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MATHEMATICAL FOUNDATIONS OF SANTILLI ISOTOPIES

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To my wife, who made this translation possible, and to the Holy Trinity, Who makes all things possible

Contents

Introduction

The main objective of this text is to present the most basic and fundamental aspects of the *isotopic Lie-Santilli theory*, more commonly known as *Lie-Santilli isotheory* or simply *isotheory*, for short. In what follows, we we will refer to it indiscriminately in any one of these three forms. In addition, we will use the term *conventional* to refer to the mathematical and physical concepts customarily used.

To achieve this goal, we will construct *isotopic liftings* of the basic mathematical structures in a general way and then arrive at the isotopic generalization of Lie algebras.

The origins of this isotheory date back to 1978, as the result of an essay by the mathematical physicist of Italian origin, Ruggero Maria Santilli (see [98]).

Santilli studied in this work and in many others later (see the bibliography at the end of this text) the way to generalize the classical mathematical theories and especially its applications in other sciences, in particular physics and engineering. He achieves this through a mathematical *lifting* of the unit element on which the theory in question is based. Through such a lifting, one obtains a new theory, which is characterized by having the same properties as the initial theory, even though the new unit on which this theory is based satisfies more general conditions than the units of the initial theory.

To perform this process, Santilli uses a particular lifting: *isotopies*. By means of them, he starts generalizing, in the first place, the basic mathematical structures, namely, groups, rings, and fields, thus building the first mathematical *isostructures*. Later and by another type of generalization, but always using isotopies, Santilli builds *isodual isostructures* and *pseudoisostructures*.

In the mid-nineties of last century, Santilli laid the foundations of an isotopic generalization which will involve a definite advance in his work: the *differential isocalculus*. With this new tool, Santilli was able to make new generalizations, now in the field of mathematical analysis: *isofunctions*, *isoderivatives*, etc.

This enabled him, in turn, to advance greatly in the development of some physical applications. In particular, one of the aims of Santilli was to apply conventional Lie theory to practical results in quantum mechanics, dynamics, and many other fields of physics, passing from studying canonical, local, and integro-differential systems to others, the most general possible, which were non-canonical, non-local, and non-integro-differential.

Our personal contribution to this text, which shows the different mathematical isostructures, as well as how to construct them using liftings through isotopies, has been to incorporate a large number of examples; to systematize and order the entire knowledge of a single isostructure, usually rather scattered in the literature (most of them due to Santilli himself, while there are others of different authors); to unify the different notations in which these results appear in the literature so far; to give new demonstrations of some of them (in some cases, they did not even exist, and the facts are taken for granted) and ultimately, to bring some original results, allowing us to provide this isotheory a proper mathematical foundation, all of which, in our opinion, contributes to improve, especially from the point of view of current mathematics, existing knowledge of the same.

To this end, special emphasis is put on the importance not only of the isotopic lifting of the unit element on which any mathematical structure is based, but also on the isotopic lifting of all operations defined in them. In this way, it incorporates a series of results that generalize part of those already existing in the literature, because it not only works with regular fields in physics (real, complex, quaternions, and octonions), but also considers sets of elements and operations that they associate, which are the most general possible.

The content of this text is structured in five chapters. Chapter 1 shows the definitions and most important properties of the well-known algebraic structures, whose subsequent lifting will result in the emergence of *isostructures*. While these definitions and properties are already known, we believe that its exposé is essential for facilitating a proper understanding of the foundations of the isotheory. Special

emphasis is placed on the foundations of the structures of *algebra*, in general, and *Lie algebra* in particular, since the lifting of the latter will lead to the isostructure called the *Lie isotopic isoalgebra*.

Chapter 2 of this text gives some biographical notes on the scientific work of the two mathematicians who have contributed, the first indirectly, and the second directly, to the birth of the isotheory: Marius Sophus Lie and the already frequently cited Ruggero Maria Santilli.

Chapters 3, 4, and 5 are primarily dedicated to the study of the Lie-Santilli isotheory. Chapter 3 begins with the definition of the concept of *isotopy*. However, as the sense of this concept is too general for what is intended, we will restrict ourselves to the case of the *Santilli isotopy*, which will be a basic tool for the development of the *Lie-Santilli isotheory*. The basic definitions of the elements and tools that will be used in the rest of the work are also introduced: isounit, isotopic element, etc.

Generalization, step by step, of the basic mathematical structures is then performed. To do this, in the first place, shows how elements of any mathematical structure are generalized isotopically, taking for example the elements of any field K, we then study the isostructures, which are gaining increasingly greater complexity in their construction. In this chapter the *isogroups*, *isorings*, and *isofields* are studied in particular.

Chapter 4 continues the study of the Lie-Santilli isotheory, performing the isotopic lifting of more complex algebraic structures than those seen before. Thus, *isovector spaces* and *metric isovector spaces* are studied, followed by *isomodules*. In addition, considering it of great interest, because of the important consequences that are derived from them, we also felt it appropriate to include a section dedicated to the study of *isotransformations* from *isovector spaces*.

Finally, Chapter 5 will consider the isotopic lifting of a new structure: algebras. The first section looks at *isoalgebras* and their associated substructures: *isosubalgebras*. The second section treats the particular case of the *Lie-Santilli algebras*, and, to finish the study, some types of Lie isotopic isoalgebras, including *isosimples*, *isosemisimples*, *isoresolvables*, *isonilpotents*, and *isofiliforms*.

Also, in all these chapters, copious examples, almost entirely original, which we understand are fundamental to a proper understanding of the isotheory, are included, since they have shaped it.

The final part of the text includes an extensive bibliography which, apart from those texts directly referenced therein, almost all them corresponding to the higher mathematical content of the isotheory, also includes others (suggested at the behest of the Institute for Basic Research itself) relating to the various physical applications deriving from the same, which in our opinion helps the interested reader have a better global understanding of the contents and importance of this isotheory in the current development of the sciences in general, and of mathematics and physics in particular.

We wish finally to put on record the thanks to our respective families for the support they have given us all the time, as well as professors R. M. Santilli and G. F. Weiss, of the Institute for Basic Research (IBR) in Florida (USA), for the help they have provided for the drafting of this work from the beginning.

Chapter 1
Preliminaries

In order to facilitate a proper understanding of this text, this chapter presents the definitions and more important properties of all those *algebraic structures* to be *lifted*, giving rise to the corresponding *isostructures* of the *Lie-Santilli isotheory*.

In the first section and in different subsections, we review, within the algebraic structures that we could call elementary, the concepts of group, ring, and field, as well as their most important properties.

