Chapter 2

ISODUAL THEORY OF POINT-LIKE ANTIPARTICLES

2.1 ELEMENTS OF ISODUAL MATHEMATICS

2.1.1 Isodual Unit, Isodual Numbers and Isodual Fields

The first comorehensive study of the isodual theory for point-like antiparticles has been presented by the author in monograph [34]. However, the field is subjected to continuous developments following its first presentation in papers [1] of 1985. Hence, it is important to review the most recent formulation of the isodual mathematics in sufficient details to render this monograph selfsufficient.

In this section, we identify only those aspects of isodual mathematics that are essential for the understanding of the physical profiles presented in the subsequent sections of this chapter. We begin with a study of the most fundamental elements of all mathematical and physical formulations, units, numbers and fields, from which all remaining formulations can be uniquely and unambiguously derived via simple compatibility arguments. To avoid un-necessary repetitions, we assume the reader has a knowledge of the basic mathematics used for the classical and operator treatment of matter.

DEFINITION 2.1.1: Let $F = F(a, +, \times)$ be a field (of characteristic zero), namely a ring with elements given by real number $a = n, F = R(n, +, \times)$, complex numbers $A = c, F = C(c, +, \times)$, or quaternionic numbers $a = q, F = Q(q, +, \times)$, with conventional sum $a + b$ verifying the commutative law

$$a + b = b + a = c \in F, \tag{2.1.1}$$

the associative law

$$(a + b) + c = a + (b + c) = d \in F, \tag{2.1.2}$$

conventional product $a \times b$ *verifying the associative law*

$$(a \times b) \times c = a \times (b \times c) = e \in F, \tag{2.1.3}$$

(but not necessarily the commutative law, $a \times b \neq b \times a$ *since the latter is violated by quaternions), and the right and left distributive laws*

$$(a + b) \times c = a \times c + b \times c = f \in F, \tag{2.1.4a}$$

$$a \times (b + c) = a \times b + a \times c = g \in F, \tag{2.1.4b}$$

left and right additive unit 0*,*

$$a + 0 = 0 + a = a \in F, \tag{2.1.5}$$

and left and right multiplicative unit I*,*

$$a \times I = I \times a = a \in F, \tag{2.1.6}$$

$\forall a, b, c \in F$*. Santilli's isodual fields (first introduced in* Refs. [1] *and then presented in details in* Ref. [2]*) are rings* $F^d = F^d(a^d, +^d, \times^d)$ *with elements given by isodual numbers*

$$a^d = -a^\dagger, \ a^d \in F, \tag{2.1.7}$$

with associative and commutative isodual sum

$$a^d +^d b^d = -(a + b)^\dagger = c^d \in F^d, \tag{2.1.8}$$

associative and distributive isodual product

$$a^d \times^d b^d = a^d \times (I^d)^{-1} \times b^d = c^d \in F^d, \tag{2.1.9}$$

additive isodual unit $0^d = 0$*,*

$$a^d +^d 0^d = 0^d +^d a^d = a^d, \tag{2.1.10}$$

and multiplicative isodual unit $I^d = -I^\dagger$*,*

$$a^d \times^d I^d = I^d \times^d a^d = a^d, \forall a^d, b^d \in F^d. \tag{2.1.11}$$

The proof of the following property is elementary.

LEMMA 2.1.1 [1,2]: *Isodual fields are fields, namely, if* F *is a field, its image* F^d *under the isodual map is also a field.*

The above lemma establishes the property (first identified in Refs. [1]) that *the axioms of a field do not require that the multiplicative unit be necessarily positive-definite, because the same axioms are also verified by negative-definite units.* The proof of the following property is equally simple.

LEMMA 2.1.2 [1,2]: *Fields F and their isodual images F^d are anti-isomorphic to each other.*

Lemmas 2.1.1 and 1.2.2 illustrate the origin of the name "isodual mathematics". In fact, to represent antimatter the needed mathematics must be a suitable "dual" of conventional mathematics, while the prefix "iso" is used in its Greek meaning of preserving the original axioms.

It is evident that for real numbers we have

$$n^d = -n, \tag{2.1.12}$$

while for complex numbers we have

$$c^d = (n_1 + i \times n_2)^d = -n_1 + i \times n_2 = -\bar{c}, \tag{2.1.13}$$

with a similar formulation for quaternions.

It is also evident that, for consistency, *all operations on numbers must be subjected to isoduality when dealing with isodual numbers.* This implies: the *isodual powers*

$$(a^d)^{n^d} = a^d \times^d a^d \times^d a^d ... \tag{2.1.14}$$

(n times, with n an integer); the *isodual square root*

$$a^{d(1/2)d} \; = \; -\sqrt{-a^\dagger}^\dagger, a^{d(1/2)d} \times^d a^{d(1/2)d} \; = \; a^d, \quad 1^{d(1/2)d} \; = \; -i; \tag{2.1.15}$$

the *isodual quotient*

$$a^d /^d b^d \; = \; -(a^\dagger / b^\dagger) \; = \; c^d, \;\; b^d \times^d c^d \; = \; a^d; \tag{2.1.16}$$

etc.

An important property for the characterization of antimatter is the following:

LEMMA 2.1.3. [2]: *isodual fields have a negative–definite norm, called isodual norm,*

$$|a^d|^d \; = \; |a^\dagger| \times I^d \; = \; -(a a^\dagger)^{1/2} \; < \; 0, \tag{2.1.17}$$

where $|...|$ denotes the conventional norm.

For isodual real numbers we therefore have the isodual isonorm

$$|n^d|^d \; = \; -|n| < 0, \tag{2.1.18}$$

and for isodual complex numbers we have

$$|c^d|^d = -|\bar{c}| = -(c\bar{c})^{1/2} = -(n_1^2 + n_2^2)^{1/2}. \qquad (2.1.19)$$

LEMMA 2.1.4 [2]: *All quantities that are positive–definite when referred to positive units and related fields of matter (such as mass, energy, angular momentum, density, temperature, time, etc.) became negative–definite when referred to isodual units and related isodual fields of antimatter.*

As recalled Chapter 1, antiparticles have been discovered in the *negative-energy solutions* of Dirac's equation and they were originally thought to evolve *backward in time* (Stueckelberg, Feynman, and others, see Refs. [1,2] of Chapter 1). The possibility of representing antiparticles via isodual methods is therefore visible already from these introductory notions.

The main novelty is that the conventional treatment of negative–definite energy and time was (and still is) referred to the conventional unit +1. This leads to a number of contradictions in the physical behavior of antiparticles.

By comparison, *negative-definite physical quantities of isodual theories are referred to a negative–definite unit* $I^d < 0$. This implies a mathematical and physical equivalence between *positive–definite quantities referred to positive–definite units, characterizing matter, and negative–definite quantities referred to negative–definite units, characterizing antimatter.* These foundations then permit a novel characterization of antimatter beginning at the *Newtonian* level, and then persisting at all subsequent levels.

DEFINITION 2.1.2 [2]: *A quantity is called isoselfdual when it coincides with its isodual.*

It is easy to verify that the imaginary unit is isoselfdual because

$$i^d = -i^\dagger = -\bar{i} = -(-i) = i. \qquad (2.1.20)$$

This property permits a better understanding of the isoduality of complex numbers that can be written explicitly

$$c^d = (n_1 + i \times n_2)^d = n_1^d + i^d \times^d n_2^d = -n_1 + i \times n_2 = -\bar{c}. \qquad (2.1.21)$$

The above property will be important to prove the equivalence of isoduality and charge conjugation at the operator level.

As we shall see, *isoselfduality is a new fundamental view of nature* with deep physical implications, not only in classical and quantum mechanics but also in cosmology. For instance we shall see that Dirac's gamma matrices are isoselfdual, thus implying a basically new interpretation of this equation that has remained unidentified for about one century. We shall also see that, when applied to cosmology, isoselfduality implies equal distribution of matter and antimatter in the universe, with identically null total physical characteristic, such as identically null total time, identically null total mass, etc.

We assume the reader is aware of the emergence here of *new numbers,* those with a negative unit, that have no connection with ordinary negative numbers and are the true foundations of the isodual theory of antimatter.

2.1.2 Isodual Functional Analysis

All conventional and special functions and transforms, as well as functional analysis at large, must be subjected to isoduality for consistent applications, resulting in the simple, yet unique and significant *isodual functional analysis*, studied by Kadeisvili [3], Santilli [4] and others.

We here mention the *isodual trigonometric functions*

$$\sin^d \theta^d \ = \ -\sin(-\theta), \ \ \cos^d \theta^d \ = \ -\cos(-\theta), \tag{2.1.22}$$

with related basic property

$$\cos^{d\,2d} \theta^d +^d \sin^{d\,2d} \theta^d \ = \ 1^d \ = \ -1, \tag{2.1.23}$$

the *isodual hyperbolic functions*

$$\sinh^d w^d \ = \ -\sinh(-w), \ \cosh^d w^d \ = \ -\cosh(-w), \tag{2.1.24}$$

with related basic property

$$\cosh^{d\,2d} w^d -^d \sinh^{d\,2d} w^d \ = \ 1^d \ = \ -1, \tag{2.1.25}$$

the *isodual logarithm* and the *isodual exponentiation* defined respectively by

$$\log^d n^d \ = \ -\log(-n), \tag{2.1.26a}$$

$$e_d^{X^d} = 1^d + X^d/^d 1!^d + X^{d^{2d}}/^d 2!^d + ... = -e^X, \tag{2.1.26b}$$

etc. Interested readers can then easily construct the isodual image of special functions, transforms, distributions, etc.

2.1.3 Isodual Differential and Integral Calculus

Contrary to a rather popular belief, the differential calculus is indeed dependent on the assumed unit. This property is not so transparent in the conventional formulation because the basic unit is the trivial number $+1$. However, the dependence of the unit emerges rather forcefully under its generalization.

The *isodual differential calculus*, first introduced by Santilli in Ref. [5a], is characterized by the *isodual differentials*

$$d^d x^k \ = \ I^d \times dx^k \ = \ -dx^k, \ \ d^d x_k \ = \ -dx_k, \tag{2.1.27}$$

with corresponding *isodual derivatives*

$$\partial^d /^d \partial^d x^k \ = \ -\partial/\partial x^k, \ \ \partial^d /^d \partial^d x_k \ = \ -\partial/\partial x, \tag{2.1.28}$$

and related isodual properties.

Note that *conventional differentials are isoselfdual*, i.e.,

$$(dx^k)^d \ = \ d^d x^{kd} \equiv d\, x^k, \tag{2.1.29}$$

but *derivatives are not isoselfdual*,

$$[\partial f/\partial x^k]^d \ = \ -\partial^d f^d /^d \partial^d x^{kd}. \tag{2.1.30}$$

The above properties explain why the isodual differential calculus remained undiscovered for centuries.

Other notions, such as the *isodual integral calculus,* can be easily derived and shall be assumed as known hereon.

2.1.4 Lie-Santilli Isodual Theory

Let $\mathbf{L}$ be an n–dimensional Lie algebra in its regular representation with universal enveloping associative algebra $\xi(\mathbf{L})$, $[\xi(\mathbf{L})]^- \approx \mathbf{L}$, n-dimensional unit $I = Diag.(1, 1, ..., 1)$, ordered set of Hermitean generators $X = X^\dagger = \{X_k\}$, $k = 1, 2, ..., n$, conventional associative product $X_i \times X_j$, and familiar Lie's Theorems over a field $F(a, +, \times)$.

The *Lie-Santilli isodual theory* was first submitted in Ref. [1] and then studied in Refs. [4-7] as well as by other authors [23-31]. *The isodual universal associative algebra* $[\xi(\mathbf{L})]^d$ is characterized by the *isodual unit* I^d, *isodual generators* $X^d = -X$, and isodual associative product

$$X_i^d \times^d X_j^d \ = \ -X_i \times X_j, \tag{2.1.31}$$

with corresponding infinite–dimensional basis characterized by the *Poincaré–Birkhoff–Witt-Santilli isodual theorem*

$$I^d, X_i^d \times^d X_j^d, \ \ i \leq j; \ \ X_i^d \times^d X_j^d \times X_k^d, \ i \leq j \leq k, ... \tag{2.1.32}$$

and related *isodual exponentiation* of a generic quantity A^d

$$e^{d A^d} \ = \ I^d + A^d /^d 1!^d + A^d \times^d A^d /^d 2!^d + ... \ = \ -e^{A^\dagger}, \tag{2.1.33}$$

where e is the conventional exponentiation.

The attached *Lie-Santilli isodual algebra* $\mathbf{L}^d \approx (\xi^d)^-$ over the isodual field $F^d(a^d, +^d, \times^d)$ is characterized by the *isodual commutators* [1]

$$[X_i^d, {}^d X_j^d] \ = \ -[X_i, X_j] \ = \ C_{ij}^{k^d} \times^d X_k^d. \tag{2.1.34}$$

with classical realizations given in Section 2.2.6.

Let G be a conventional, connected, n–dimensional Lie transformation group on a metric (or pseudo-metric) space $S(x,\, g,\, F)$ admitting $\mathbf{L}$ as the Lie algebra in the neighborhood of the identity, with generators X_k and parameters $w \ = \ \{w_k\}$.

The *Lie-Santilli isodual transformation group* G^d admitting the isodual Lie algebra $\mathbf{L}^d$ in the neighborhood of the isodual identity I^d is the n–dimensional group with generators $X^d \ = \ \{-X_k\}$ and parameters $w^d \ = \ \{-w_k\}$ over the isodual field F^d with generic element [1]

$$U^d(w^d) \ = \ e^{d^{i^d \times^d w^d \times^d X^d}} \ = \ -e^{i\times(-w)\times X} \ = \ -U(-w). \tag{2.1.35}$$

The *isodual symmetries* are then defined accordingly via the use of the isodual groups G^d and they are anti–isomorphic to the corresponding conventional symmetries, as desired. For additional details, one may consult Ref. [4,5b].

