{T}(t, \dots) = \hat{T}^{>}(t, \dots) = \hat{T}(t, \dots) = \hat{T}^{\dagger}(t, \dots) \neq \hat{T}(-t, \dots), \tag{4.4.13b}$$

for which the Hamiltonian is manifestly conserved. Nevertheless the system is manifestly irreversible. Note also the first and only known observability of the Hamiltonian (due to its iso-Hermiticity) under irreversibility.

As one can see, brackets (A, B) of Eqs. (4.4.6) are jointly Lie- and Jordan-admissible.

Note also that finite genotransforms (4.4.5) verify the condition of genohermiticity, Eq. (4.4.4).

We should finally mention that, as it was the case for isotheories, *genotheories are also admitted by the abstract axioms of quantum mechanics, thus providing a broader realization.* This can be seen, e.g., from the invariance under a complex number C

$$< \psi|x|\psi > \times I =< \psi|xC^{-1} \times |\psi > \times(C \times I) =< \psi| > |\psi > \times I^{>}. \quad (4.4.14)$$

Consequently, *genomechanics provide another explicit and concrete realization of "hidden variables" [26], thus constituting another "completion" of quantum mechanics in the E-P-R sense [27].* For the studies of these aspects we refer the interested reader to Ref. [28].

The above formulation must be completed with three additional Lie-admissible formulations, the backward formulation for matter under time reversal and the two additional isodual formulations for antimatter. Their study is left to the interested reader for brevity.

4.4.2 Simple Construction of Lie-Admissible Theories

As it was the case for the isotopies, a simple method has been identified in Ref. [44] for the construction of Lie-admissible (geno-) theories from any given conventional, classical or quantum formulation. It consists in *identifying the genounits as the product of two different nonunitary transforms,*

$$\hat{I}^{>} = ({}^{<}\hat{I})^{\dagger} = U \times W^{\dagger}, \quad {}^{<}\hat{I} = W \times U^{\dagger}, \quad (4.4.15a)$$

$$U \times U^{\dagger} \neq 1, \quad W \times W^{\dagger} \neq 1, \quad U \times W^{\dagger} = \hat{I}^{>}, \quad (4.4.15b)$$

and subjecting the totality of quantities and their operations of conventional models to said dual transforms,

$$I \rightarrow \hat{I}^{>} = U \times I \times W^{\dagger}, \quad I \rightarrow^{<} \hat{I} = W \times I \times U^{\dagger}, \quad (4.4.16a)$$

$$a \rightarrow \hat{a}^{>} = U \times a \times W^{\dagger} = a \times \hat{I}^{>}, \quad (4.4.16b)$$

$$a \rightarrow^{<} \hat{a} = W \times a \times U^{\dagger} =^{<} \hat{I} \times a, \quad (4.4.16c)$$

$$a \times b \rightarrow \hat{a}^{>} > \hat{b}^{>} = U \times (a \times b) \times W^{>} =$$

$$= (U \times a \times W^{\dagger}) \times (U \times W^{\dagger})^{-1} \times (U \times b \times W^{\dagger}), \quad (4.4.16d)$$

$$\partial/\partial x \rightarrow \hat{\partial}^{>}/\hat{\partial}^{>}\hat{x}^{>} = U \times (\partial/\partial x) \times W^{\dagger} = \hat{I}^{>} \times (\partial/\partial x), \tag{4.4.16e}$$

$$< \psi| \times |\psi > \rightarrow <^{<} \psi| > |\psi^{>} >= U \times (< \psi| \times |\psi >) \times W^{\dagger}, \tag{4.4.16f}$$

$$H \times |\psi > \rightarrow \hat{H}^{>} > |\psi^{>} >=$$
$$= (U \times H \times W^{\dagger}) \times (U \times W^{\dagger})^{-1} \times (U \times \psi > W^{\dagger}), \text{ etc.} \tag{4.4.16g}$$

As a result, any given conventional, classical or quantum model can be easily lifted into the genotopic form.