In the second section, we review, as more general algebraic structures, vector spaces and modules, also indicating their most important properties.

The third section will have special emphasis on the definition and most important properties of *algebras*, in particular of *Lie algebras*.

1.1 Elementary algebraic structures

1.1.1 Groups

We recall that a *group* is an algebraic structure consisting of a pair $(G, \circ)$, where G is a set of elements $\{\alpha, \beta, \gamma, \ldots\}$ and $\circ$ is a binary operation on G satisfying the following properties, $\forall \alpha, \beta, \gamma \in G$:

1. *Associative*: $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$.
2. *Existence of the Elemental Unit*: $\exists I \in G$ such that $\alpha \circ I = I \circ \alpha = \alpha$.
3. *Existence of the Inverse Element*: Given $\alpha \in G$, $\exists \alpha^{-I} \in G$ such that $\alpha \circ \alpha^{-I} = \alpha^{-I} \circ \alpha = I$.

If in addition $\circ$ is commutative, i.e., it satisfies $\alpha \circ \beta = \beta \circ \alpha$ for all $\alpha, \beta \in G$, then G is called an *Abelian group* or *commutative group*.

Let $(G, \circ)$ be a group. A set H is called a *subgroup* of G if the following conditions are satisfied:

1. $H \subseteq G$.
2. The binary operation $\circ$ is closed over H, i.e., $\alpha \circ \beta \in H$, for all $\alpha, \beta \in H$.
3. $(H, \circ)$ has a group structure.

Let $(G, \circ)$ and $(G', \bullet)$ be any two groups. A function $f : G \to G'$ is called the *group homomorphism* if $f(\alpha \circ \beta) = f(\alpha) \bullet f(\beta)$, $\forall \alpha, \beta \in G$.

If f is bijective, it is called the group *isomorphism*. If $G = G'$, f is called an *endomorphism*, and if it is also an isomorphism, it is called an *automorphism*.

1.1.2 Rings

A *ring* is a triplet $(A, \circ, \bullet)$, where A is a set of elements $\{\alpha, \beta, \gamma, \ldots\}$, equipped with two binary operations, $\circ$ and $\bullet$, on A satisfying $\forall \alpha, \beta, \gamma \in A$ the following properties:

1. $(A, \circ)$ is an Abelian group.
2. Associativity of $\bullet$: $(\alpha \bullet \beta) \bullet \gamma = \alpha \bullet (\beta \bullet \gamma)$.
3. Existence of the elementary unit of $\bullet$: $\exists e \in A$ such that $\alpha \bullet e = e \bullet \alpha = \alpha$.
4. Left and right distributivity:
   $\alpha \bullet (\beta \circ \gamma) = (\alpha \bullet \beta) \circ (\alpha \bullet \gamma)$
   $(\alpha \circ \beta) \bullet \gamma = (\alpha \bullet \gamma) \circ (\beta \bullet \gamma)$.

If in addition the *commutative* property of $\bullet$ is satisfied, i.e., if $\alpha \bullet \beta = \beta \bullet \alpha$, $\forall \alpha, \beta \in A$, then A is called an *Abelian* or *commutative* ring.

Let $(A, \circ, \bullet)$ be any ring. The set B is called a *subring* of A if:

1. B is closed for laws $\circ$ and $\bullet$, also satisfying the conditions of associativity of $\bullet$ and distributivity over both operations.
2. $(B, \circ)$ is a subgroup of $(A, \circ)$.
3. $e \in B$.

Let $(A, \circ, \bullet)$ and $(A', +, \times)$ be two rings with units e and e' with respect to operations $\bullet$ and $\times$, respectively. A function $f : A \to A'$ is called a ring *homomorphism*, if $\forall \alpha, \beta \in A$, then:

1. $f(\alpha \circ \beta) = f(\alpha) + f(\beta)$.
2. $f(\alpha \bullet \beta) = f(\alpha) \times f(\beta)$.
3. $f(e) = e'$.

In the particular case that f is bijective, it is called an *isomorphism*. If $A = A'$, f is called an *endomorphism*, and in this case, if f is bijective, it is then called an *automorphism*.

Among the subrings there are ones which possess special properties: the *ideals*.

Let $(A, \circ, \bullet)$ be a ring. The set $\Im$ is called an *ideal* of A if:

1. $(\Im, \circ)$ is a subgroup of $(A, \circ)$.
2. $\Im \bullet A \subseteq \Im$ and $A \bullet \Im \subseteq \Im$, i.e., $x \bullet \alpha \in \Im$ and $\alpha \bullet x \in \Im$, $\forall x \in \Im$ and $\forall \alpha \in A$.

Let $(A, \circ, \bullet)$ be any ring and $\Im$ an ideal of A. $J \subseteq \Im$ is called a *subideal* of $\Im$ if $(J, \circ, \bullet)$ has the structure of an ideal of A.

The concept of an ideal of a ring allows you to establish at the same time the concept of a quotient ring, as follows: Let $(A, \circ, \bullet)$ be a ring and $\Im$ its ideal. We call a *quotient ring* that which is associated with A and $\Im$ for the quotient set $A/\Im$, endowed with operations $+$ and $\times$, and satisfying that

1. $(\alpha + \Im) + (\beta + \Im) = (\alpha \circ \beta) + \Im, \forall \alpha, \beta \in A.$
2. $(\alpha + \Im) \times (\beta + \Im) = (\alpha \bullet \beta) + \Im, \forall \alpha, \beta \in A.$

1.1.3 Fields

We call a *field with an associative product* that which has an algebraic structure consisting of a triplet $(K, +, \times)$ (which henceforth and for reasons of the subsequent lifting to which it is going to be subjected we will denote by $K(a, +, \times)$), where K is a set of elements $\{a, b, c, \ldots\}$ (which are usually called *numbers*), equipped with two binary operations, $+$ and $\times$, over K satisfying the following properties:

1. Additive properties:

   a. $(K, +)$ is closed: $a + b \in K, \forall a, b \in K.$
   b. $+$ is commutative: $a + b = b + a, \forall a, b \in K.$
   c. $+$ associative: $(a + b) + c = a + (b + c), \forall a, b, c \in K.$
   d. Neutral element for $+$: $\exists\, S \in K$ such that $a + S = S + a = a$, $\forall a \in K.$
   e. Inverse element for $+$: Given $a \in K$, $\exists a^{-S} \in K$, such that $a + a^{-S} = a^{-S} + a = S.$

2. Multiplicative properties:

   a. $(K, \times)$ is closed: $a \times b \in K, \forall a, b \in K.$
   b. $\times$ is commutative: $a \times b = b \times a\ \forall a, b \in K.$
   c. $\times$ is associative: $(a \times b) \times c = a \times (b \times c), \forall a, b, c \in K.$
   d. Unit element for $\times$: $\exists e \in K$ such that $a \times e = e \times a = a, \forall a \in K.$

e. Inverse element for $\times$: Given $a \in K$, $\exists a^{-e} \in K$, such that $a \times a^{-e} = a^{-e} \times a = e$.