In this chapter we shall therefore use the *conventional Poincaré, internal and other symmetries* for the characterization of *matter*, and the *Poincaré-Santilli, internal and other isodual symmetries* for the characterization of *antimatter.*

2.1.5 Isodual Euclidean Geometry

Conventional (vector and) metric spaces are defined over conventional fields. It is evident that the isoduality of fields requires, for consistency, a corresponding isoduality of (vector and) metric spaces. The need for the isodualities of all quantities acting on a metric space (e.g., conventional and special functions and transforms, differential calculus, etc.) becomes then evident.

DEFINITION 2.1.3: Let $S \ = \ S(x, g, R)$ be a conventional N-dimensional metric or pseudo-metric space with local coordinates $x = \{x^k\}$, $k = 1, 2, ..., N$, nowhere degenerate, sufficiently smooth, real–valued and symmetric metric $g(x, ...)$ and related invariant

$$x^2 \ = \ (x^i \times g_{ij} \times x^j) \times I, \tag{2.1.36}$$

over the reals R. The isodual spaces, first introduced in Ref. [1] (see also Refs. [4,5] and, for a more recent account, Ref. [22]), *are the spaces $S^d(x^d, g^d, R^d)$ with isodual coordinates $x^d \ = \ x^d = -x^t$ (where t stands for transposed), isodual metric*

$$g^d(x^d, ...) \ = \ -g^\dagger(-x^\dagger, ...) \ = \ -g(-x^t, ...), \tag{2.1.37}$$

and isodual interval

$$(x-y)^{d2\,d} \ = \ [(x-y)^{id} \times^d g^d_{ij} \times^d (x-y)^{jd}] \times I^d =$$

$$= [(x-y)^i \times g^d_{ij} \times (x-y)^j] \times I^d, \quad (2.1.38)$$

defined over the isodual field $R^d = R^d(n^d, +^d, \times^d)$ *with the same isodual isounit* I^d.

The basic nonrelativistic space of our analysis is the three–dimensional *isodual Euclidean space* [1,9],

$$E^d(r^d, \delta^d, R^d) : r^d = \{r^{kd}\} = \{-r^k\} = \{-x, -y, -z\}, \quad (2.1.39a)$$

$$\delta^d = -\delta = diag.(-1,-1,-1),$$
$$I^d = -I = Diag.(-1,-1,-1). \quad (2.1.39b)$$

The *isodual Euclidean geometry* is the geometry of the isodual space E^d over R^d and it is given by a step–by–step isoduality of all the various aspects of the conventional geometry (see monograph [5a] for details).

By recalling that the norm on R^d is negative–definite, the *isodual distance* among two points on an isodual line is also negative definite and it is given by

$$D^d = D \times I^d = -D, \quad (2.1.40)$$

where D is the conventional distance. Similar isodualities apply to all remaining notions, including the notions of parallel and intersecting isodual lines, the Euclidean axioms, etc.

The *isodual sphere* with radius $R^d = -R$ is the perfect sphere on E^d over R^d and, as such, it has *negative radius* (Figure 2.1),

$$R^{d2d} = (x^{d2d} + y^{d2d} + z^{d2d}) \times I^d =$$
$$= (x^2 + y^2 + z^2) \times I = R^2. \quad (2.1.41)$$

Note that the above expression coincides with that for the conventional sphere. This illustrates the reasons, following about one century of studies, the isodual rotational group and symmetry where identified for the first time in Refs. [1]. Note, however, that the latter result required the prior discovery of *new numbers,* those with a negative unit.

A similar characterization holds for other isodual shapes characterizing antimatter in our isodual theory.

LEMMA 2.1.5: The isodual Euclidean geometry on E^d *over* R^d *is anti–isomorphic to the conventional geometry on* E *over* R.

The group of isometries of E^d over R^d is the *isodual Euclidean group* $E^d(3) = \mathcal{R}^d(\theta^d) \times^d T^d(3)$ where $\mathcal{R}^d(\theta)$ is the isodual group of rotations first introduced in Ref. [1], and $T^(3)$ is the isodual group of translations (see also Ref. [5a] for details).

2.1.6 Isodual Minkowskian Geometry

Let $M(x, \eta, R)$ be the conventional Minkowski spacetime with local coordinates $x = (r^k, t) = (x^\mu), k = 1, 2, 3,\ \mu = 1, 2, 3, 4$, metric $\eta = Diag.(1, 1, 1, -1)$ and basic unit $I = Diag.(1, 1, 1, 1)$ on the reals $R = R(n, +, \times)$.

The *Minkowski-Santilli isodual spacetime*, first introduced in Ref. [7] and studied in details in Ref. [8], is given by

$$M^d(x^d, \eta^d, R^d) : x^d \ = \ \{x^{\mu d}\} \ = \ \{x^\mu \times I^d\} \ = \ \{-r, -c_o t\} \times I, \tag{2.1.42}$$

with isodual metric and isodual unit

$$\eta^d \ = \ -\eta \ = \ diag.(-1, -1, -1, +1), \tag{2.1.43a}$$

$$I^d \ = \ Diag.(-1, -1, -1, -1). \tag{2.1.43b}$$

The *Minkowski-Santilli isodual geometry* [8] is the geometry of isodual spaces M^d over R^d. The new geometry is also characterized by a simple isoduality of the conventional Minkowskian geometry as studied in details in memoir.

The fundamental symmetry of this chapter is given by the group of isometries of M^d over R^d, namely, the *Poincaré-Santilli isodual symmetry* [7,8]

$$P^d(3.1) \ = \ \mathcal{L}^d(3.1) \times T^d(3.1) \tag{2.1.44}$$

where $\mathcal{L}^d(3.1)$ is the Lorentz-Santilli isodual group and $T^d(3.1)$ is the isodual group of translations.

2.1.7 Isodual Riemannian Geometry

Consider a Riemannian space $\Re(x, g, R)$ in (3+1) dimensions [32] with basic unit $I \ = \ Diag.(1, 1, 1, 1)$, nowhere singular and symmetric metric $g(x)$ and related Riemannian geometry in local formulation (see, e.g., Ref. [27]).

The *Riemannian-Santilli isodual spaces* (first introduced in Ref. [11]) are given by

$$\begin{aligned} &\Re^d(x^d, g^d, R^d) : \ \ x^d \ = \ \{-x^\mu\}, \\ &g^d \ = -g(x), \ g \in \Re(x, g, R), \\ &I^d \ = \ Diag.(-1, -1, -1, -1) \end{aligned} \tag{2.1.45}$$

with interval

$$\begin{aligned} x^{2d} \ &= \ [x^{dt} \times^d g^d(x^d) \times^d x^d] \times I^d = \\ &= [x^t \times g^d(x^d) \times x] \times I^d \in R^d, \end{aligned} \tag{2.1.46}$$

where t stands for transposed.

The *Riemannian-Santilli isodual geometry* [8] is the geometry of spaces $\Re^d$ over R^d, and it is also given by step–by–step isodualities of the conventional

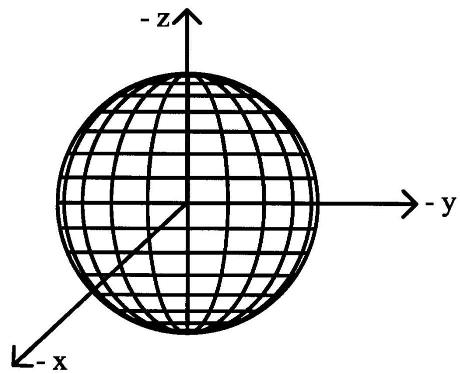

Figure 2.1. A schematic view of the isodual sphere on isodual Euclidean spaces over isodual fields. The understanding of the content of this chapter requires the knowledge that the isodual sphere and the conventional sphere coincide when inspected by an observer either in the Euclidean or in the isodual Euclidean space, due to the identity of the related expressions (2.1.36) and (2.1.38). This identity is at the foundation of the perception that antiparticles "appear" to exist in our space, while in reality they belong to a structurally different space coexisting within our own, thus setting the foundations of a "multidimensional universe" coexisting in the same space of our sensory perception. The reader should keep in mind that the isodual sphere is the idealization of the shape of an antiparticle used in this monograph.

geometry, including, most importantly, the isoduality of the differential and exterior calculus.

As an example, an *isodual vector field* $X^d(x^d)$ on $\Re^d$ is given by $X^d(x^d) = -X^t(-x^t)$. The *isodual exterior differential* of $X^d(x^d)$ is given by

$$D^d X^{kd}(x^d) \;=\; d^d X^{kd}(x^d) + \Gamma^{dk}_{i\,j} \times^d X^{id} \times^d d^d x^{jd} \;=\; DX^k(-x), \quad (2.1.47)$$

where the Γ^d's are the components of the *isodual connection*. The *isodual covariant derivative* is then given by

$$X^{id}(x^d)_{|^dk} \;=\; \partial^d X^{id}(x^d)/^d \partial^d x^{kd} + \Gamma^{di}_{jk} \times^d X^{jd}(x^d) \;=\; -X^i(-x)_{|^k}. \quad (2.1.48)$$

The interested reader can then easily derive the isoduality of the remaining notions of the conventional geometry.

It is an instructive exercise for the interested reader to work out in detail the proof of the following:

LEMMA 2.1.6 [8]: *The isodual image of a Riemannian space* $\Re^d(x^d, g^d, R^d)$ *is characterized by the following maps*:

Basic Unit

$$I \;\rightarrow\; I^d \;=\; -I,$$

$$Metric$$

$$g \;\rightarrow\; g^d \;=\; -g, \tag{2.1.49a}$$

$$Connection \quad Coefficients$$

$$\Gamma_{klh} \;\rightarrow\; \Gamma^d_{klh} \;=\; -\Gamma_{klh}, \tag{2.1.49b}$$

$$Curvature \quad Tensor$$

$$R_{lijk} \;\rightarrow\; R^d_{lijk} \;=\; -R_{lijk}, \tag{2.1.49c}$$

$$Ricci \quad Tensor$$

$$R_{\mu\nu} \;\rightarrow R^d_{\mu\nu} \;=\; -R_{\mu\nu}, \tag{2.1.49d}$$

$$Ricci \quad Scalar$$

$$R \;\rightarrow\; R^d \;=\; R, \tag{2.1.49e}$$

$$Einstein - Hilbert \quad Tensor$$

$$G_{\mu\nu} \;\rightarrow\; G^d_{\mu\nu} \;=\; -G_{\mu\nu}, \tag{2.1.49f}$$

$$Electromagnetic \quad Potentials$$

$$A_\mu \;\rightarrow\; A^d_\mu \;=\; -A_\mu, \tag{2.1.49g}$$

$$Electromagnetic \quad Field$$

$$F_{\mu\nu} \;\rightarrow\; F^d_{\mu\nu} \;=\; -F_{\mu\nu}, \tag{2.1.49h}$$

$$ElmEnergy - Momentum \quad Tensor$$

$$T_{\mu\nu} \;\rightarrow\; T^d_{\mu\nu} \;=\; -T_{\mu\nu}, \tag{2.1.49i}$$

In summary, the geometries significant for this study are: the *conventional Euclidean, Minkowskian and Riemannian geometries* used for the characterization of *matter*; and the *isodual Euclidean, Minkowskian and Riemannian geometries* used for the characterization of *antimatter*.

The reader can now begin to see the achievement of axiomatic compatibility between gravitation and electroweak interactions that is permitted by the isodual theory of antimatter. In fact, the latter is treated via negative-definite energy-momentum tensors, thus being compatible with the negative-energy solutions of electroweak interactions, therefore setting correct axiomatic foundations for a true grand unification studied in the next chapter.

2.2 CLASSICAL ISODUAL THEORY OF POINT-LIKE ANTIPARTICLES

2.2.1 Basic Assumptions

Thanks to the preceding study of isodual mathematics, we are now sufficiently equipped to resolve the scientific impasse caused by the absence of a classical theory of antimatter studied in Section 1.1.

As it is well known, the contemporary treatment of matter is characterized by *conventional mathematics*, here referred to ordinary numbers, fields, spaces, etc. with *positive units and norms*, thus having positive characteristics of mass, energy, time, etc.

In this chapter we study the *characterization of antimatter via isodual numbers, fields, spaces, etc., thus having negative–definite units and norms*. In particular, all characteristics of matter (and not only charge) change sign for antimatter when represented via isoduality.

The above characterization of antimatter evidently provides the correct conjugation of the charge at the desired classical level. However, by no means, the sole change of the sign of the charge is sufficient to ensure a consistent classical representation of antimatter. To achieve consistency, the theory must resolve the main problematic aspect of current classical treatments, the fact that their operator image is not the correct charge conjugate state (Section 2.1).

The above problematic aspect is indeed resolved by the isodual theory. The main reason is that, jointly with the conjugation of the charge, isoduality also conjugates *all* other physical characteristics of matter. This implies *two* channels of quantization, the conventional one for matter and a new *isodual quantization* for antimatter (see Section 2.3) in such a way that its operator image is indeed the charge conjugate of that of matter.

In this section, we study the physical consistency of the theory in its classical formulation. The novel isodual quantization, the equivalence of isoduality and charge conjugation and related operator issues are studied in the next section.

Beginning our analysis, we note that the isodual theory of antimatter resolves the traditional obstacles against negative energies and masses. In fact, *particles with negative energies and masses measured with negative units are fully equivalent to particles with positive energies and masses measured with positive units*. This result has permitted the elimination of sole use of second quantization for the characterization of antiparticles because antimatter becomes treatable at *all* levels, including second quantization.

The isodual theory of antimatter also resolves the additional, well known, problematic aspects of motion backward in time. In fact, *time moving backward measured with a negative unit is fully equivalent on grounds of causality to time moving forward measured with a positive unit.*

This confirms the plausibility of the first conception of antiparticles by Stueckelberg and others as moving backward in time (see the historical analysis in Ref. [1] of Chapter 1), and creates new possibilities for the ongoing research on the so-called "spacetime machine" studied in Chapter 5.

In this section, we construct the classical isodual theory of antimatter at the Newtonian, Lagrangian, Hamiltonian, Galilean, relativistic and gravitational levels; we prove its axiomatic consistency; and we verify its compatibility with available classical experimental evidence (that dealing with electromagnetic interactions only). Operator formulations and their experimental verifications will be studied in the next section.