Note that the above construction implies that *all conventional physical quantities acquire a well defined direction of time.* For instance, the correct genotopic formulation of energy, linear momentum, etc., is given by

$$\hat{H}^{>} = U \times H \times W^{\dagger}, \ \ \hat{p}^{>} = U \times p \times W^{>}, \text{ etc.} \tag{4.4.17}$$

In fact, under irreversibility, the value of a nonconserved energy at a given time t for motion forward in time is generally different than the corresponding value of the energy for $-t$ for motion backward in past times.

This explains the reason for having represented in this section energy, momentum and other quantities with their arrow of time $>$. Such an arrow can indeed be omitted for notational simplicity, but only after the understanding of its existence.

Note finally that a conventional, one dimensional, unitary Lie transformation group with Hermitean generator X and parameter w can be transformed into a covering Lie-admissible group via the following nonunitary transform

$$Q(w) \times Q^{\dagger}(w) = Q^{\dagger}(w) \times Q(w) = I, \ w \in R, \tag{4.4.18a}$$

$$U \times U^{\dagger} \neq I, \quad W \times W^{\dagger} \neq 1, \tag{4.4.18b}$$

$$A(w) = Q(w) \times A(0) \times Q^{\dagger}(w) = e^{X \times w \times i} \times A(0) \times e^{-i \times w \times X} \rightarrow$$
$$\rightarrow U \times (e^{X \times w \times i} \times A(0) \times e^{-i \times w \times X}) \times U^{\dagger} =$$
$$\equiv [U \times (e^{X \times w \times i}) \times W^{\dagger} \times (U \times W^{\dagger})^{-1} \times A \times A(0) \times$$
$$\times U^{\dagger} \times (W \times U^{\dagger})^{-1} \times [W \times (e^{-i \times w \times X}) \times U^{\dagger}] =$$
$$= (e^{i \times X \times X})^{>} > A(0) <^{<} (e^{-1 \times w \times X}) = \hat{U}^{>} > A(0) <^{<} \hat{U}, \tag{4.4.18c}$$

which confirm the property of Section 4.2, namely, that under the necessary mathematics *the Lie-admissible theory is indeed admitted by the abstract Lie axioms, and it is a realization of the latter broader than the isotopic form.*

4.4.3 Invariance of Lie-Admissible Theories

Recall that a fundamental axiomatic feature of quantum mechanics is the invariance under time evolution of all numerical predictions and physical laws, which invariance is due to the *unitary structure* of the theory.

However, quantum mechanics is reversible and can only represent in a scientific way beyond academic beliefs reversible systems verifying total conservation laws due to the antisymmetric character of the brackets of the time evolution.

As indicated earlier, the representation of irreversibility and nonconservation requires theories with a *nonunitary structure*. However, the latter are afflicted by the catastrophic inconsistencies of Theorem 1.5.2.

The only resolution of such a basic impasse known to the author has been the achievement of invariance under nonunitarity and irreversibility via the use of genomathematics, provided that such genomathematics is applied to the *totality* of the formalism to avoid evident inconsistencies caused by mixing different mathematics for the selected physical problem.[5]

Such an invariance was first achieved by Santilli in Ref. [44] of 1997 and can be illustrated by reformulating any given nonunitary transform in the *genounitary form*

$$U = \hat{U} \times \hat{T}^{>1/2}, W = \hat{W} \times \hat{T}^{>1/2}, \tag{4.4.19a}$$

$$U \times W^\dagger = \hat{U} > \hat{W}^\dagger = \hat{W}^\dagger > \hat{U} = \hat{I}^> = 1/\hat{T}^>, \tag{4.4.19b}$$

and then showing that genounits, genoproducts, genoexponentiation, etc., are indeed invariant under the above genounitary transform in exactly the same way as conventional units, products, exponentiations, etc. are invariant under unitary transforms,

$$\hat{I}^> \to \hat{I}^{>\prime} = \hat{U} > \hat{I}^> > \hat{W}^\dagger = \hat{I}^>, \tag{4.4.20a}$$