3. Additive and multiplicative properties:

   a. $(K, +, \times)$ is closed: $a \times (b + c)$, $(a + b) \times c \in K$, $\forall a, b, c \in K$.
   b. Distributivity of both operations: $a \times (b + c) = (a \times b) + (a \times c)$, $(a + b) \times c = (a \times c) + (b \times c)$, $\forall a, b, c \in K$.

In the particular case that the associative property of multiplication is replaced by the following two (called *alternation* properties): $a \times (b \times b) = (a \times b) \times b$ and $(a \times a) \times b = a \times (a \times b)$, $\forall a, b, c \in K$, the field will said to be *with an alternate product*, rather than with an associative product.

1.2 More general algebraic structures

1.2.1 Vector spaces

We call a *vector space* that which has over a field $K(a, +, \times)$ a triplet $(U, \circ, \bullet)$, where U is a set of elements $\{X, Y, Z, \ldots\}$ (which are usually called *vectors*, equipped with two binary operations, $\circ$ and $\bullet$, on U satisfying $\forall a, b \in K, \forall X, Y, Z \in U$, the following properties:

1. $(U, \circ, \bullet)$ is closed, $(U, \circ)$ being a group.
2. The 4 axioms of the external operations:

   a. $a \bullet (b \bullet X) = (a \times b) \bullet X$.
   b. $a \bullet (X \circ Y) = (a \bullet X) \circ (a \bullet Y)$.
   c. $(a + b) \bullet X = (a \bullet X) \circ (b \bullet X)$.
   d. $e \bullet X = X$, e being the unit element associated with K.

Let $(U, \circ, \bullet)$ be a vector space over the the field $K(a, +, \times)$ and consider n vectors $e_1, e_2, \ldots, e_n \in U$. We say that the set $\beta = \{e_1, \ldots, e_n\}$ is a basis U (and thus, U is *n-dimensional*) if:

1. β is a set of generators, i.e. $\forall X \in U, \exists \lambda_{1,2}\, \lambda, \ldots, \lambda_n \in K$ such that $X = (\lambda_1 \bullet e_1) \circ (\lambda_2 \bullet e_2) \circ \ldots \circ (\lambda_n \bullet e_n)$.
2. β is a linearly independent system, i.e., given $\lambda_1, \ldots,$ $\lambda_n \in K$ such that $(\lambda_1 \bullet e_1) \circ (\lambda_2 \bullet e_2) \circ \ldots \circ (\lambda_n \bullet e_n) = S$ (S is the unit element associated with U with respect to $\circ$), then $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$ (where 0 is the unit element associated with K with respect to $+$).

Let $(U, \circ, \bullet)$ be a vector space over the field $K(a, +, \times)$. The set W is called a *vector subspace* of U if $W \subseteq U$ and $(W, \circ, \bullet)$ has the structure of vector space on $K(a, +, \times)$.

With respect to functions between vector spaces, we recall that if $(U, \circ, \bullet)$ and $(U', \triangle, \triangledown)$ are two vector spaces over the same field $K(a, +, \times)$, a function $f : U \to U'$ is called a *vector space homomorphism* if, $\forall a \in K$ and $\forall X, Y \in U$, it satisfies:

1. $f(X \circ Y) = f(X) \triangle f(Y)$.
2. $f(a \bullet X) = a \triangledown f(X)$.

If f is also bijective, it is called an *isomorphism*. If $U = U'$, it is then called an *endomorphism* or *linear operator*. In the latter case, if f is also bijective, it is called an *automorphism*.

1.2.2 Modules

Let $(A, \circ, \bullet)$ be a ring. We call an *A-module* that which has a pair $(M, +)$, where M is a set of elements $\{m, n, \ldots\}$ endowed with a binary operation $+$, which is endowed in turn with an external product $\times$ on A, given by $\times : A \times M \to M$ and satisfying that:

1. $(M, +)$ is a group, with, in addition, $a \times m \in M$, for all $a \in A$ and $m \in M$.
2. $a \times (b \times m) = (a \bullet b) \times m$, for all $a, b \in A$ and $\forall m \in M$.
3. $a \times (m + n) = (a \times m) + (a \times n)$, for all $a \in A$ and $m, n \in M$.
4. $(a \circ b) \times m = (a \times m) + (b \times m)$, for all $a, b \in A$ and $m \in M$.

5. $e \times m = m, \forall m \in M$, e being the unit element associated with A with respect to the operation $\bullet$.

The notion of *submodule* is analogous in its definition to the other previous substructures: Let $(A, \circ, \bullet)$ be a ring and $(M, +)$ a ring and let $(M, +)$ be an A-module, with external product $\times$ on K. The set N is called a *submodule* of M if $N \subseteq M$ and $(N, +)$ has the structure of an A-module with external product $\times$ on K.

The definitions of some functions between these structures, as well as a definition of *distance* in vector spaces, will be presented below.

Let $(A, \circ, \bullet)$ be a ring and $(M, +)$ and $(M', \triangle)$ two A-modules, with respective external products $\times$ and $\bigtriangledown$ on K. A function $f : M \to M'$ is called a *homomorphism* of A-modules if for all $a \in A$ and for all $m, n \in M$:

1. $f(m + n) = f(m) \triangle f(n)$.
2. $f(a \times m) = a \bigtriangledown f(m)$.

If f is also bijective, it is called an *isomorphism*. If $M = M'$, then it is called an *endomorphism*; in this latter case, if f is also bijective, it is called an *automorphism*.

Let $(U, \circ, \bullet)$ be a vector space over a field $K(a, +, \times)$. A function $f : U \times U \to K$ is called a *bilinear form* if $\forall a, b \in K$ and $\forall X, Y, Z \in U$:

1. $f((a \bullet X) \circ (b \bullet Y), Z) = (a \times f(X, Z)) + (b \times f(Y, Z))$.
2. $f(X, (a \bullet Y) \circ (b \bullet Z)) = (a \times f(X, Y)) + (b \times f(X, Z))$.