2.2.2 Need for Isoduality to Represent All Time Directions

It is popularly believed that time has only two directions, the celebrated *Eddington's time arrows.* In reality, *time has four different directions* depending on whether motion is forward or backward and occurs in the future or in the past, as illustrated in Figure 2.2. In turn, the correct use of all four different directions of time is mandatory, for instance, in serious studies of bifurcations, as we shall see.

It is evident that theoretical physics of the 20-th century could not explain all four directions of time, since it possessed only one conjugation, time reversal, and this explains the reason the two remaining directions of time were ignored.

It is equally evident that isoduality does indeed permit the representation of the two missing directions of time, thus illustrating its need.

We assume the reader is now familiar with the differences between time reversal and isoduality. Time reversal changes the direction of time while keeping the underlying space and units unchanged, while isoduality changes the direction of time while mapping the underlying space and units into different forms.

Unless otherwise specified, through the rest of this volume time t will be indicate *motion forward toward in future times,* $-t$ will indicate *motion backward in past times,* t^d will indicate motion backward from future times, and $-t^d$ will indicate *motion forward from past times.*

2.2.3 Experimental Verification of the Isodual Theory of Antimatter in Classical Physics

The experimental verification of the isodual theory of antimatter at the *classical* level is provided by the compliance of the theory with the only available experimental data, those on Coulomb interactions.

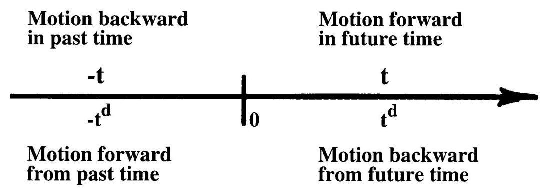

Figure 2.2. A schematic view of the "four different directions of time", depending on whether motion is forward or backward and occurs in the future or in the past. Due to the sole existence of one time conjugation, time reversal, the theoretical physics of the 20-th century missed two of the four directions of time, resulting in fundamental insufficiencies ranging from the lack of a deeper understanding of antiparticles to basic insufficiencies in biological structures and excessively insufficient cosmological views. It is evident that isoduality can indeed represent the two missing time arrows and this illustrates a basic need for the isodual theory.

For that purpose, let us consider the Coulomb interactions under the customary notation that *positive (negative) forces represent repulsion (attraction)* when formulated in conventional Euclidean space.

Under such an assumption, the *repulsive* Coulomb force among two *particles* of negative charges $-q_1$ and $-q_2$ in Euclidean space $E(r, \delta, R)$ is given by

$$F \ = \ K \times (-q_1) \times (-q_2)/r \times r > 0, \tag{2.2.1}$$

where K is a positive constant whose explicit value (here irrelevant) depends on the selected units, the operations of multiplication $\times$ and division $/$ are the conventional ones of the underlying field $R(n, +, \times)$.

Under isoduality to $E^d(r^d, \delta^d, R^d)$ the above law is mapped into the form

$$F^d \ = \ K^d \times^d (-q_1)^d \times^d (-q_2)^d /^d r^d \times^d r^d \ = \ -F < 0, \tag{2.2.2}$$

where $\times^d = -\times$ and $/^d = -/$ are the isodual operations of the underlying field $R^d(n^d, +, \times^d)$.

But the isodual force $F^d \ = \ -F$ occurs in the isodual Euclidean space and it is, therefore, defined with respect to the unit -1. This implies that the reversal of the sign of a repulsive force measured with a negative unit also describes repulsion. As a result, isoduality correctly represents the *repulsive* character of the Coulomb force for two *antiparticles* with *positive* charges, a result first achieved in Ref. [9].

The formulation of the cases of two particles with positive charges and their antiparticles with negative charges is left to the interested reader.

The Coulomb force between a *particle* and an *antiparticle* can only be computed by *projecting the antiparticle in the conventional space of the particle* *or vice-versa.* In the former case we have

$$F \;=\; K \times (-q_1) \times (-q_2)^d / r \times r < 0, \tag{2.2.3}$$

thus yielding an *attractive* force, as experimentally established. In the projection of the particle in the isodual space of the antiparticle, we have

$$F^d \;=\; K^d \times^d (-q_1) \times^d (-q_2)^d /^d r^d \times^d r^d > 0. \tag{2.2.4}$$

But this force is now measured with the unit -1, thus resulting in being again *attractive.*

The study of Coulomb interactions of magnetic poles and other classical experimental data is left to the interested reader.

In conclusion, the isodual theory of antimatter correctly represents all available classical experimental evidence in the field.

2.2.4 Isodual Newtonian Mechanics

A central objective of this section is to show that the isodual theory of antimatter resolves the scientific imbalance of the 20-th century between matter and antimatter, by permitting the study of antimatter at *all* levels as occurring for matter. Such an objective can only be achieved by first establishing the existence of a *Newtonian* representation of antimatter subsequently proved to be compatible with known operator formulations.

As it is well known, the Newtonian treatment of N *point-like particles* is based on a $7N$–dimensional representation space given by the Kronecker products of the Euclidean spaces of time t, coordinates r and velocities v (for the conventional case see Refs. [33,34]),

$$S(t, r, v) = E(t, R_t) \times E(r, \delta, R_r) \times E(v, \delta, R_v), \tag{2.2.5}$$

where

$$r = (r^k_a) = (r^1_a, r^2_a, r^3_a) = (x_a, y_a, z_a), \tag{2.2.6a}$$

$$v = (v_{ka}) = (v_{1a}, v_{2a}, v_{3a}) = (v_{xa}, v_{ua}, v_{za}) = dr/dt, \tag{2.2.6b}$$

$$\delta = Diag.(1, 1, 1), k = 1, 2, 3, a = 1, 2, 3, ..., N, \tag{2.2.6c}$$

and the base fields are trivially identical, i.e., $R_t = R_r = R_v$, since all units are assumed to have the trivial value $+1$, resulting in the trivial total unit

$$I_{tot} = I_t \times I_r \times I_v = 1 \times 1 \times 1 = 1. \tag{2.2.7}$$

The resulting basic equations are then given by the celebrated *Newton's equations for point-like particles*

$$m_a \times dv_{ka}/dt = F_{ka}(t, r, v), k = 1, 2, 3, a = 1, 2, 3, ..., N. \tag{2.2.8}$$

The basic space for the treatment of n antiparticles is given by the $7N-$ dimensional *isodual space* [9]

$$S^d(t^d, r^d, v^d) \ = \ E^d(t^d, R^d_t) \times E^d(r^d, \delta^d, R^d) \times E^d(v^d, \delta^d, R^d), \qquad (2.2.9)$$

with *isodual unit* and *isodual metric*

$$I^d_{Tot} \ = \ I^d_t \times I^d_r \times I^d_v, \qquad (2.2.10a)$$

$$I^d_t \ = \ -1, \ I^d_r \ = \ I^d_v \ = Diag.(-1, -1, -1), \qquad (2.2.10b)$$

$$\delta^d = Diag.(1^d, 1^d, 1^d) = Diag.(-1, -1, -1). \qquad (2.2.10c)$$

We reach in this way the basic equations of this chapter, today known as the *Newton-Santilli isodual equations for point-like antiparticles,* first introduced in Ref. [4],[1]

$$m^d_a \times^d d^d v^d_{ka} /^d d^d t^d \ = \ F^d_{ka}(t^d, r^d, v^d), \qquad (2.2.11)$$

$$k \ = \ x, y, z, \ a \ = \ 1, 2, ..., n.$$

whose experimental verification has been provided in the preceding section.

It is easy to see that the isodual formulation is anti-isomorphic to the conventional version, as desired, to such an extent that the two formulations actually coincide at the abstract, realization-free level.

Despite this axiomatic simplicity, the physical implications of the isodual theory of antimatter are rather deep. To begin their understanding, note that throughout the 20-th century it was believed that matter and antimatter exist in the same spacetime. In fact, as recalled earlier, charge conjugation is a map of our physical spacetime into itself.

One of the first physical implications of the Newton-Santilli isodual equations is that *antimatter exists in a spacetime co-existing, yet different than our own.* In fact, the isodual Euclidean space $E^d(r^d, \delta^d, R^d)$ co-exist within, but it is physically distinct from our own Euclidean space $E(r, \delta, R)$, and the same occurs for the full representation spaces $S^d(t^d, r^d, v^d)$ and $S(t, r, v)$.

The next physical implication of the Newton-Santilli isodual equations is the confirmation that *antimatter moves backward in time in a way as causal as the motion of matter forward in time* (again, because negative time is measured with a negative unit). In fact, the *isodual time* t^d is necessarily negative whenever t is our ordinary time. Alternatively, we can say that *the Newton-Santilli isodual equations provide the only known causal description of particles moving backward in time.*

[1]Note as necessary pre-requisites of the new Newton's equations, the prior discovery of isodual numbers, spaces and differential calculus.

Yet another physical implication is that *antimatter is characterized by negative mass, negative energy and negative magnitudes of other physical quantities.* As we shall see, these properties have the important consequence of eliminating the necessary use of Dirac's "hole theory."

The rest of this chapter is dedicated to showing that the above novel features are necessary to achieve a consistent representation of antimatter at all levels of study, from Newton to second quantization.

As we shall see, the physical implications are truly at the edge of imagination, such as: the existence of antimatter in a new spacetime different from our own constitutes the first known evidence of multi-dimensional character of our universe despite our sensory perception to the contrary; the achievement of a fully equivalent treatment of matter and antimatter implies the necessary existence of antigravity for antimatter in the field of matter (and vice-versa); the motion backward in time implies the existence of a causal spacetime machine (although restricted for technical reasons only to isoselfdual states); and other far reaching advances.

2.2.5 Isodual Lagrangian Mechanics

The second level of treatment of *matter* is that via the conventional *classical Lagrangian mechanics.* It is, therefore, essential to identify the corresponding formulation for *antimatter,* a task first studied in Ref. [4] (see also Ref. [9]).

A conventional (first–order) Lagrangian $L(t,r,v) = \frac{1}{2} \times m \times v^k \times v_k + V(t,r,v)$ on configuration space (2.2.5) is mapped under isoduality into the *isodual Lagrangian*

$$L^d(t^d, r^d, v^d) \ = \ -L(-t, -r, -v), \tag{2.2.12}$$

defined on isodual space (2.2.9).

In this way we reach the basic analytic equations of this chapter, today known as *Lagrange-Santilli isodual equations,* first introduced in Ref. [4]

$$\frac{d^d}{d^d t^d} d \frac{\partial^d L^d(t^d, r^d, v^d)}{\partial^d v^{kd}} d - \frac{\partial^d L^d(t^d, r^d, v^d)}{\partial^d r^{kd}} d = 0, \tag{2.2.13}$$

All various aspects of the *isodual Lagrangian mechanics* can then be readily derived.

It is easy to see that isodual equations (2.3.13) provide a *direct analytic representation* (i.e., a representation without integrating factors or coordinate transforms) of the isodual equations (2.2.11),

$$\frac{d^d}{d^d t^d} d \frac{\partial^d L^d(t^d, r^d, v^d)}{\partial^d v^{kd}} d - \frac{\partial^d L^d(t^d, r^d, v^d)}{\partial^d x^{kd}} d =$$

$$= m_k^d \times^d d^d v_k^d /^d d^d t^d - F_k^{d\,SA}(t, r, v) \ = \ 0. \tag{2.2.14}$$

The compatibility of the isodual Lagrangian mechanics with the primitive Newtonian treatment then follows.

2.2.6 Isodual Hamiltonian Mechanics

The *isodual Hamiltonian* is evidently given by [4,9]

$$H^d = p^d_k \times^d p^{dk} /^d 2^d \times^d m^d + V^d(t^d, r^d, v^d) = -H. \tag{2.2.15}$$

It can be derived from (nondegenerate) isodual Lagrangians via a simple isoduality of the Legendre transforms and it is defined on the $7N$-dimensional *isodual phase space (isocotangent bundle)*

$$S^d(t^d, r^d, p^d) = E^d(t^d, R^d_t) \times E^d(r^d, \delta^d, R^d) \times E^d(p^d, \delta^d, R^d). \tag{2.2.16}$$

The *isodual canonical action* is given by [4,9]

$$A^{\circ d} = \int_{t_1}^{t_2} (p^d_k \times^d d^d r^{kd} - H^d \times^d d^d t^d) =$$

$$= \int_{t_1}^{t_2} [R^{\circ d}_\mu (b^d) \times^d d^d b^{\mu d} - H^d \times^d d^d t^d], \tag{2.2.17a}$$

$$R^\circ = \{p, 0\},\ b = \{x, p\},\ \mu = 1, 2, ..., 6. \tag{2.2.17b}$$

Conventional variational techniques under simple isoduality then yield the fundamental canonical equations of this chapter, today known as *Hamilton-Santilli isodual equations* [4,24-31] that can be written in the disjoint r and p notation

$$\frac{d^d x^{kd}}{d^d t^d} = \frac{\partial^d H^d(t^d, x^d, p^d)}{\partial^d p^d_k}, \ \frac{d^d p^d_k}{d^d t^d} = -\frac{\partial^d H^d(t^d, x^d, p^d)}{\partial^d x^{dk}}, \tag{2.2.18}$$

or in the unified notation

$$\omega^d_{\mu\nu} \times^d \frac{d^d b^{d\nu}}{d^d t^d} = \frac{\partial^d H^d(t^d, b^d)}{\partial^d b^{d\mu}}, \tag{2.2.19}$$

where $\omega^d_{\mu\nu}$ is the *isodual canonical symplectic tensor*

$$(\omega^d_{\mu\nu}) = (\partial^d R^{\circ d}_\nu /^d \partial^d b^{d\mu} - \partial^d R^{\circ d}_\mu /^d \partial^d b^{d\nu}) =$$

$$= \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = (\omega^{\mu\nu}). \tag{2.2.20}$$

Note that isoduality maps the canonical symplectic tensor into the canonical Lie tensor, with intriguing geometric and algebraic implications.