$$\begin{aligned} \hat{A} > \hat{B} \to \hat{U} > (A > B) > \hat{W}^\dagger = \\ = (\hat{U} \times \hat{T}^> \times A \times T^> \times \hat{W}^\dagger) \times (\hat{T}^> \times W^\dagger)^{-1} \times \hat{T}^> \times \\ \times (\hat{U} \times \hat{T}^>)^{-1} \times (\hat{U} \times T^> \times \hat{A} \times T^> \times \hat{W}^>) = \\ = \hat{A}' \times (\hat{U} \times \hat{W}^\dagger)^{-1} \times \hat{B} = \hat{A}' \times \hat{T}^> \times B' = \hat{A}' > \hat{B}', \text{ etc.} \end{aligned} \tag{4.4.20b}$$

[5] Due to decades of protracted use it is easy to predict that physicists and mathematicians may be tempted to treat the Lie-admissible branch of hadronic mechanics with conventional mathematics, whether in part or in full. Such a posture would be fully equivalent, for instance, to the elaboration of the spectral emission of the hydrogen atom with the genodifferential calculus, resulting in an evident nonscientific setting.

from which all remaining invariances follow, thus resolving the catastrophic inconsistencies of Theorem 1.5.2.

Note the *numerical invariances of the genounit* $\hat{I}^{>} \rightarrow \hat{I}^{>\prime} \equiv \hat{I}^{>}$, *of the genotopic element* $\hat{T}^{>} \rightarrow \hat{T}^{>\prime} \equiv \hat{T}^{>}$, *and of the genoproduct* $>\rightarrow>' \equiv>$ that are necessary to have invariant numerical predictions.

4.5 LORENTZIAN AND GALILEAN GENORELATIVITIES AND THEIR ISODUALS

TO BE COMPLETED

Another important implication of genomathematics is the construction of yet another lifting of special relativity, this time intended for the invariant characterization of irreversible classical, quantum and gravitational processes, today known as *Santilli's genorelativity.*

Studies in the new relativity were initiated with memoir [23] of 1978[6] and continued in monographs [49,50]. The studies were then continued via the genotopies of: the background Euclidean topology [14]; the Minkowski space [15]; the Poincaré symmetry [29]; the physical laws; etc. The geno-Galilean case is treated in monographs [52,53] which appeared prior to the advent of the genodifferential calculus [14]. The relativistic case is outlined in Ref. [29].

Regrettably, we cannot review genorelativity in details to avoid a prohibitive length. For the limited scope of this presentation it is sufficient to indicate that genorelativity can be also constructed from the isorelativity of the preceding section via the lifting of the isounits into time dependent and/or nonsymmetric forms, with consequential selection of an ordering of the product to identify the selection direction of time.

Alternatively, all aspects of genorelativity can be explicitly constructed by subjecting the corresponding aspects of conventional special relativity to the dual noncanonical or nonunitary transform, as of Section 3.6.

The result is a fully invariant description of irreversible and nonconservative processes in classical mechanics, particle physics and gravitation. Note that the latter is achieved thanks to the first known admission of a nonsymmetric metric in the genotopic realization of the Minkowskian axioms, as necessary for a credible representation of irreversible gravitational events, such as the explosion of a star.

Note finally that, as it was the case for isorelativity, all distinctions between special and general relativity are lost also for genorelativity because the two

[6]This memoir contains the first generalization of Noether's Theorem on Lie symmetries and conservation laws to Lie-admissible symmetries and nonconservation laws. The indication by interested colleagues of any prior representation of nonconservation laws via any symmetry would be appreciated.

relativities are again unified into one single relativity verifying the same basic axioms, and merely differentiated via different realizations of the basic unit.

As it is well known, throughout the 20-th century thermodynamics has been basically disjoint from Hamiltonian mechanics precisely because the former is strictly irreversible, e.g., for the increase of the entropy in realistic systems, while the latter is strictly reversible.

It appears that the Lie-admissible classical and operator genomechanics presented in this section change the above setting and offer, apparently for the first time, realistic possibility for an interconnection between thermodynamics and mechanics, according to studies left to the interested reader.

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