Let $K(a, +, \times)$ be a field endowed with an order $\leq$ and let $0 \in K$ be the unit element of K with respect to the operation $+$. Let $(U, \circ, \bullet)$ be a vector space on K. U is called a *Hilbert vector space* if it is equipped with of a *scalar product* $\langle ., . \rangle : U \times U \to K$, satisfying $\forall a, b \in K$ and $\forall X, Y, Z \in U$ the following conditions:

1. $0 \leq \langle X, X \rangle$; $\langle X, X \rangle = 0 \Leftrightarrow X = 0$.
2. $\langle X, Y \rangle = \overline{\langle Y, X \rangle}$, $\overline{a}$ being the set of a in K, $\forall a \in K$.
3. $\langle X, (a \bullet Y) \circ (b \bullet Z) \rangle = (a \times \langle X, Y \rangle) + (b \times \langle X, Z \rangle)$.

Let $(U, \circ, \bullet)$ be a vector space (of elements $X, Y, Z, \ldots$) over a field $K(a, +, \times)$, endowed with an order $\leq$ and $0 \in K$ being the unit element associated with K with respect to $+$. U is called a *metric vector space* if it is equipped with a *distance metric d*, satisfying $\forall X, Y, Z \in U$ that:

1. $0 \leq d(X, X)$ and $d(X, Y) = 0 \Leftrightarrow X = Y$.
2. $d(X, Y) = d(Y, X)$.
3. *Triangle inequality*: $d(X, Y) \leq d(X, Z) + d(Z, Y)$.

If instead of the first condition we have the following:

$$0 \leq d(X, Y) \quad \text{and} \quad d(X, X) = 0,$$

then d is called a *pseudometric distance* and U is a *pseudometric vector space*.

If $\beta = \{e_1, \ldots, e_n\}$ is a basis of U and we consider the n^2 numbers $d_{ij} = d(e_i, e_j)$, $\forall i, j \in \{1, \ldots, n\}$, the matrix $g \equiv (g_{ij})_{i,j \in \{1,\ldots,n\}} \equiv (d_{ij})_{i,j \in \{1,\ldots,n\}}$ is said to constitute a *metric* of the metric vector space U if d is a distance metric, or that it constitutes a *pseudometric* of U if d is a pseudometric distance. The said metric vector space is often denoted by $U(X, g, K)$.

1.3 Algebras

1.3.1 Algebras in general

Let $K(a, +, \times)$ be a field. We call an *algebra* on K that which is a quaternion $(U, \circ, \bullet, \cdot)$, where U is a set of elements $\{X, Y, Z, \ldots\}$ endowed with two binary operations, $\circ$ and $\cdot$, and an external product $\bullet$ on K, satisfying $\forall a, b \in K$ and $\forall X, Y, Z \in U$, the following conditions:

1. $(U, \circ, \bullet)$ has a vector space structure on $K(a, +, \times)$.
2. $(a \bullet X) \cdot Y = X \cdot (a \bullet Y) = a \bullet (X \cdot Y)$.

3. a. $X \cdot (Y \circ Z) = (X \cdot Y) \circ (X \cdot Z)$
   b. $(X \circ Y) \cdot Z = (X \cdot Z) \circ (Y \cdot Z)$

If the operation $\cdot$ is commutative, i.e. if $\forall X, Y \in U$, $X \cdot Y = Y \cdot X$ is satisfied, then U is called a *commutative algebra*. If the operation $\cdot$ is associative, i.e. if $\forall X, Y, Z \in U$, $X \cdot (Y \cdot Z) = (X \cdot Y) \cdot Z$ is satisfied, then U is called an *associative algebra*. If $\forall X, Y \in U$, $X \cdot (Y \cdot Y) = (X \cdot Y) \cdot Y$ is satisfied and $(X \cdot X) \cdot Y = X \cdot (X \cdot Y)$, then U is an *alternate algebra*. Finally, if $S \in U$ is the unit element of U with respect to the operation $\circ$, then U is a *division algebra* if $\forall A, B \in U$, with $A \neq S$, the equation $A \cdot X = B$ always has a solution.

The concept of *subalgebra* is defined analogously to that of the other substructures already seen; thus, if $(U, \circ, \bullet, \cdot)$ is an algebra on $K(a, +, \times)$, a set W is called a *subalgebra* of U if $W \subseteq U$ and $(W, \circ, \bullet, \cdot)$ has the structure of an algebra on $K(a, +, \times)$.

Finally, let $(U, \circ, \bullet, \cdot)$ and $(U', \triangle, \triangledown, \triangleright)$ be two algebras defined over a field $K(a, +, \times)$. A function $f : U \to U'$ is called a *homomorphism of algebras* if, $\forall X, Y \in U$:

1. f is a homomorphism of vector spaces restricted to the operations $\circ$ and $\bullet$.
2. $f(X \cdot Y) = f(X) \triangleright f(Y)$.

Likewise, analogously to the previous homomorphisms are the concepts of isomorphism, endomorphism, and automorphism defined for algebras.

1.3.2 Lie algebras

Let $(U, \circ, \bullet, \cdot)$ be an algebra over a field $K(a, +, \times)$. U is called a *Lie algebra* if $\forall a, b \in K$ and $\forall X, Y, Z \in U$:

1. $\cdot$ is a bilinear operation, i.e.:
   a. $((a \bullet X) \circ (b \bullet Y)) \cdot Z = (a \bullet (X \cdot Z)) \circ (b \bullet (Y \cdot Z))$.

b. $X \cdot ((a \bullet Y) \circ (b \bullet Z)) = (a \bullet (X \cdot Y)) \circ (b \bullet (X \cdot Z))$.

2. $\cdot$ is anticommutative, i.e. $X \cdot Y = -(Y \cdot X)$.
3. *Jacobi's identity*: $((X \cdot Y) \cdot Z) \circ ((Y \cdot Z) \cdot X) \circ ((Z \cdot X) \cdot Y) = S$, where S is the unit element of U with respect to $\circ$.

Let $(U, \circ, \bullet, \cdot)$ be an algebra over a field $K(a, +, \times)$. U is called a *admissible Lie algebra* if the product commutator $[.,.]$ associated with $\cdot$ is a Lie algebra, this product being defined according to: $[X, Y] = (X \cdot Y) - (Y \cdot X)$, for all $X, Y \in U$.

To facilitate the reading of the remainder of this section, in what follows, we will agree that $\mathcal{L} \equiv (U, \circ, \bullet, \cdot)$ will represent an algebra over a field $K(a, +, \times)$.