The *Hamilton–Jacobi-Santilli isodual equations* are then given by [4,9]

$$\partial^d A^{\circ d} /^d \partial^d t^d + H^d \; = \; 0, \tag{2.2.21a}$$

$$\partial^d A^{\circ d} /^d \partial^d x^d_k - p^d_k \; = \; 0, \; \partial^d A^{\circ d} /^d \partial^d p^d_k \equiv 0. \tag{2.2.21b}$$

The *Lie-Santilli isodual brackets* among two isodual functions A^d and B^d on $S^d(t^d, x^d, p^d)$ then become

$$[A^d ,^d B^d] \; = \; \frac{\partial^d A^d}{\partial^d b^{d\mu}} d \; \times^d \omega^{d\mu\nu} \times^d \frac{\partial^d B^d}{\partial^d b^{d\nu}} d \; = \; -[A, B] \tag{2.2.22}$$

where

$$\omega^{d\mu\nu} \; = \; (\omega_{\mu\nu}), \tag{2.2.23}$$

is the *Lie-Santilli isodual tensor* (that coincides with the conventional canonical tensor). The direct representation of isodual equations in first–order form is self–evident.

In summary, all properties of the isodual theory at the Newtonian level carry over at the level of isodual Hamiltonian mechanics.

2.2.7 Isodual Galilean Relativity

As it is well known, the Newtonian, Lagrangian and Hamiltonian treatment of matter are only the pre-requisites for the characterization of physical laws via basic relativities and their underlying symmetries. Therefore, no equivalence in the treatment of matter and antimatter can be achieved without identifying the relativities suitable for the *classical* treatment of antimatter.

To begin this study, we introduce the *Galilei-Santilli isodual symmetry* $G^d(3.1)$ [7,5,9,22-31] as the step-by-step isodual image of the conventional *Galilei symmetry* G (3.1) (herein assumed to be known[2]). By using conventional symbols for the Galilean symmetry of a Keplerian system of N point particles with non-null masses m_a, $a \; = \; 1, 2, ..., n$, $G^d(3.1)$ is characterized by *isodual parameters and generators*

$$w^d \; = \; (\theta^d_k, r^{kd}_o, v^{kd}_o, t^d_o) = -w, \tag{2.2.24a}$$

$$J^d_k \; = \; \sum{}_{aijk} r^d_{ja} \times^d p^k_{ja} \; = \; -J_k \tag{2.2.24b}$$

$$P^d_k \; = \; \sum{}_a p^d_{ka} \; = \; -P_k, \tag{2.2.24c}$$

$$G^d_k \; = \; \sum{}_a (m^d_a \times^d r^d_{ak} - t^d \times p^d_{ak}), \tag{2.2.24d}$$

[2]The literature on the conventional Galilei and special relativities and related symmetries is so vast to discourage discriminatory quotations.

$$H^d \;=\; \frac{1}{2}^d \times^d \sum{}_a p^d_{ak} \times^d p^{kd}_a + V^d(r^d) \;=\; -H, \tag{2.2.24e}$$

equipped with the *isodual commutator*

$$[A^d,^d B^d] \;=\; \sum{}_{a,k}[(\partial^d A^d/^d \partial^d r^{kd}_a) \times^d (\partial^d B^d/^d \partial^d p^d_{ak}) -$$

$$-(\partial^d B^d/^d \partial^d r^{kd}_a) \times^d (\partial^d A^d/^d \partial^d p^d_{ak})]. \tag{2.2.25}$$

In accordance with rule (2.1.34), the structure constants and Casimir invariants of the isodual algebra $G^d(3.1)$ are negative–definite. If $g(w)$ is an element of the (connected component) of the Galilei group $G(3.1)$, its isodual is characterized by

$$g^d(w^d) \;=\; e^{d^{-i^d \times^d w^d \times^d X^d}} \;=$$

$$= -e^{i\times(-w)\times X} \;=\; -g(-w) \in G^d(3.1). \tag{2.2.26}$$

The *Galilei-Santilli isodual transformations* are then given by

$$t^d \;\rightarrow\; t'^d \;=\; t^d + t^d_o \;=\; -t', \tag{2.2.27a}$$

$$r^d \;\rightarrow\; r'^d \;=\; r^d + r^d_o \;=\; -r' \tag{2.2.27b}$$

$$r^d \;\rightarrow\; r'^d \;=\; r^d + v^d_o \times^d t^d_o \;=\; -r', \tag{2.2.27c}$$

$$r^d \;\rightarrow\; r'^d \;=\; R^d(\theta^d) \times^d r^d \;=\; -R(-\theta) \times r. \tag{2.2.27d}$$

where $R^d(\theta^d)$ is an element of the *isodual rotational symmetry* first studied in the original proposal [1].

The desired classical nonrelativistic characterization of antimatter is therefore given by imposing the $G^d(3.1)$ invariance to the considered isodual equations. This implies, in particular, that the equations admit a representation via isodual Lagrangian and Hamiltonian mechanics.

We now confirm the classical experimental verification of the above isodual representation of antimatter already treated in Section 2.2.2. Consider a conventional, classical, massive *particle* and its *antiparticle* in exterior dynamical conditions in vacuum. Suppose that the particle and antiparticle have charge $-e$ and $+e$, respectively (say, an *electron* and a *positron*), and that they enter into the gap of a magnet with constant magnetic field $\mathbf{B}$.

As it is well known, visual experimental observation establishes that particles and antiparticles under the same magnetic field have spiral trajectories of *opposite orientation*. But this behavior occurs for the *representation of both the particle and its antiparticle in the same Euclidean space*. The situation under isoduality is different, as described by the following:

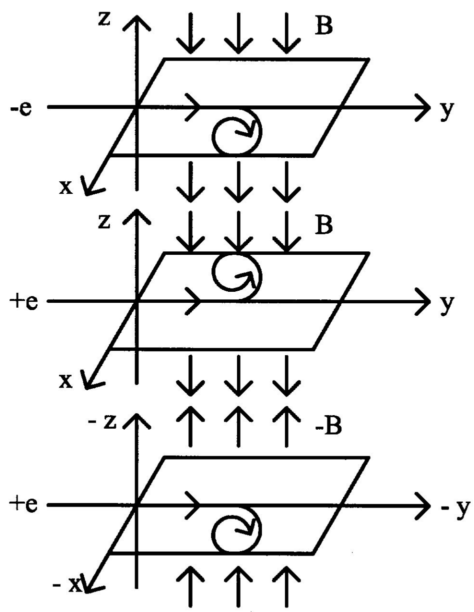

Figure 2.3. A schematic view of the trajectories of an electron and a positron with the same kinetic energy under the same magnetic field. The trajectories "appear" to be the reverse of each other when inspected by one observer, such as that in our spacetime (top and central views). However, when the two trajectories are represented in their corresponding spacetimes they coincide, as shown in the text (top and bottom views).

LEMMA 2.2.1 [5a]: *The trajectories under the same magnetic field of a charged particle in Euclidean space and of the corresponding antiparticle in isodual Euclidean space coincide.*

Proof: Suppose that the particle has negative charge $-e$ in Euclidean space $E(r, \delta, R)$, i.e., the value $-e$ is defined with respect to the positive unit $+1$ of the underlying field of real numbers $R = R(n, +, \times)$. Suppose that the particle is under the influence of the magnetic field $\mathbf{B}$.

The characterization of the corresponding antiparticle via isoduality implies the reversal of the sign of all physical quantities, thus yielding the charge $(-e)^d = +e$ in the isodual Euclidean space $E^d(r^d, \delta^d, R^d)$, as well as the reversal of the magnetic field $B^d = -B$, although now defined with respect to the negative unit $(+1)^d = -1$.

It is then evident that the trajectory of a particle with charge $-e$ in the field B defined with respect to the unit $+1$ in Euclidean space and that for the antiparticle of charge $+e$ in the field $-B$ defined with respect to the unit -1 in isodual Euclidean space coincide (Figure 2.3). **q.e.d.**

An aspect of Lemma 2.2.1, which is particularly important for this monograph, is given by the following:

COROLLARY 2.2.1A: Antiparticles reverse their trajectories when projected from their own isodual space into our own space.

Lemma 2.2.1 assures that isodualities permit the representation of the correct trajectories of antiparticles as physically observed, despite their negative energy, thus providing the foundations for a consistent representation of antiparticles at the level of *first* quantization studied in the next section. Moreover, Lemma 2.2.1 tells us that the trajectories of antiparticles *appear* to exist in our space while in reality they belong to an independent space.

2.2.8 Isodual Special Relativity

We now introduce *isodual special relativity* for the classical relativistic treatment of point-like antiparticles (for the conventional case see Ref. [32]).

As it is well known, conventional special relativity is constructed on the fundamental 4-dimensional unit of the Minkowski space

$$I = Diag.(1, 1, 1, 1),$$

representing the dimensionless units of space, e.g., (+1 cm, +1 cm, +1 cm), and the dimensionless unit of time, e.g., +1 sec, and constituting the basic unit of the conventional *Poincarè symmetry* $P(3.1)$ (hereon assumed to be known).

It then follows that *isodual special relativity* is characterized by the map

$$I = Diag.(\{1, 1, 1\}, 1) > 0 \quad \rightarrow$$
$$\rightarrow I^d = Diag.(\{-1, -1, -1\}, -1) < 0. \tag{2.2.28}$$

namely, the antimatter relativity is based on *negative units of space and time*, e.g., $I^d = Diag.(-1\,\text{cm}, -1\,\text{cm}, -1\,\text{cm}, -1\,\text{sec})$. This implies the reconstruction of the entire mathematics of the special relativity with respect to the common, isodual unit I^d, including: the *isodual field* $R^d = R^d(n^d, +^d, \times^d)$ of *isodual numbers* $n^d = n \times I^d$; the *isodual Minkowski spacetime* $M^d(x^d, \eta^d, R^d)$ with isodual coordinates $x^d = x \times I^d$, isodual metric $\eta^d = -\eta$ and basic invariant over R^d

$$(x - y)^{d2d} = [(x^\mu - y^\mu) \times \eta^d_{\mu\nu} \times (x^\nu - y^\nu)] \times I^d \in R^d. \tag{2.2.29}$$

This procedure yields the central symmetry of this chapter indicated in Section 2.2.6, today known as the *Poincaré-Santilli isodual symmetry* [7]

$$P^d(3.1) \ = \ \mathcal{L}^d(3.1) \times^d T^d(3.1), \qquad (2.2.30)$$

where $\mathcal{L}^d(3.1)$ is the *Lorentz-Santilli isodual symmetry*, $\times^d$ is the *isodual direct product* and $T^d(3.1)$ represents the *isodual translations.*

The algebra of the connected component $P_+^{\uparrow d}(3.1)$ of $P^d(3.1)$ can be constructed in terms of the isodual parameters $w^d \ = \ \{-w_k\} \ = \ \{-\theta, -v, -a\}$ and isodual generators $X^d \ = \ -X \ = \ \{-X_k\} \ = \ \{-M_{\mu\nu}, -P_\mu\}$. The isodual commutator rules are given by [7]

$$[M^d_{\mu\nu},^d M_{\alpha\beta}]^d \ =$$

$$= i^d \times^d (\eta^d_{\nu\alpha} \times^d M^d_{\mu\beta} - \eta^d_{\mu\alpha} \times^d M^d_{\nu\beta} - \eta^d_{\nu\beta} \times^d M^d_{\mu\alpha} + \eta^d_{\mu\beta} \times^d \hat{M}^d_{\alpha\nu}), \quad (2.2.31a)$$

$$[M^d_{\mu\nu},^d p^d_\alpha] \ = \ i^d \times^d (\eta^d_{\mu\alpha} \times^d p^d_\nu - \eta^d_{\nu\alpha} \times^d p^d_\mu), \qquad (2.2.31b)$$

$$[p^d_\alpha, p^d_\beta]^d \ = \ 0. \qquad (2.3.31c)$$

The *Poincarè-Santilli isodual transformations* are given by[3]

$$x^{1d\prime} \ = \ x^{1d} = -x^1, \qquad (2.2.32a)$$

$$x^{2d\prime} \ = \ x^{2d} = -x^2, \qquad (2.2.32b)$$

$$x^{3d\prime} \ = \ \gamma^d \times^d (x^{3d} - \beta^d \times^d x^{4d}) = -x^{3\prime}, \qquad (2.2.32c)$$

$$x^{4d\prime} \ = \ \gamma^d \times^d (x^{4d} - \beta^d \times^d x^{3d}) = -x^{4\prime}, \qquad (2.2.32d)$$

$$x^{d\mu\prime} \ = \ x^{d\mu} + a^{d\mu} \ = \ -x^{\mu\prime}, \qquad (2.3.32e)$$

where

$$\beta^d \ = \ v^d /^d c^d_\circ \ = \ -\beta, \ \ \beta^{d2d} \ = \ -\beta^2, \ \ \gamma^d \ = \ -(1-\beta^2)^{-1/2}. \qquad (2.2.33)$$

and the use of the isodual operations (quotient, square roots, etc.), is assumed.

The *isodual spinorial covering*

$$\mathcal{P}^d(3.1) = \mathcal{SL}^d(2.C^d) \times^d \mathcal{T}^d(3.1) \qquad (2.2.34)$$

can then be constructed via the same methods.

[3]It should be indicated that, contrary to popular beliefs, the conventional Poincaré symmetry will be shown in Chapter 3 to be *eleven* dimensional, the 11-th dimension being given by a new invariant under change of the unit. Therefore, the isodual symmetry $P^d(3.1)$ is also 11-dimensional.

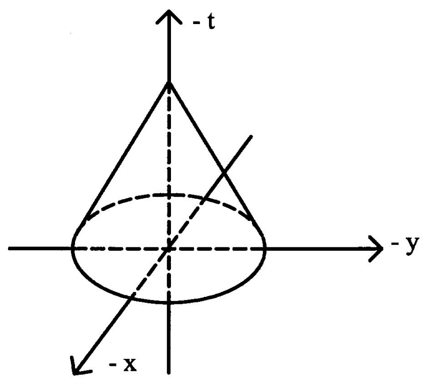

Figure 2.4. A schematic view of the "isodual backward light cone" as seen by an observer in our own spacetime with a time evolution reversed with respect to the "conventional forward light cone."

The basic postulates of the isodual special relativity are also a simple isodual image of the conventional postulates [7]. For instance, the *maximal isodual causal speed in vacuum* is the speed of light in M^d, i.e.,

$$V^d_{max} \;=\; c^d_\circ \;=\; -c_\circ, \tag{2.2.35}$$

with the understanding that it is measured with a *negative–definite unit*, thus being fully equivalent to the conventional maximal speed c_o referred to a positive unit. A similar situation occurs for all other postulates.