The Lie algebra $\mathcal{L}$ will be called *real* or *complex* depending on what the field K associated with it is. Also, the concepts of *dimension* and *basis* of $\mathcal{L}$ are defined as those corresponding to the vector space underlying $\mathcal{L}$.

If $\{e_1, \ldots e_n\}$ is a basis of $\mathcal{L}$, then we have $e_i \cdot e_j = \sum c^h_{i,j} \cdot e_h$, for all $1 \leq i, j \leq n$. By definition, the coefficients $c^h_{i,j}$ are called the *structure constants* or *Maurer-Cartan constants* of the algebra. These structure constants define the algebra and satisfy the following two properties:

1. $c^h_{i,j} = -c^h_{j,i}$
2. $\sum (c^r_{i,j} c^s_{r,h} \circ c^r_{j,h} c^s_{r,i} \circ c^r_{h,i} c^s_{r,j}) = 0$.

From both of these it can be deducted that the operation $\cdot$ is distributive and not associative.

The following results are easily proved:

1. If K is a field of characteristic zero, then $X \cdot X = \overrightarrow{0}$, for all $X \in \mathcal{L}$, where $\overrightarrow{0}$ is the unit element of $\mathcal{L}$ with respect to $\circ$.
2. $X \cdot \overrightarrow{0} = \overrightarrow{0} \cdot X = \overrightarrow{0}$, for all $X \in \mathcal{L}$.
3. If the three vectors that form a Jacobi identity are equal or proportional, each addend of this identity is zero.

Let $\mathcal{L}$ and $\mathcal{L}'$ be two Lie algebras over the same field K. $\Phi : \mathcal{L} \to \mathcal{L}'$ is called a *homomorphism* of Lie algebras if Φ is a linear function

such that $\Phi(X \cdot Y) = \Phi(X) \cdot \Phi(Y)$, for all $X, Y \in \mathcal{L}$. The *kernel* of the homomorphism Φ is the set whole of the elements X of the algebra such that $\Phi(X) = \overrightarrow{0}$.

Let $\mathcal{L}$ be a Lie algebra. We call a Lie *subalgebra* of $\mathcal{L}$ all that is a vector subspace $W \subset \mathcal{L}$ such that $X \cdot Y \in W$, for all $X, Y \in W$ and $\Im$ is called an *ideal* of $\mathcal{L}$ if $\Im$ is a subalgebra of $\mathcal{L}$ such that $X \cdot Y \in \Im$, for all $X \in \Im$ and for all $Y \in \mathcal{L}$ (i.e., if $\Im \cdot \mathcal{L} \subset \Im$). It is proved that given a Lie algebra $\mathcal{L}$, both the set constituted by its unit element and the algebra itself are ideals of itself. Likewise, they are both also ideals of the algebra as of its *center* (i.e., the set of elements $X \in \mathcal{L}$ such that $X \cdot Y = \overrightarrow{0}$, for all $Y \in \mathcal{L}$) as the kernel of any homomorphism from the algebra.

Given two Lie algebras $\mathcal{L}$ and $\mathcal{L}'$, we call the set $\{S = X \circ X' \mid X \in \mathcal{L} \text{ and } X' \in \mathcal{L}'\}$ the *sum* of both; it is called a *direct* sum if $\mathcal{L} \cap \mathcal{L}' = \{\overrightarrow{0}\} = \mathcal{L} \cdot \mathcal{L}'$ is satisfied. The direct sum of algebras will be denoted by $\mathcal{L} \oplus \mathcal{L}'$, and it is proved that in a direct sum $\mathcal{L}'' = \mathcal{L} \oplus \mathcal{L}'$ of Lie algebras, every element $X \in \mathcal{L}''$ can be written uniquely as $X = X_1 \circ X_2$, with $X_1 \in \mathcal{L}$ and $X_2 \in \mathcal{L}'$. It is easy to see that both the sum and the intersection and the the product (bracket) of ideals of a Lie algebra are also ideals of the algebra.

If $\mathcal{L}$ is a Lie algebra, it is called a *derived algebra* of $\mathcal{L}$ and is represented by $\mathcal{L} \cdot \mathcal{L}$, the set of elements of the form $X \cdot Y$ with $X, Y \in \mathcal{L}$.

An ideal $\Im$ of a Lie algebra $\mathcal{L}$ is called *commutative* if $X \cdot Y = \overrightarrow{0}$, for all $X \in \Im$ and for all $Y \in \mathcal{L}$. In turn, a Lie algebra is called *commutative* if, considered as an ideal, it is commutative. From the two definitions above, it follows immediately that a Lie algebra is commutative if and only if its derived algebra is null.

There are several types of Lie algebras. A Lie algebra is *simple* if it is not commutative and the unique ideals that it contains are trivial ones, while it will be called *semisimple* if it does not contain non-trivial commutative ideals. Obviously, any simple Lie algebra is semisimple.

It is easy to prove that any semisimple Lie algebra is a direct sum of simple Lie algebras and that every semisimple Lie algebra $\mathcal{L}$ satisfies that $\mathcal{L} \cdot \mathcal{L} = \mathcal{L}$.

The *classification* of complex simple Lie algebras dates back to the late 19th century (Killing, Cartan, etc.) and is as follows:

1. Lie algebras of the *special linear* set.
2. *Odd orthogonal* Lie algebras.
3. *Simplectic* Lie algebras.
4. *Even orthogonal* Lie algebras.
5. In addition, there are five simple algebras that are not contained in any of these groups and that are referred to as *exotic*.

Other types of Lie algebras are *resolvable* and *nilpotent*. The former are defined in the following way: Let $\mathcal{L}$ be a Lie algebra. $\mathcal{L}$ is called *resolvable* if it satisfies that in the sequence

$$\mathcal{L}_1 = \mathcal{L}, \quad \mathcal{L}_2 = \mathcal{L} \cdot \mathcal{L}, \quad \mathcal{L}_3 = \mathcal{L}_2 \cdot \mathcal{L}_2, \ldots, \mathcal{L}_i = \mathcal{L}_{i-1} \cdot \mathcal{L}_{i-1}, \ldots$$

(called the *resolvability sequence*), there is a natural number n such that $\mathcal{L}_n = \{\overrightarrow{0}\}$. The least of these numbers satisfying this condition is called the *resolvability index* of the algebra.