The *isodual light cone* is evidently given by (Figure 2.4)

$$x^{d\,2\,d} \;=\; (x^{\mu d} \times^d \eta^d_{\mu\nu} \times^d x^{\nu d}) \times I^d \;=$$
$$=\; (-x \times x - y \times y - z \times z + t \times c^2_\circ \times t) \times (-I) \;=\; 0. \tag{2.2.36}$$

As one can see, the above cone formally coincides with the conventional light cone, although the two cones belong to different spacetimes. The isodual light cone is used in these studies as *the cone of light emitted by antimatter in empty space (exterior problem).*

Note that the *two* Minkowskian metrics $\eta = Diag.(+1, +1, +1, -1)$ and $\eta = Diag.(-1, -1, -1, +1)$ have been popular since Minkowski's times, although both referred to the *same* unit I. We have learned here that these two popular metrics are connected by isoduality.

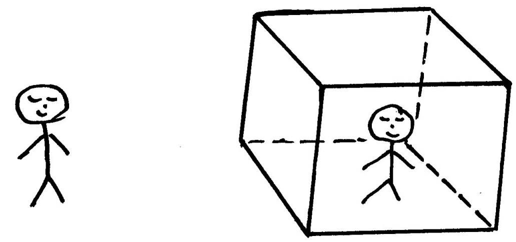

Figure 2.5. A schematic view of the "isodual cube," here defined as a conventional cube with two observers, an external observer in our spacetime and an internal observer in the isodual spacetime. The first implication of the isodual theory is that the same cube coexist in the two spacetimes and can, therefore, be detected by both observers. A most intriguing implication of the isodual theory is that each observer sees the other becoming younger. This occurrence is evident for the behavior of the internal observer with respect to the exterior one, since the former evolves according to a time opposite that of the latter. The same occurrence is less obvious for the opposite case, the behavior of the external observer with respect to the internal one, and it is due to the fact that the projection of our positive time into the isodual spacetime is indeed a motion backward in that spacetime.

We finally introduce the *isodual electromagnetic waves* and related *isodual Maxwell's equations* [9]

$$F^d_{\mu\nu} = \partial^d A^d_\mu /^d \partial^d x^{\nu d} - \partial^d A^d_\nu /^d \partial^d x^{d\mu}, \tag{2.2.37a}$$

$$\partial^d_\lambda F^d_{\mu\nu} + \partial^d_\mu F^d_{\nu\lambda} + \partial^d_\nu F^d_{\lambda\mu} = 0, \tag{2.2.37b}$$

$$\partial^d_\mu F^{d\mu\nu} = -J^{d\nu}. \tag{2.2.37c}$$

As we shall see, the nontriviality of the isodual special relativity is illustrated by the fact that isodual electromagnetic waves experience gravitational repulsion when in the field of matter.

2.2.9 Inequivalence of Isodual and Spacetime Inversions

As it is well known (see, the fundamental spacetime symmetries of the 20-th century are the continuous (connected) component of the Poincaré symmetry plus discrete symmetries characterized by *space reversal* (also called *parity*) and *time reversal.*

As noted earlier, antiparticles are assumed in the above setting to exist in the same representation spacetime and to obey the same symmetries as those of particles. On the contrary, according to the isodual theory, antiparticles are represented in a spacetime and possess symmetries distinct from those of particles, although connected to the latter by the isodual transform.

The latter occurrence requires the introduction of the *isodual spacetime inversions,* that is, the isodual images of space and time inversions, first identified in Ref. [9], that can be formulated in unified coordinate form as follows

$$x^{d\mu\prime} \ = \ \pi^d \times^d x^d \ = \ -\pi \times x =$$

$$= (-r, x^4), \ \ \tau^d \times^d x^d = -\tau \times x \ = \ -(r, -x^4), \tag{2.2.38}$$

with field theoretical extension (here expressed for simplicity for a scalar field)

$$\pi^d \times^d \phi^d(x^d) \times^d \pi^{d\dagger} = \phi^d(x'^d, x'^d = (-r^d, t^d) = (r, -t), \tag{2.2.39a}$$

$$\tau^d \times^d \phi^d(x^d) \times^d \tau^{d\dagger} = \bar{\phi}^d(x"^d, x"^d = (r^d, -t^d) = (-r, t), \tag{2.2.39b}$$

where $r^d(= -r)$ is the *isodual coordinate* on space $E^d(r^d, \delta^d, R^d)$, and t^d is the *isodual time* on $E^d(t^d, 1, R^d_t)$.

LEMMA 2.2.2 [9]: *Isodual inversions and spacetime inversions are inequivalent.*

Proof. Spacetime inversions are characterized by the change of sign $x \rightarrow -x$ *by always preserving the original metric measured with positive units*, while isodual inversions imply the map $x \rightarrow x^d \ = \ -x$ *but now measured with an isodual metric* $\eta^d \ = \ -\eta$ *with negative units* $I^d \ = \ -I$, thus being inequivalent. **q.e.d.**

Despite their simplicity, isodual inversions (or isodual discrete symmetries) are not trivial (Figure 2.6). In fact, all measurements are done in our spacetime, thus implying the need to consider the *projection* of the isodual discrete symmetries into our spacetime which are manifestly different than the conventional forms.

In particular, they imply a sort of interchange, in the sense that the conventional *space* inversion $(r, t) \rightarrow (-r, t)$ emerges as belonging to the projection in our spacetime of the isodual *time* inversion, and vice-versa.

Note that the above “interchange” of parity and time reversal of isodual particles projected in our spacetime could be used for experimental verifications, but this aspect is left to interested readers.

In closing this subsection, we point out that the notion of isodual parity has intriguing connections with the parity of antiparticles in the $(j, 0) + (0, j)$ representation space more recently studied by Ahluwalia, Johnson and Goldman [10]. In fact, the latter parity results in being opposite that of particles which is fully in line with isodual space inversion (isodual parity).

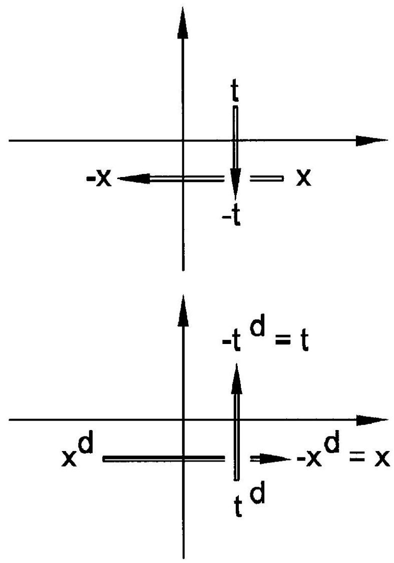

Figure 2.6. A schematic view of the additional peculiar property that the projection in our spacetime of the isodual space inversion appears as a time inversion and vice versa. In fact, a point in the isodual spacetime is given by $(x^d, t^d) = (-x, -t)$. The projection in our spacetime of the isodual space inversion $(x^d, t^d) \rightarrow (-x^d, t^d)$ is then given by $(x, -t)$, thus appearing as a time (rather than a space) inversion. Similarly, the projection in our spacetime of the isodual time inversion $(x^d, t^d) \rightarrow (x^d, -t^d)$ appears as $(-x, t)$, that is, as a space (rather than time) inversion. Despite its simplicity, the above occurrence has rather deep implications for all discrete symmetries in particle physics indicated later on.

2.2.10 Isodual Thermodynamics of Antimatter

An important contribution to the isodual theory has been made by J. Dunning-Davies [11] who introduced in 1999 the first, and only known consistent thermodynamics for antimatter with intriguing results and implications.

As conventionally done in the field, let us represent heat with Q, internal energy with U, work with W, entropy with S, and absolute temperature with T. *Dunning-Davies isodual thermodynamics of antimatter* is evidently defined via the isodual quantities

$$Q^d = -Q, U^d = -U, W^d = -W, S^d = -S, T^d = -T \tag{2.2.40}$$

on isodual spaces over the isodual field of real numbers $R^d = R^d(n^d, +^d, \times^d)$ with isodual unit $I^d = -1$.

Recall from Section 2.1.3 that *differentials are isoselfdual* (that is, invariant under isoduality). Dunning-Davies then has the following:

THEOREM 2.2.1 [21]: *Thermodynamical laws are isoselfdual.*

Proof. For the *First Law of thermodynamics* we have

$$dQ = dU - dW \ \equiv\ d^dQ^d = d^dU^d - d^dW^d. \tag{2.2.41}$$

Similarly, for the *Second Law of thermodynamics* we have

$$dQ = T \times dS \ \equiv\ d^dQ^d = T^d \times^d S^d, \tag{2.2.42}$$

and the same occurs for the remaining laws. **q.e.d.**

Despite their simplicity, Dunning-Davies results [21] have rather deep implications. First, the identity of thermodynamical laws, by no means, implies the identity of the thermodynamics of matter and antimatter. In fact, *in Dunning-Davies isodual thermodynamics the entropy must always decrease in time*, since the isodual entropy is always negative and is defined in a space with evolution backward in time with respect to us. However, these features are fully equivalent to the conventional increase of the entropy tacitly referred to positive units.

Also, Dunning-Davies results indicate that *antimatter galaxies and quasars cannot be distinguished from matter galaxies and quasars via the use of thermodynamics,* evidently because their laws coincide, in a way much similar to the identity of the trajectories of particles and antiparticles of Lemma 2.2.1.

This result indicates that the only possibility known at this writing to determine whether far-away galaxies and quasars are made up of matter or of antimatter is that via the predicted gravitational repulsion of the light emitted by antimatter called *isodual light* (see next section and Chapter 5).

2.2.11 Isodual General Relativity

For completeness, we now introduce the *isodual general relativity* for the classical gravitational representation of antimatter. A primary motivation for its study is the incompatibility with antimatter of the positive-definite character of the energy-momentum tensor of the conventional general relativity studied in Chapter 1.

The resolution of this incompatibility evidently requires a structural revision of general relativity [33] for a consistent treatment of antimatter. The *only* solution known to the author is that offered by isoduality.[4]

It should be stressed that this study is here presented merely for completeness, since the achievement of a consistent treatment of negative-energies, by

[4]The author would be grateful to colleagues who care to bring to his attention other "classical" gravitational theories of antimatter compatible with the negative-energy solutions needed by antimatter.

no means, resolves the serious inconsistencies of gravitation on a Riemannian space caused by curvature, as studied in Section 1.2, thus requiring new geometric vistas beyond those permitted by the Riemannian geometry (see Chapters 3 and 4).

As studied in Section 2.1.7, the *isodual Riemannian geometry* is defined on the isodual field $R^d(n^d, +^d, \times^d)$ for which *the norm is negative–definite*, Eq. (2.1.18). As a result, *all quantities that are positive in Riemannian geometry become negative under isoduality, thus including the energy-momentum tensor.*

In fact, the energy-momentum tensor of isodual electromagnetic waves (2.2.37) is negative-definite [8,9]

$$T^d_{\mu\nu} \;=\; (4\times\pi)^{-1d}\times^d (F^d_{\mu\alpha}\times^d F^d_{\alpha_\nu} + (1/4)^{-1d}\times^d g^d_{\mu\nu}\times^d F^d_{\alpha\beta}\times^d F^{d\alpha\beta}). \quad (2.2.43)$$

The *Einstein-Hilbert isodual equations for antimatter in the exterior conditions in vacuum* are then given by [6, 9]

$$G^d_{\mu\nu} \;=\; R^d_{\mu\nu} - \frac{1}{2}^d \times^d g^d_{\mu\nu}\times^d R^d \;=\; k^d\times^d T^d_{\mu\nu}, \qquad (2.2.44)$$

The rest of the theory is then given by the use of the isodual Riemannian geometry of Section 2.1.7.

The explicit study of this gravitational theory of antimatter is left to the interested reader due to the indicated inconsistencies of gravitational theories on a Riemannian space for the *conventional case of matter* (Section 1.2). These inconsistencies multiply when treating antimatter, as we shall see.

2.3 OPERATOR ISODUAL THEORY OF POINT-LIKE ANTIPARTICLES

2.3.1 Basic Assumptions

In this section we study the operator image of the classical isodual theory of the preceding section; we prove that the operator image of isoduality is equivalent to charge conjugation; and we show that isodual mathematics resolves all known objections against negative energies.

A main result of this section is the identification of a simple, structurally new formulation of quantum mechanics known as *isodual quantum mechanics* or, more properly, as the *isodual branch of hadronic mechanics* first proposed by Santilli in Refs. [5]. Another result of this section is the fact that all numerical predictions of operator isoduality coincide with those obtained via charge conjugation on a Hilbert space, thus providing the experimental verification of the isodual theory of antimatter at the operator level.

Despite that, the isodual image of quantum mechanics is not trivial because of a number of far reaching predictions we shall study in this section and in the next chapters, such as: the prediction that antimatter emits a new light distinct from that of matter; antiparticles in the gravitational field of matter experience antigravity; bound states of particles and their antiparticles can move backward in time without violating the principle of causality; and other predictions.

Other important results of this section are a new interpretation of the conventional Dirac equation that escaped detection for about one century, as well as the indication that the isodual theory of antimatter originated from the Dirac equation itself, not so much from the negative-energy solutions, but more properly from their two-dimensional unit that is indeed negative-definite, $I_{2\times 2} = Diag.(-1, -1)$.

As we shall see, Dirac's "hole theory", with the consequential restriction of the study of antimatter to the sole second quantization and resulting scientific imbalance indicated in Section 1.1, were due to Dirac's lack of knowledge of a *mathematics based on negative units.*

Intriguingly, had Dirac identified the quantity $I_{2\times 2} = Diag.(-1, -1)$ as the unit of the *mathematics* treating the negative energy solutions of his equation, the physics of the 20-th century would have followed a different path because, despite its simplicity, the unit is indeed the most fundamental notion of all mathematical and physical theories.