Similarly, an ideal of the algebra is called *resolvable* if, in forming the corresponding resolvability sequence, there is an $n \in \mathbf{N}$ such that $\Im_n = \{\overrightarrow{0}\}$. In this regard, the following results are proved:

1. If $\mathcal{L}$ is a Lie algebra, then $\mathcal{L}_i$ is an ideal of $\mathcal{L}$ and $\mathcal{L}_{i-1}$, for all $i \in \mathbf{N}$.
2. Every subalgebra of a resolvable Lie algebra is resolvable.
3. The intersection, the sum, and product of resolvable ideals of $\mathcal{L}$ are also resolvable ideals of $\mathcal{L}$.

The consequence of this last result is that the sum of all resolvable ideals of a Lie algebra is also a resolvable ideal of the algebra, which is called the *radical* of $\mathcal{L}$ and is denoted by $\mathrm{rad}(\mathcal{L})$. In the particular case of $\mathcal{L}$ being semisimple, $\mathrm{rad}(\mathcal{L}) = \{\overrightarrow{0}\}$.

With respect to the second type of Lie algebras mentioned above, a Lie algebra $\mathcal{L}$ is called *nilpotent* if in the sequence

$$\mathcal{L}^1 = \mathcal{L}, \quad \mathcal{L}^2 = \mathcal{L} \cdot \mathcal{L}, \quad \mathcal{L}^3 = \mathcal{L}^2 \cdot \mathcal{L}, \ldots, \mathcal{L}^i = \mathcal{L}^{i-1} \cdot \mathcal{L}, \ldots$$

(called the *nilpotency sequence*), there is a natural number n such that $\mathcal{L}^n = \{\overrightarrow{0}\}$. The least of these natural numbers satisfying this condition is called *index of nilpotency*.

Analogous to what happened with resolvable Lie algebras, an ideal $\Im$ of $\mathcal{L}$ is called *nilpotent* if in forming the sequence of nilpotency of $\Im$, there exists an $n \in \mathbf{N}$ such that $\Im^n = \{\overrightarrow{0}\}$ and it is also satisfied that the sum of two nilpotent ideals of a Lie algebra is another nilpotent ideal and that every nilpotent subalgebra of a Lie algebra is nilpotent. Moreover, every non-null nilpotent Lie algebra must have a non-null center.

The sum of all the nilpotent ideals of Lie algebra $\mathcal{L}$ is called the *nil radical* of $\mathcal{L}$ and is denoted by nil-rad($\mathcal{L}$). In this respect, it is proved that the nil-radical of a (not necessarily nilpotent) Lie algebra is also a nilpotent ideal of the algebra and that the nil-radical is contained in the radical of the algebra.

A result that relates some of the concepts defined above is as follows: a complex Lie algebra is resolvable if and only if its derived algebra is nilpotent. We will make use of it when we consider the isotopic lifting of Lie algebras.

Finally, we recall the definition and some properties of a particularly important subset of nilpotent Lie algebras, which will also be discussed in the subsequent lifting that is carried out. They are the *filiform* Lie algebras (obtained by Vegné in 1966). Their definition is as follows: Let $\mathcal{L}$ be a nilpotent Lie algebra. $\mathcal{L}$ is called *filiform* if it satisfies that

$$\dim \mathcal{L}^2 = n-2\,, \ldots\,, \dim \mathcal{L}^i = n-i, \ldots\,, \dim \mathcal{L}^n = 0,$$

where $\dim \mathcal{L} = n$.

In all filiform Lie algebras, the existence of a basis $\{e_1, \ldots, e_n\}$, known as an *adapted basis*, is proved. It satisfies that

$$e_1 \cdot e_2 = 0, \quad e_1 \cdot e_h = e_{h-1} \quad (h = 3, \ldots, n), \quad e_3 \cdot e_h = 0 \quad (h = 2, \ldots, n).$$

Let $\mathcal{L}$ be a filiform Lie algebra. The *invariants* i and j of the algebra (invariant in the sense of not relying on the adapted basis chosen in

the algebra), are defined according to:

$$i = \inf\{k \in \mathbf{Z}^+ \mid e_k \cdot e_n \neq \overrightarrow{0} \quad k > 1\},$$

$$j = \inf\{\, k \in \mathbf{Z}^+ \mid e_k \cdot e_{k+1} \neq \overrightarrow{0}\},$$

both invariants being related by the following inequalities:

$$4 \leq i \leq j < n \leq 2j - 2$$

and likewise also satisfying that all complex filiform Lie algebras are defined with respect to a suitable basis, if the products $e_h \cdot e_n$ for $i \leq h \leq n-1$ or the products $e_k \cdot e_{k+1}$ for $j \leq k < n$ are satisfied.

Chapter 2
HISTORICAL EVOLUTION OF THE LIE AND LIE-SANTILLI THEORIES

This chapter lists some biographical notes on the life and scientific work of two distinguished mathematicians who with their contributions have contributed to a great development not only in mathematics, but also in other related sciences, primarily physics and engineering.

The first of them, perhaps not so well-known as he should be and who, as a general rule, has not been given the importance which he really deserves, is **Marius Sophus Lie** (Norfjordeid (Norway) 1842 - Christiania (today Oslo) 1899), undoubtedly one of the greatest mathematicians of the 19th century. He is notable not only for his mathematical discoveries, which gave rise to what is now known as *Lie theory*, but also for their countless applications to physics and engineering, which have made extraordinary progress in the further development of these disciplines. It is said that the great Albert Einstein affirmed that "*without the discoveries of Lie, the Theory of Relativity would probably never have been born.*"

The second of these authors, our contemporary, is **Maria Ruggero Santilli**, an American mathematician of Italian origin who is devoting his life to the study of a generalization of Lie theory, a generalization which is currently known as *Lie-Santilli theory*, which tries to give a satisfactory answer to certain questions of various types—physical,

chemical, biological, astronomical, etc.—that today are not adequately explained, neither in Lie theory itself nor in the current knowledge of science.

2.1 Lie theory

In this section we will discuss some aspects of Lie theory: its origins, its later development and some attempts at generalizing it, in particular the generalizations of groups and of Lie algebras.

2.1.1 Origin of Lie theory

The theory of *permutation groups* of a finite set is developed and begins to be used (by Serret, Kronecker, and Jordan, among others) around 1860. On the other hand, the theory of *invariants*, then in full development, acquainted mathematicians with certain infinite sets of geometric transformations established by composition (linear or projective transformations). However, it was not until 1868 when both theories are unified, thanks to a work of Jordan on *motion groups* (closed subgroups of the group of displacements of the Euclidean space in three dimensions) (see [45]).

In 1869, Felix Klein and Sophus Lie are admitted to the University of Berlin. There, Lie conceived the notion of *invariant* in analysis and differential geometry from the conservation of the differential equations by means of a *continuous* family of transformations. A year later, both travel to Paris, where they developed together a work (see [55]) in which they studied the connected commutative subgroups of the projective group of the plane, along with the geometric properties of their orbits. In 1871 Klein would begin to be interested in non-Euclidean geometries and a first classification of all the known geometries, while Lie (who already used the term *transformation group*) explicitly pre-

sented the problem of the determination of all continuous or discontinuous subgroups of $Gl(n, \mathbf{C})$ ([62]).

Since 1872, Lie seems to abandon the theory of transformations groups for the study of contact transformations, integration of first-order partial differential equations, and relationships between these two theories. In the course of his investigations, Lie became familiar with the so-called *Poisson brackets* (expressions of type $(f, g) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x_i} \cdot \frac{\partial f}{\partial p_i} \right)$, where f and g are two transformations and x_i and p_i are the canonical coordinates in the cotangent space $T(\mathbf{C}^n)$) and also began to work with the *brackets of differential operators* (brackets of type $[X, Y] = XY - YX$), which had already appeared in the theory of Jacobi-Clebsch on complete systems of partial differential equations of first order (a notion equivalent to the completely integrable systems of Frobenius). Lie then applies the theory of Jacobi to the Poisson bracket, considering that these are associated with differential operators of Jacobi brackets. Thus, Lie examines the set of functions $(u_j)_{1 \leq j \leq n}$, depending on the variables x_i and p_i, such that the parentheses (u_j, u_k) be functions of u_h, calling these *groups* (which were considered already by Jacobi, implicitly).

In the fall of 1873, Lie continues his studies of transformation groups. On the basis of a *continuous group* of transformations over n variables, he shows that these transformations form a stable set by composition (see [63]). On the other hand, he links the theory of continuous groups with his previous research on contact transformations and differential equations. All of this begins to consolidate the new theory of transformation groups.

In the following years, Lie continued this study, obtaining a certain number of more specific results, among others: the determination of transformation groups of straight lines in a plane, subgroups of small codimension of projective groups, groups with at least six parameters, etc. However, he does not abandon the theory of differential equations. On the contrary, he thinks that the theory of transformation groups should be an instrument for integrating differential equations, where the transformation groups would play an analogous role to the Galois

groups with an algebraic equation. These investigations lead him to introduce certain sets of transformations with an infinite number of parameters, which he called *continuous and infinite groups* (today called *Lie pseudo-groups,* to differentiate them from Banach-Lie groups).

With all of the above results, Lie already introduces what will later be called *Lie groups and algebras.* However, given that his first works were written in Norwegian and published in the reviews of the Academy of Christiania, which had very little external dissemination, his ideas were not well known at first.

From 1886 until 1898, however, Lie comes to occupy the post that Klein leaves vacant at Leipzig, which will encourage the start of a great diffusion of their works. This also contributed to having Engel as an assistant associate, who worked with Lie on the ideas of the latter. This permits the appearance of the treatise *Teorie der transformationsgruppe* ([67]), written by both between 1888 and 1893. It addresses transformation groups, the *variable space,* x_i, and the *the parameter space,* a_j, being vitally important. These variables and parameters were considered at first as belonging to the complex field, with which the composition of transformations sometimes presented serious difficulties, so he had to establish the local point of view whenever it was necessary. Both authors also treated in this text the *mixed groups* (groups with a finite number of connected components).

In short, the general theory developed in this treatise constituted a real dictionary on the properties of continuous and finite groups with their infinitesimal transformations. Also, in addition, three fundamental theorems of Lie appeared, on which the theory of Lie groups is based. These results are completed with the study of isomorphisms between groups. Thus, two transformation groups are said to be *similar* if one could pass from one to another by an invertible coordinate transformation over variables and an invertible coordinate transformation over parameters. Lie proves that two groups are similar if using a variable transform one can arrive at infinitesimal transformations from one group to another. A necessary condition to make this so was that the Lie algebras of these two groups be isomorphic. However this condi-

tion was not sufficient, so that Lie had to devote a whole chapter of his treatise to obtain additional conditions to ensure that the groups were similar.

By analogy with the theory of permutation groups, Lie also introduced in this treatise the notions of *subgroup* and *distinguished subgroup*, proving that they corresponded to the subalgebras and ideals of a Lie algebra, respectively. For these results, as for the three fundamental theorems, Lie used primarily the *Jacobi-Clebsch theorem*, which gives the integrability of a differential system.

Notions of *transitivity* and *primitivity*, so important for permutation groups, are presented naturally for finite and continuous groups of transformations, being also studied in detail in this treatise, as well as the relationships between the *subgroup stabilizers* of a point and the notion of *homogeneous space*. Finally, the treatise finishes with the introduction of the notions of *derived group* and *resolvable group* (called an *integrable group* by Lie). This terminology, suggested by the theory of differential equations, would remain in use until a later work of Hermann Weyl, in 1934, who first introduced the term *Lie algebra* as a substitute for *infinitesimal algebra*, which had been used until then and which he himself introduced in 1925.

2.1.2 Further development of Lie theory

The period between 1888 and 1894 is marked by the work of Engel and his student Umlauf and, above all, of Killing and E. Cartan. They all arrived at a series of spectacular results regarding complex Lie algebras.

Killing is an example of this advance in [54]. It had been Lie himself who introduced (see vol. I, p. 270 of [67]) the notion of *resolvable* Lie algebras and who proved the theorem of reduction of linear Lie algebras resolvable to the triangular form (in the complex case). Killing noted that in any Lie algebra there is a resolvable ideal (today called

the *radical*) and that the quotient of a Lie algebra by its radical has null radical. Killing then called Lie algebras of null radical *semi-simples* and proved that they are products of simple algebras (this had already been studied by Lie (see vol. 3, p. 682 of [67]), who proved the simplicity of *classic* Lie algebras).

In addition, although Lie had already studied something about the subject (treating the subalgebras of Lie of dimension two that contain a given element of a Lie algebra), Killing also introduced the characteristic equation $\det(\mathrm{ad}(x) - \omega \cdot 1) = 0$ into Lie algebras and made use of the roots of this equation for a semi-simple algebra to obtain the classification of the complex simple Lie algebras (later, Umlauf in 1891 will classify nilpotent Lie algebras of dimension ≤ 6).