2.3.2 Isodual Quantization

The isodual Hamiltonian mechanics (and its underlying *isodual symplectic geometry* [5a] not treated in this chapter for brevity) permit the identification of a new quantization channel, known as the *naive isodual quantization* [6] that can be readily formulated via the use of the Hamilton-Jacobi-Santilli isodual equations (2.2.21) as follows

$$A^{\circ d} \rightarrow -i^d \times^d \hbar^d \times^d Ln^d \psi^d(t^d, r^d), \tag{2.3.1a}$$

$$\partial^d A^{\circ d} /^d \partial^d t^d + H^d \ = \ 0 \rightarrow i^d \times^d \partial^d \psi^d /^d \partial^d t^d \ = \ = \ H^d \times^d \psi^d \ = \ E^d \times^d \psi^d, \tag{2.3.1b}$$

$$\partial^d A^{\circ d} /^d \partial^d x^{dk} - \hat{p}_k \ = \ 0 \rightarrow p^d_k \times^d \psi^d \ = \ -i^d \times^d \partial^d_k \psi^d, \tag{2.3.1c}$$

$$\partial^d A^{\circ d} /^d \partial^d p^d_k = 0 \rightarrow \partial^d \psi^d /^d \partial^d p^d_k = 0. \tag{2.3.1d}$$

Recall that the fundamental unit of quantum mechanics is Planck's constant $\hbar = +1$. It then follows that the fundamental unit of the isodual operator theory is the new quantity

$$\hbar^d = -1. \tag{2.3.2}$$

It is evident that the above quantization channel identifies the new mechanics known as *isodual quantum mechanics,* or the *isodual branch of hadronic mechanics.*

2.3.3 Isodual Hilbert Spaces

Isodual quantum mechanics can be constructed via the anti-unitary transform

$$U \times U^{\dagger} = \hbar^d = I^d = -1, \tag{2.3.3}$$

applied, for consistency, to the *totality* of the mathematical and physical formulations of quantum mechanics. We recover in this way the isodual real and complex numbers

$$n \to n^d = U \times n \times U^{\dagger} = n \times (U \times U^{\dagger}) = n \times I^d, \tag{2.3.4}$$

isodual operators

$$A \to U \times A \times U^{\dagger} = A^d, \tag{2.3.5}$$

the isodual product among generic quantities A, B (numbers, operators, etc.)

$$A \times B \to U \times (A \times B) \times U^{\dagger} =$$

$$= (U \times A \times U^{\dagger}) \times (U \times U^{\dagger})^{-1} \times (U \times B \times U^{\dagger}) = A^d \times^d B^d, \tag{2.3.6}$$

and similar properties.

Evidently, isodual quantum mechanics is formulated in the *isodual Hilbert space* $\mathcal{H}^d$ with *isodual states* [6]

$$|\psi>^d = -|\psi>^{\dagger} = - < \psi|, \tag{2.3.7}$$

where $< \psi|$ is a conventional dual state on $\mathcal{H}$, and *isodual inner product*

$$< \psi|^d \times (-1) \times |\psi>^d \times I^d, \tag{2.3.8}$$

with *isodual expectation values* of an operator A^d

$$< A^d >^d = (< \psi|^d \times^d A^d \times^d |\psi>^d /^d < \psi|^d \times^d |\psi>^d), \tag{2.3.9}$$

and *isodual normalization*

$$< \psi|^d \times^d |\psi>^d = -1. \tag{2.3.10}$$

defined on the *isodual complex field* C^d with unit -1 (Section 2.1.1).

The isodual expectation values can also be reached via anti-unitary transform (2.3.3),

$$< \psi| \times A \times |\psi> \to U \times (< \psi| \times A \times |\psi>) \times U^{\dagger} =$$

$$= (< \psi| \times U^{\dagger}) \times (U \times U^{\dagger})^{-1} \times (U \times A \times U^{\dagger}) \times (U \times U^{\dagger})^{-1} \times$$

$$\times (U \times |\psi>) \times (U \times U^{\dagger}) = < \psi|^d \times^d A^d \times^d |\psi>^d \times I^d. \tag{2.3.11}$$

The proof of the following property is trivial.

LEMMA 2.3.1 [5b]: *The isodual image of an operator A that is Hermitean on $\mathcal{H}$ over* C *is also Hermitean on $\mathcal{H}^d$ over C^d (isodual Hermiticity).*

It then follows that *all quantities that are observables for particles are equally observables for antiparticles represented via isoduality.*

LEMMA 2.3.2 [5b]: *Let* H *be a Hermitean operator on a Hilbert space $\mathcal{H}$ over* C *with positive-definite eigenvalues E,*

$$H \times |\psi >= E \times |\psi >, H = H^{\dagger}, E => 0. \tag{2.3.12}$$

Then, the eigenvalues of the isodual operator H^d on the isodual Hilbert space $\mathcal{H}^d$ over C^d are negative-definite,

$$H^d \times^d |\psi >^d = E^d \times^d |\psi >^d, H^d = H^{d\dagger^d}, E^d < 0. \tag{2.3.13}$$

This important property establishes an evident compatibility between the classical and operator formulations of isoduality.

We also mention the *isodual unitary laws*

$$U^d \times^d U^{d\dagger} = U^{d\dagger} \times^d U^d = I^d, \tag{2.3.14}$$

the *isodual trace*

$$Tr^d A^d = (Tr A^d) \times I^d \in C^d, \tag{2.3.15a}$$

$$Tr^d (A^d \times^d B^d) = Tr^d A^d \times^d Tr^d B^d, \tag{2.3.15b}$$

the *isodual determinant*

$$Det^d A^d = (Det A^d) \times I^d \in C^d, \tag{2.3.16a}$$

$$Det^d (A^d \times^d B^d) = Det^d \times^d Det^d B^d, \tag{2.3.16b}$$

the *isodual logarithm* of a real number n

$$Log^d n^d = -(Log\, n^d) \times I^d, \tag{2.3.17}$$

and other isodual operations.

The interested reader can then work out the remaining properties of the isodual theory of linear operators on a Hilbert space.

2.3.4 Isoselfduality of Minkowski's Line Elements and Hilbert's Inner Products

A most fundamental new property of the isodual theory, with implications as vast as the formulation of a basically new cosmology, is expressed by the following lemma whose proof is a trivial application of transform (2.3.3).

LEMMA 2.3.3 [23]: *Minkowski's line elements and Hilbert's inner products are invariant under isoduality (or they are isoselfdual according to Definition 2.1.2),*

$$x^2 = (x^\mu \times \eta_{\mu\nu} \times x^\nu) \times I \equiv$$

$$\equiv (x^{d\mu} \times^d \eta^d_{\mu\nu} \times^d x^{d\nu}) \times I^d = x^{d2^d}, \quad (2.3.18a)$$

$$< \psi| \times |\psi > \times I \equiv < \psi|^d \times^d |\psi >^d \times I^d. \quad (2.3.18b)$$

As a result, *all relativistic and quantum mechanical laws holding for matter also hold for antimatter under isoduality.* The equivalence of charge conjugation and isoduality then follows, as we shall see shortly.

Lemma 2.3.3 illustrates the reason why isodual special relativity and isodual Hilbert spaces have escaped detection for about one century. Note, however, that invariances (2.3.18) require the prior discovery of *new numbers,* those with negative unit.

2.3.5 Isodual Schrödinger and Heisenberg's Equations

The fundamental dynamical equations of isodual quantum mechanics are the isodual images of conventional dynamical equations. They are today known as the *Schrödinger-Santilli isodual equations* [4] (where we assume hereon $\hbar^d = -1$, thus having $\times^d \hbar^d = 1$)

$$i^d \times^d \partial|\psi >^d /^d \partial^d t^d = H^d \times^d |\psi >^d, \quad (2.3.19a)$$

$$p^d_k \times^d |\psi >^d = -i^d \times^d \partial^d|\psi >^d /^d \partial^d r^d, \quad (2.3.19b)$$

and the *Heisenberg-Santilli isodual equations*

$$i^d \times^d d^d A^d /^d d^d t^d = A^d \times^d H^d - H^d \times^d A^d = [A^d, H^d]^d, \quad (2.3.20a)$$

$$[r^d_i, p^d_j]^d = i^d \times^d \delta^{di}_j, [r^d, r^{dj}]^d = [p^d_i, p^d_j]^d = 0. \quad (2.3.20b)$$

Note that, when written explicitly, Eq. (2.3.19a) is based on an associative modular action *to the left,*

$$- < \psi| \times^d H^d = (\partial^d < \psi|\partial^d t^d) \times^d i^d, \quad (2.3.21)$$

It is an instructive exercise for readers interested in learning the new mechanics to prove the equivalence of the isodual Schrödinger and Heisenberg equations via the anti-unitary transform (2.3.3).

2.3.6 Isoselfdual Re-Interpretation of Dirac's Equation

Isoduality has permitted a novel interpretation of the conventional *Dirac equation* (we shall here used the notation of Ref. [12]) in which the negative-energy states are reinterpreted as belonging to the isodual images of positive energy states, resulting in the first known *consistent representation of antiparticles in first quantization.*

This result should be expected since the isodual theory of antimatter applies at the Newtonian level, let alone that of first quantization. Needless to say, the treatment via isodual first quantization does not exclude that via isodual second quantization. The point is that the treatment of antiparticles is no longer restricted to second quantization, as a condition to resolve the scientific imbalance between matter and antimatter indicated earlier.

Consider the conventional Dirac equation [2]

$$[\gamma^\mu \times (p_\mu - e \times A_\mu/c) + i \times m] \times \Psi(x) = 0, \tag{2.3.22}$$

with realization of Dirac's celebrated gamma matrices

$$\gamma_k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}, \ \gamma^4 = i \times \begin{pmatrix} I_{2\times 2} & 0, \\ 0 & -I_{2\times 2} \end{pmatrix}, \tag{2.3.23a}$$

$$\{\gamma_\mu, \tilde{\gamma}_\nu\} \ = \ 2\times\eta_{\mu\nu}, \ \ \Psi = i \times \begin{pmatrix} \Phi \\ -\Phi^\dagger \end{pmatrix}, \tag{2.3.23b}$$

At the level of first quantization here considered, the above equation is rather universally interpreted as representing an electron under an external electromagnetic field.

The above equations are generally defined in the 6-dimensional space given by the Kronecker product of the conventional Minkowski spacetime and an internal spin space

$$M_{Tot} = M(x, \eta, R) \times S_{spin}, \tag{2.3.24}$$

with total unit

$$\begin{aligned} I_{Tot} = I_{orb} \times I_{spin} = \\ = Diag.(1,1,1,1) \times Diag.(1,1), \end{aligned} \tag{2.3.25}$$

and total symmetry

$$P(3.1) = SL(2.C) \times T(3.1). \tag{2.3.26}$$

The proof of the following property is recommended to interested readers.

THEOREM 2.3.1 [5b]: *Pauli's sigma matrices and Dirac's gamma matrices are isoselfdual,*

$$\sigma_k \equiv \sigma_k^d, \tag{2.3.27a}$$

$$\gamma_\mu \equiv \gamma_\mu^d. \tag{2.3.27b}$$

The above properties imply an important re-interpretation of Eqs. (2.3.22), first identified in Ref. [9] and today known as the *Dirac-Santilli isoselfdual equation,* that can be written

$$[\tilde{\gamma}^\mu \times (p_\mu - e \times A_\mu/c) + i \times m] \times \tilde{\Psi}(x) = 0, \tag{2.3.28}$$

with re-interpretation of the gamma matrices

$$\tilde{\gamma}_k = \begin{pmatrix} 0 & \sigma_k^d \\ \sigma_k & 0 \end{pmatrix}, \quad \tilde{\gamma}^4 = i \begin{pmatrix} I_{2\times 2} & 0, \\ 0 & I_{2\times 2}^d \end{pmatrix}, \tag{2.3.29a}$$

$$\{\tilde{\gamma}_\mu, \tilde{\gamma}_\nu\} = 2^d \times^d \eta_{\mu\nu}^d, \quad \tilde{\Psi} = -\tilde{\gamma}_4 \times \Psi = i \times \begin{pmatrix} \Phi \\ \Phi^d \end{pmatrix}, \tag{2.3.29b}$$

By recalling that isodual spaces coexist with, but are different from conventional spaces, we have the following:

THEOREM 3.3.2 [9]: *The Dirac-Santilli isoselfdual equation is defined on the 12-dimensional isoselfdual representation space*

$$M_{Tot} = \{M(x, \eta, R) \times S_{spin}\} \times \{M^d(x^d, \eta^d, R^d) \times^d S_{spin}^d\}, \tag{2.3.30}$$

with isoselfdual total 12-dimensional unit

$$I_{Tot} = \{I_{orb} \times I_{spin}\} \times \{I_{orb}^d \times^d I_{spin}^d\}, \tag{2.3.31}$$

and its symmetry is given by the isoselfdual product of the Poincaré symmetry and its isodual

$$S_{Tot} = \mathcal{P}(3.1) \times \mathcal{P}^d(3.1) =$$
$$= \{SL(2.C) \times T(3.1)\} \times \{SL^d(2.C^d) \times^d T^d(3.1)\}. \tag{2.3.32}$$

A direct consequence of the isoselfdual structure can be expressed as follows.

COROLLARY 2.3.2a [9]: *The Dirac-Santilli isoselfdual equation provides a joint representation of an electron and its antiparticle (the positron) in first quantization,*

$$Dirac\ \ Equation = Electron \times Positron \tag{2.3.33}$$

In fact, the two-dimensional component of the wave function with positive-energy solution represents the electron and that with negative-energy solutions represent the positron without any need for second quantization, due to the physical behavior of negative energies in isodual treatment established earlier.

Note the complete democracy and equivalence in treatment of the electron and the positron in equation (2.3.28), in the sense that the equation can be equally used to represent an electron or its antiparticle. By comparison, according to the original Dirac interpretation, the equation could only be used to represent the electron [12], since the representation of the positron required the "hole theory".

It has been popularly believed throughout the 20-th century that Dirac's gamma matrices provide a "four-dimensional representation of the SU(2)-spin symmetry". This belief is disproved by the isodual theory, as expressed by the following

THEOREM 3.3.3 [5b]: *Dirac's gamma matrices characterize the direct product of an irreducible two-dimensional (regular) representation of the* SU(2)-*spin symmetry and its isodual,*

$$Dirac's\ \ Spin\ \ Symmetry : SU(2) \times SU^d(2). \tag{2.3.34}$$

In fact, the gamma matrices are characterized by the conventional, 2-dimensional Pauli matrices σ_k and related identity $I_{2\times 2}$ as well as other matrices that have resulted in being the exact isodual images σ_k^d with isodual unit $I_{2\times 2}^d$.

It should be recalled that the isodual theory was born precisely out of these issues and, more particularly, from the incompatibility between the popular interpretation of gamma matrices as providing a "four-dimensional" representation of the SU(2)-spin symmetry and the *lack of existence of such a representation in Lie's theory.*

The sole possibility known to the author for the reconciliation of Lie's theory for the SU(2)-spin symmetry and Dirac's gamma matrices was to assume that $-I_{2\times 2}$ is the unit of a dual-type representation. The entire theory studied in this chapter then followed.

It should also be noted that, as conventionally written, Dirac's equation *is not* isoselfdual because not sufficiently symmetric in the two-dimensional states and their isoduals.

In summary, Dirac's was forced to formulate the "hole theory" for antiparticles because he referred the *negative* energy states to the conventional *positive* unit, while their reformulation with respect to *negative* units yields fully physical results.

It is easy to see that the same isodual reinterpretation applies for Majorana's spinorial representations [13] (see also [14,15]) as well as Ahluwalia's broader spinorial representations $(1/2, 0) + (0, 1/2)$ [16] (see also the subsequent paper [17]), that are reinterpreted in the isoselfdual form $(1, 2, 0) + (1, 2, 0)^d$, thus extending their physical applicability to first quantization.

In the latter reinterpretation the representation (1/2,0) is evidently done conventional spaces over conventional fields with unit +1, while the isodual representation $(1/2, 0)^d$ is done on the corresponding isodual spaces defined on isodual fields with unit -1. As a result, all quantities of the representation (1/2,0) change sign under isoduality.

It should be finally indicated that Ahluwalia treatment of Majorana spinors has a deep connection with isoduality because the underlying Class II spinors have a *negative norm* [16] precisely as it is the case for isoduality. As a result, the isodual reinterpretation under consideration here is quite natural and actually warranted for mathematical consistency, e.g., to have the topology characterized by a negative norm be compatible with the underlying fields.

2.3.7 Equivalence of Isoduality and charge conjugation

We come now to another fundamental point of this chapter, the proof that isoduality is equivalent to charge conjugation. This property is crucial for the experimental verification of isoduality at the particle level too. This equivalence was first identified by Santilli in Ref. [6] and can be easily expressed today via the following:

LEMMA 2.3.4 [6,5b,18]: *The isodual transform is equivalent to charge conjugation.*

Proof. Charge conjugation is characterized by the following transform of wavefunctions (see, e.g., Ref. [12], pages 109 and 176)

$$\Psi(x) \rightarrow C\Psi(x) = c \times \Psi^{\dagger}(x), \tag{2.3.35}$$

where

$$|c| = 1, \tag{2.3.36}$$

thus being manifestly equivalent to the isodual transform

$$\Psi(x) \rightarrow \Psi^{d}(x^{d}) = -\Psi^{\dagger}(-x^{t}), \tag{2.3.37}$$

where t denotes transpose.

A reason why the two transforms are equivalent, rather than identical, is the fact that charge conjugation maps spacetime into itself, while isoduality maps spacetime into its isodual. **q.e.d.**

Let us illustrate Lemma 2.3.4 with a few examples. As well known, the Klein-Gordon equation for a free particle

$$\partial^{\mu}\partial_{\mu}\Psi - m^{2} \times \Psi = 0, \tag{2.3.38}$$

is invariant under charge conjugation, in the sense that it is turned into the form

$$c \times [\bar{\Psi}\partial^{\mu}\partial_{\mu} - \bar{\Psi} \times m^{2}] = 0, \;\; |c| = 1, \tag{2.3.39}$$

where the upper bar denotes complex conjugation (since $\bar{\Psi}$ is a scalar), while the Lagrangian density

$$L = -(\hbar \times \hbar/2 \times m) \times \{\partial^\mu \bar{\Psi} - i \times e \times A^\mu/\hbar \times c) \times \bar{\Psi}] \times$$

$$\times [\partial \Psi + (i \times e \times A_\mu/\hbar \times c) \times \Psi] + m \times m \times \bar{\Psi} \times \Psi, \tag{2.3.40}$$

is left invariant and the four-current

$$J_\mu = -(i \times \hbar/2 \times m) \times [\bar{\psi} \times \partial_\mu \Psi - (\partial_\mu \bar{\Psi}) \times \Psi], \tag{2.3.41}$$

changes sign

$$J_\mu \rightarrow C J_\mu = -J_\mu. \tag{2.3.42}$$

By recalling the selfduality of ordinary derivatives, Eq. (2.1.30), under isoduality the Klein-Gordon Equation becomes

$$[\partial^\mu \partial_\mu \Psi - m^2 \times \Psi]^d = \Psi^d \partial^{d\mu} \partial^d_\mu - \Psi^d \times^d m^d \times^d m^d =$$

$$= -[\bar{\Psi} \partial^\mu \partial_\mu - \bar{\Psi} \times m^2] = 0, \tag{2.3.43}$$

thus being equivalent to Eq. (2.3.39), while the Lagrangian changes sign and the four-current changes sign too,

$$J^d_\mu = -(i \times \hbar/2 \times m) \times [\bar{\Psi} \times \partial_\mu \Psi - (\partial_\mu \bar{\Psi}) \times \Psi]^d =$$

$$= (i \times \hbar/2 \times m) \times [\bar{\Psi} \times \partial_\mu \Psi - (\partial_\mu \bar{\Psi}) \times \Psi], \tag{2.3.44}$$

(where we have used the isoselfduality of the imaginary number i).

The above results confirm Lemma 2.3.4 because of the equivalent behavior of the equations of motion and the four-current, while the change of sign of the Lagrangian does not affect the numerical results.

As it is also well known, the Klein-Gordon equation for a particle under an external electromagnetic field [12]

$$[(\partial_\mu + i \times e \times A_\mu/\hbar \times c) \times$$

$$\times (\partial^\mu + i \times e \times A^\mu/\hbar \times c) - m^2] \times \Psi = 0, \tag{2.3.45}$$

is equally invariant under charge conjugation in which *either* e *or* A_μ change sign, in view of the known invariance

$$C(i \times e \times A_\mu/\hbar \times c) = i \times e \times A_\mu/\hbar \times c, \tag{2.3.46}$$

while the four-current also changes sign. By noting that the preceding invariance persists under isoduality,

$$(i \times e \times A_\mu/\hbar \times c)^d = i \times e \times A_\mu/\hbar \times c, \tag{2.3.47}$$

Eq. (2.3.45) remains invariant under isoduality, while the Lagrangian density changes sign and the four-current, again, changes sign.

Similarly, consider Dirac equation (see also Ref. [12], pp. 176-177)

$$[\gamma^\mu \times (\partial_\mu \Psi - (i \times e \times A_\mu/\hbar \times c) \times \Psi + m \times \Psi = 0, \tag{2.3.48}$$

with Lagrangian density

$$L = (\hbar \times c/2) \times \{\tilde{\Psi} \times \gamma^\mu \times [\partial_\mu \Psi + (i \times e \times A_\mu/\hbar \times c) \times \Psi] -$$

$$-(\partial^\mu \tilde{\Psi} - (i \times e \times A^\mu/\hbar \times c) \times \tilde{\Psi}] \times \gamma_\mu - m \times \tilde{\Psi} \times \Psi \tag{2.3.49a}$$

$$\tilde{\Psi} = \Psi^\dagger \times \gamma_4, \tag{2.3.49b}$$

and four-current

$$J_\mu = i \times c \times \tilde{\Psi} \times \gamma_\mu \times \Psi = i \times c \times \Psi^\dagger \times \gamma_4 \times \gamma_\mu \times \Psi \tag{2.3.50}$$

The charge conjugation for Dirac's equations is given by the transform [12]

$$\Psi \rightarrow C\Psi = c \times S_C^{-1} \times \tilde{\Psi}^t \tag{2.3.51}$$

where S_C is a unitary matrix such that

$$\gamma_\mu \rightarrow -\gamma_\mu^t = S_C \times \gamma_\mu \times S_c^{-1}, \tag{2.3.52}$$

and there is the change of sign *either* of e *or* of A_μ, under which the equation is transformed into the form

$$[\partial_\mu \tilde{\Psi} - (i \times e \times A_\mu/\hbar \times c) \times \tilde{\Psi}] \times \gamma^\mu - m \times \tilde{\Psi} = 0, \tag{2.3.53}$$

while the Lagrangian density changes sign and the four-current remains the same,

$$L \rightarrow CL = -L, J_\mu \rightarrow CJ_\mu = J_\mu. \tag{2.3.54}$$

It is easy to see that isoduality provides equivalent results. In fact, for Eq. (2.3.48) we have

$$\{[\gamma^\mu \times (\partial_\mu \Psi - i \times e \times A_\mu/\hbar \times c) \times \Psi + m \times \Psi\}^d =$$

$$= [\partial_\mu \Psi^\dagger - (i \times e \times A_\mu/\hbar \times c) \times \Psi^\dagger] \times \gamma^\mu - m \times \Psi^\dagger = 0, \tag{2.3.55}$$

that, when multiplied by γ_4 reproduces Eq. (2.3.53) identically. Similarly, by recalling that Dirac' s gamma matrices are isoselfdual (Theorem 2.3.1), and by noting that

$$\tilde{\Psi}^d = (\Psi^\dagger \times \gamma_4)^d = \gamma_4 \times \Psi, \tag{2.3.56}$$

we have

$$L^d = L \tag{2.3.57}$$

while for the four-current we have

$$J^d_\mu = -i \times c \times \Psi^\dagger \times \gamma_\mu \times \gamma_4 \times \psi. \tag{2.3.58}$$

But the γ_μ and γ_4 anticommute. As a consequence, the four-current does not change sign under isoduality as in the conventional case.

Note that the lack of change of sign under isoduality of Dirac's four-current J_μ confirms reinterpretation (2.3.28) since, for the latter equation, the total charge is null.

The equivalence between isoduality and charge conjugation of other equations, such as those by Weyl, Majorana, etc., follows the same lines.

2.3.8 Experimental Verification of the Isodual Theory of Antimatter in Particle Physics

In Section 2.2.3. we have established the experimental verification of the isodual theory of antimatter in classical physics. That in particle physics requires no detailed elaboration since it is established by the equivalence of charge conjugation and isoduality (Lemma 2.3.4), and we can write:

LEMMA 2.3.5 [6,5b,18], [7]: *All experimental data currently available for antiparticles represented via charge conjugation are equally verified by the isodual theory of antimatter.*

2.3.9 Elementary Particles and their Isoduals

We assume the reader is familiar with the conventional definition of *elementary particles* as irreducible unitary representations of the spinorial covering of the Galilei symmetry $G(3.1)$ for nonrelativistic treatments and those of the Poincaré symmetry $P(3.1)$ for relativistic treatments. We therefore introduce the following:

DEFINITION 2.3.1: Elementary isodual particles (antiparticles) are given by irreducible unitary representations of the spinorial covering of the Galilei-Santilli's isodual symmetry $G^d(3.1)$ *for nonrelativistic treatments and those of the Poincaré-Santilli isodual symmetry* $P^d(3.1)$ *for relativistic treatments.*

A few comments are now in order. First, one should be aware that "isodual particles" and "antiparticles" do not represent the same notion, evidently because of the negative mass, energy and time of the former compared to positive mass, energy and time of the latter. In the rest of this chapter, unless otherwise stated, the word "antiparticle" will be referred to the "isodual particle."

For instance the word "positron" e^+ is more appropriately intended to represent the "isodual electron" with symbol e^{-d}. Similarly the, "antiproton" p^- is intended to represent the "isodual proton" p^{+d}.

Second, the reader should note the insistence on the *elementary* character of the antiparticles here admitted. The reason is that the antigravity studied in Chapter 4 is specifically formulated for "elementary" isodual particles, such as the isodual electron, due to a number of unsettled aspects pertaining to composite particles.

Consider, as an illustration, the case of mesons. If the π° is a bound state of a particle and its isodual, the state is isoselfdual and, as such, it *cannot* experience antigravity, as illustrated in the next section. A number of ambiguities then follow for the study of the gravity of the charged mesons $\pi^\pm$, such as the problem of ascertaining which of the two mesons is a particle and which is its isodual or, whether the selected antiparticle is indeed the isodual image of the particle as a necessary condition for meaningful study of their gravity.

Note that essentially the same ambiguities prohibit the use of muons for a serious theoretical and experimental studies of the gravity of antiparticles, again, because of unsettled problems pertaining to the structure of the muons themselves. Since the muons are naturally unstable, they cannot be credibly believed to be elementary. Therefore, serious theoretical and experimental studies on the gravity of muons require the prior identification of their constituents with physical particles.

Finally, the reader should be aware that *Definition 2.3.1 excludes the use of quark conjectures for the gravitational studies of this monograph.* This is due to the well-known basic inconsistency of quark conjecture of not admitting any gravitation at all (see, e.g., the Appendix of Ref. [18]). In fact, gravity can only be defined in our spacetime while quarks can only be defined in their mathematical unitary internal space with no known connection with our spacetime due to the O'Rafearthaigh theorem.[5]

Also, the only "masses" that can be credibly claimed as possessing inertia are the eigenvalues of the second-order Casimir invariant of the Poincaré symmetry $p_\mu \times p^\mu = m^2$. Quarks cannot be characterized via such a fundamental symmetry, as well known. It then follows that "quark masses" are mere mathematical parameters defined in the mathematical internal complex-unitary space that cannot possibly be used as serious basis for gravitational tests.

[5]The possible connection between internal and spacetime symmetries offered by supersymmetric theories cannot be credibly used for gravitational tests due to their highly unsettled character and the prediction of a zoo of new particles none of which has been experimentally detected to the author's best knowledge.

2.3.10 Photons and their Isoduals

As it is well known, photons have no charge and, therefore, they are invariant under charge conjugation, as transparent from the simple plane-wave representation

$$\Psi(t,r) = N \times e^{i\times(k\times r - E\times t)}, \quad N \in R, \tag{2.3.59}$$

with familiar relativistic form

$$\Psi(x) = N \times e^{i\times k_\mu \times x^\mu}, \tag{2.3.60}$$

and familiar expression for the energy

$$E = h \times \nu. \tag{2.3.61}$$

As a result, matter and antimatter have been believed throughout the 20-th century to emit the same light. In turn, this belief has left fundamentally unsettled basic questions in astrophysics and cosmology, such as the lack of quantitative studies as to whether far-away galaxies and quasars are made up of matter or of antimatter.

One of the most intriguing and far reaching implications of the isodual theory is that, while remaining evidently invariant under charge conjugation, *the photon is not invariant under isoduality,* thus admitting a conjugate particle first submitted by Santilli in Ref. [18] under the name of *isodual photon.* In particular, the isodual photon emerges as having physical characteristics that can be experimentally measured as being different from those of the photon.

Therefore, the isodual theory offers the first known possibilities of quantitative theoretical and experimental studies as to whether a far-away galaxy or quasar is made of matter or of antimatter due to detectable physical differences of their emitted light.

Note that the term "antiphoton" could be misleading because the prefix "anti" is generally assumed as referring to charge conjugation. For this reason the name of "isodual photon" appears to be preferable, also because it represents, more technically, the intended state.

In fact, the photon is mapped by isoduality into a new particle possessing all negative-definite physical characteristics, with the following simple isodual plane-wave representation

$$\Psi^d(t^d, r^d) = N^d \times^d e_d^{i^d \times^d (k^d \times^d r^d - E^d \times^d t^d)}, \quad N^d \in R^d, \tag{2.3.62}$$

with relativistic expression on isodual Minkowski space

$$\Psi^d(x^d) = N^d \times^d e_d^{i^d \times^d k_\mu^d \times^d x^{d\mu}}, \tag{2.3.63}$$

and isodual expression for the energy

$$E^d \ = \ h^d \times^d \nu^d, \tag{2.3.64}$$

where e_d is the isodual exponentiation (2.1.26b).

Note that, since i is isoselfdual, Eq. (2.1.20), *the exponent of the plane-wave representation is invariant under both charge conjugation and isoduality,* as illustrated by the following expression

$$i^d \times^d (k^d \times^d r^d - E^d \times^d t^d) \equiv i \times (k \times r - E \times t), \tag{2.3.65}$$

or its relativistic counterpart

$$i^d \times^d k^d_\mu \times^d x^{d\mu} \equiv i \times k_\mu \times x^\mu. \tag{2.3.66}$$

thus confirming the lack of contradiction between charge conjugation and isoduality.

Moreover, both the photon and the isodual photon travel in vacuum with the same (absolute) speed $|c|$, for which we have the additional identity

$$k^d_\mu \times^d k^{d\mu} \equiv k_\mu \times k^\mu = 0. \tag{2.3.67}$$

Despite the above identities, energy and time are positive-definite for the photon, while they are negative-definite for the isodual photon. As we shall see, the latter property implies that photons are attracted by the gravitational field of matter while isodual photons are repelled, thus providing a physically detectable difference.

Additional differences between light emitted by matter and that emitted by antimatter, such as those pertaining to parity and other discrete symmetries, require additional study.

2.3.11 Electrons and their Isoduals

The next truly elementary particles and antiparticles are the electron e^- and its antiparticle, the positron e^+ or the isodual electron e^{-d}. The differences between the "positron" and the "isodual electron" should be kept in mind. In fact, the former has positive rest energy and moved forward in time, while the latter has negative rest energy and moves backward in time.

Also, the electron is known to experience gravitational attraction in the field of matter, as experimentally established. As conventionally defined, the positron too is predicted to experience gravitational attraction in the field of matter (because its energy is positive).

However, as we shall see in Chapter 4, the isodual electron is predicted to experience antigravity when immersed in the field of matter, and this illustrates again the rather profound physical differences between the "positron" and the "isodual electron".

Note that, in view of their truly elementary character, isodual electrons are the ideal candidates for the measurement of the gravitational field of antiparticles.

2.3.12 Protons and their Isoduals

The next particles demanding comments are the proton p^+, the antiproton p^- and the isodual proton p^{+d}. In this case the differences between the "antiproton" and the "isodual proton" should be kept in mind to avoid major inconsistencies with the isodual theory, such as the study of the possible antigravity for antiprotons in the field of matter which antigravity cannot exist for the isodual theory (due, again, to the positive mass of the antiproton).

Note that these particles are not elementary and, as such, they are not admitted by Definition 2.3.1. moreover, as stressed earlier [18], when represented in term of quark conjectures both the proton and the antiproton cannot admit any gravity at all, let alone antigravity. As a result, extreme scientific care should be exercised before extending to all antimatter any possible gravitational measurements for antiprotons.

2.3.13 The Hydrogen Atom and its Isodual

The understanding of this chapter requires the knowledge that studies conducted on the *antihydrogen atom* (see, e.g., the various contributions in Proceedings [19]), even though evidently interesting per se, they have no connection with the *isodual hydrogen atom,* because the antihydrogen atom has positive mass, for which antigravity is prohibited, and emits conventional photons. Therefore, it is important to inspect the differences between these two formulations of the simplest possible atom of antimatter.

We assume as exactly valid the conventional quantum mechanical theory of *bound states of point-like particles at large mutual distances,*[6] as available in quantum mechanical books so numerous to discourage even a partial listing.

For the case of two particles denoted with the indices 1, 2, the total state in the Hilbert space is the familiar tensorial product of the two states

$$|\psi >= |\psi_1 > \times|\psi_2 > . \tag{2.3.68}$$

The total Hamiltonian H is the sum of the kinetic terms of each state plus the familiar interaction term $V(r)$ depending on the mutual distance r,

$$H = p_1 \times p_1/2 \times m_1 + p_2 \times p_2/2 \times m_2 + V(r). \tag{2.3.69}$$

The total angular momentum is computed via the familiar expressions for angular momenta and spins

$$J = J_1 \times I + I \times J_2, \;\; S = S_1 \times I + I \times S_2, \tag{2.3.70}$$

[6]We are here referring to the large mutual distances as occurring in the atomic structure and exclude the short mutual distances as occurring in the structure of hadrons, nuclei and stars since a serious study of the latter is dramatically beyond the capabilities of quantum mechanics, as shown beyond scientific doubt in Chapter 3.

where the I's are trivial units, with the usual rules for couplings, addition, etc. One should note that the unit for angular momenta is three-dimensional while that for spin has a generally different dimension.

A typical example of two-body bound states of particles is the *hydrogen atom* that experiences attraction in the gravitational field of matter with the well established emission of conventional photons.

The study of *bound states of point-like isodual particles at large mutual distances* is an important part of isodual quantum mechanics. These bound states can be studied via an elementary isoduality of the corresponding bound states for particles, that is, via the use of the isodual Hilbert spaces $\mathcal{H}^d$ studied earlier.

The *total isodual state* is the tensorial product of the two isodual states

$$|\psi^d(r^d)>^d = |\psi_1^d(r^d)>^d \times^d |\psi_2^d(r^d)>^d = -<\psi_1(-r)| \times <\psi_2(-r)|. \quad (2.3.71)$$

The *total isodual Hamiltonian* is the sum of the isodual kinetic terms of each particle plus the isodual interaction term depending on the isodual mutual distance,

$$H^d = p_1^d \times^d p_1^d /^d 2^d \times^d m_1^d + p_2^d \times^d p_2^d /^d 2^d \times^d m_2^d + V^d(r^d). \quad (2.3.72)$$

The total *isodual angular momentum* is based on the expressions for isodual angular momenta and spin

$$J^d = J_1^d \times^d I^d + I^d \times^d J_2^d, \quad (2.3.73a)$$

$$S^d = S_1^d \times^d I^d + I^d \times^d S_2^d, \quad (2.3.73b)$$

The remaining aspects (couplings, addition theory of angular momenta, etc.) are then given by a simple isoduality of the conventional theory that is here omitted for brevity.

Note that all eigenvalues that are positive for the conventional case measured with positive units become negative under isoduality, yet measured with negative units, thus achieving full equivalence between particle and antiparticle bound states.

The simplest possible application of the above isodual theory is that for the *isodual hydrogen atom* (first worked out in Ref. [18]). The novel predictions of isoduality over that of the antihydrogen atom is that the isodual hydrogen atom is predicted to experience antigravity in the field of matter and emits isodual photons that are also repelled by the gravitational field of matter.

2.3.14 Isoselfdual Bound States

Some of the most interesting and novel bound states predicted by the isodual theory are the *isoselfdual bound states,* that is, bound states that coincide with their isodual image. The simplest case is the bound state of one elementary particle and its isodual, such as the *positronium.*

The condition of isoselfduality requires that the basic symmetry must be itself isoselfdual, e.g., for the nonrelativistic case the total symmetry must be

$$G_{Tot} = G(3.1) \times G^d(3.1), \tag{2.3.74}$$

where $\times$ is the Kronecker product (a composition of states thus being isoselfdual), with a simple relativistic extension here assumed as known from the preceding sections.

The total unit must also be isoselfdual,

$$I_{Tot} = I \times I^d, \tag{2.3.75}$$

where I represents the space, time and spin units.

The total Hilbert space and related states must also be isoselfdual,

$$\mathcal{H}_{Tot} = \mathcal{H} \times \mathcal{H}^d, \tag{2.3.76a}$$

$$|\psi >_{Tot} = |\psi > + |\psi >^d = |\psi > - < \psi|, \tag{2.3.76b}$$

and so on.

A main feature is that isoselfdual states exist in both the spacetime of particles and that of antiparticles. Therefore, the computation of the total energy must be done *either* in $\mathcal{H}$, in which case the total energy is positive, *or* in $\mathcal{H}^d$, in which case the total energy is negative.

Suppose that a system of one elementary particle and its isodual is studied in our laboratory of matter. In this case the eigenvalues for both particle and its isodual must be computed in $\mathcal{H}$, in which case we have the equation

$$\begin{aligned} i \times \partial_t|\psi > &= (p \times p/2 \times m) \times |\psi > + \\ &+ (p^d \times^d p^d /^d 2^s \times^d m^d) \times^d |\psi > + V(r) \times |\psi > = \\ &= [p \times p/2 \times m + V(r)] \times |\psi > = E \times |\psi >, \end{aligned} \tag{2.3.77}$$

under which the total energy E is evidently positive.

When the same isoselfdual state is detected in the spacetime of antimatter, it must be computed with respect to $\mathcal{H}^d$, in which case the total energy is negative, as the reader is encouraged to verify.

The total angular momentum and other physical characteristics are computed along similar lines and they also result in having positive values when computed in $\mathcal{H}$, as occurring for the conventional charge conjugation.

As we shall see shortly, the positive character of the total energy of bound states of particles and their antiparticles is crucial for the removal of the inconsistencies of theories with negative energy.

The above properties of the isoselfdual bound states have the following implications:

1) Isoselfdual bound states of elementary particles and their isoduals are predicted to be attracted in both, the gravitational field of matter and that of antimatter because their total energy is positive in our world and negative in the isodual world. This renders necessary an experimental verification of the gravitational behavior of isoselfdual bound states, independently from that of individual antiparticles. Note that the prediction holds only for bound states of truly elementary particles and their isoduals, such as the positronium. No theoretical prediction for the muonium and the pionium is today feasible because the unsettled nature of their constituents.

2) Isoselfdual bound states are predicted to have a null internal total time $t + t^d = 0$ and therefore acquires the time of the matter or antimatter in which they are immersed, although the physical time t of the observer (i.e., of the bound state equation) is not null. This is readily understood by noting that the quantity t of Eq. (2.3.77) is our own time, i.e., we merely study the behavior of the state with respect to our own time. A clear understanding illustrated previously with the "isodual cube" of Section 2.1 is that the description of a state with our own time, by no means, implies that its intrinsic time necessarily coincides with our own. Note that a similar situation occurs for the energy because the intrinsic total energy of the positronium is identically null, $E + E^d = 0$. Yet the energy measured by us is $E_{part.} - E^d_{antipart.} = 2E > 0$. A similar situation occurs for all other physical quantities.

3) Isoselfdual bound states may result in being the microscopic image of the main characteristics of the entire universe. Isoselfduality has in fact stimulated a new cosmology, the *isoselfdual cosmology* [21] studied in Chapter 5, that is patterned precisely along the structure of the positronium or of Dirac's equation in our isoselfdual re-interpretation. In this case the universe results in having null total physical characteristics, such as null total energy, null total time, etc., thus implying no discontinuity at its creation.

2.3.15 Resolution of the Inconsistencies of Negative Energies

The treatment of antiparticles with negative energies was rejected by Dirac because incompatible with their physical behavior. Despite several attempts made during the 20-th century, the inconsistencies either directly or indirectly connected to negative energies have remained unresolved.

The isodual theory of antimatter resolves these inconsistencies for the reason now familiar, namely, that the inconsistencies emerge when one refers negative energies to conventional numbers with positive units, while the same inconsistencies cannot be evenly formulated when negative energies are referred to isodual numbers and their negative units.

A good illustration is given by the known objection according to which the creation of a photon from the annihilation of an electron-positron pair, with the electron having a positive energy and the positron having a negative energy, would violate the principle of conservation of the energy.

In fact, such a pair could be moved upward in our gravitational field without work and then annihilated in their new upward position. The resulting photon would then have a blueshift in our gravitational field of Earth, thus having more energy than that of the original photon.

Presumed inconsistencies of the above type cannot be even formulated within the context of the isodual theory of antimatter because, as shown in the preceding section, the electron-positron state is isoselfdual, thus having a non-null *positive* energy when observed in our spacetime. Consequently, the lifting upward of the pair does indeed require work and no violation of the principle of conservation of the energy can be expected.

A considerable search has established that all other presumed inconsistencies of negative energy known to the author cannot even be formulated within the context of the isodual theory of antimatter. Nevertheless, the author would be particularly grateful to any colleague who bring to its attention inconsistencies of negative energies that are really applicable under negative units.

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