Killing also proved that an algebra derived from a resolvable algebra is of *zero range*, that is, that $\mathrm{ad}(x)$ is nilpotent for every element x of the algebra. Soon after, Engel would prove that these algebras of zero rank are *resolvable*; Cartan, in turn, introduced in his thesis what we now call the *Killing form*, establishing the two fundamental criteria that characterize, by means of this form, resolvable Lie algebras and semisimple Lie algebras.

Killing also claimed that the algebra derived from a Lie algebra is sum of a semi-simple algebras and its radical (which is nilpotent), but his demonstration was incomplete. Subsequently, Cartan gave a demonstration proving more generally that any Lie algebra is the sum of its radical and a semi-simple subalgebra (t 1, p. 104 of [20]). However, it was Engel who, in an indisputable way, came to affirm the existence in any non-resolvable Lie algebra of a simple Lie subalgebra of dimension 3. It would be E. E. Levi who, in 1905 (see [61]), gave the first demonstration of this result, although for the complex case. Another demonstration of this, now valid for the real case, was given by Whitehead in 1936 (see [185]). This result would be completed later, in 1942, by A. Malcev through the uniqueness theorem of *Levi sections*.

2.1.3 First generalizations of Lie algebras

The exponential function was the starting point to achieve the first generalizations of Lie algebras. Thus, the early studies are due to E. Study and Engel. The latter, considering this function, gave examples of two locally isomorphic groups, but from the global point of view they were very different (see [26]).

In 1899 (see vol. 3, pp. 169-212 of [90]), Poincaré took up the study of the exponential function, providing important results regarding the adjoint group, attaining that a semi-simple element of a group G may be the exponential of an infinite number of elements of the Lie algebra $\mathbf{g} \equiv L(G)$, while a non-semi-simple element may not be an exponential. In the course of his investigations, Poincaré regarded the associative algebra of differential operators of all orders generated by the operators of a Lie algebra. He proves that if $(x_i)_{1\leq i\leq n}$ is a basis for the Lie algebra, this associated algebra (generated by the x_i) has as its basis certain functions symmetric to the x_i (sums of non-commutative *monomials*, deduced from a given monomial by all permutations of the factors). The gist of his demonstration is its algebraic nature, which allows one to obtain the structure of an *enveloping algebra*. Other similar demonstrations would also be given by Birkhoff and Witt, in 1937.

Other researchers who also used the exponential function in their research on Lie theory were Campbell in the biennium 1897-1898, Pascal, Baker in 1905 (see [11]), Haussdorff in 1906 (see [31]), and finally Dynkin in 1947 (see [24]), who took up again the question, generalizing all the previous results for Lie algebras of finite dimension over $\mathbf{R}$, $\mathbf{C}$, or an ultrametric field.

Besides the foregoing, Hilbert posed a new theoretical development in 1900. Previously, F. Schur (see [166]) had come to improve one of the results of Lie, using functions of analytic character in the transformations that he handled. Schur demonstrated, then, that if functions used were of the class $\mathbf{C}^2$, the groups that were obtained were holoedrically isomorphic to an analytical group. Lie himself had already announced in [65], based on the geometry, that the hypotheses of ana-

lyticity were not of a natural character. These findings led Hilbert to wonder whether the same conclusion was possible under the hypothesis that the functions would be continuous (this would be the fifth problem posed by Hilbert in the Mathematical Congress of 1900). This problem gave rise to numerous investigations, the most complete result being the theorem proved by A. Gleason, D. Montgomery, and L. Zippin in 1955 (see [79]): *Any locally compact topological group has an open subgroup that is the projective limit of Lie groups,* a result that implies that any locally Euclidean group is a Lie group.

On the other hand, Lie raised in his theory the problem of determining the linear representations of minimal dimension of the simple Lie algebras, solving it for the classical algebras. In his doctoral thesis, Cartan met this problem also for exceptional simple algebras. The point of view of Cartan was to study the non-trivial extensions of Lie algebras (from a simple Lie algebra and a (commutative) radical of minimal dimension). These methods would be generalized by him later for all irreducible representations of the real or complex simple Lie algebras.

In addition, and as a result of his investigations about the integration of differential systems, Cartan introduced in 1904 (see vol. II, p. 371 of [20]) the Pfaff forms: $\omega_k = \sum_{i=1}^{n} \Psi_{ki}\, da_i$, subsequently called *Maurer-Cartan forms.* With them, Cartan showed that he could develop a theory of finite and continuous groups from the ω_k, and establish the equivalence of this point of view with that of Lie. For Cartan, however, the interest of this method was due above all to its being adapted to infinite and continuous groups, thus generalizing the Lie theory.

Two other problems already raised by Lie himself were the problem of the isomorphism of every Lie algebra with a linear Lie algebra and the problem of complete reducibility of a linear representation of a Lie algebra. Regarding the first of them, Lie believed in his affirmative answer (see [64]), considering the adjoint representation. Ado was in 1935 (see [1]) the first who showed it properly.

Regarding the second problem, Study already worked on it, in a geometric form, in an unpublished manuscript, although cited by

Lie (see vol. 3, pp. 785-788 of [67]), demonstrating the verification of this property for the linear representations of Lie algebras of $\mathbf{Sl}(2,\mathbf{C}), \mathbf{Sl}(3,\mathbf{C})$ and $\mathbf{Sl}(4,\mathbf{C})$. In this regard, Lie himself and Engel had guessed a theorem of complete reducibility valid in $\mathbf{Sl}(n,\mathbf{C})$, $\forall n \in \mathbf{N}$. Already in 1925 (see [183]) H. Weyl established the complete reducibility of linear representations of semi-simple Lie algebras, using an argument of a global nature (which had already been used by Cartan in irreducible representations, according to Weyl himself). But it was not until 1935 when Casimir and Van der Waerden ([21]) obtained an algebraic proof of the result. Other algebraic demonstrations would be given by R. Brauer in 1936 (see [15]) and J. H. C. Whitehead in 1937 (see [185]).

Most of the aforementioned works were limited to real or complex Lie algebras which only corresponded with Lie groups in the usual sense. The study of Lie algebras over a field other than $\mathbf{R}$ or $\mathbf{C}$ was tackled by Nathan Jacobson in 1935 (see [36]), showing that most of the classic findings remained valid for any field of zero characteristic.

A new branch in the theory of Lie algebras was opened in 1948 by Chevalley and Eilenberg, who introduced the *cohomology* of Lie algebras in terms of invariant differential forms. In 1951, Hochschild discussed the relationship between the cohomology of Lie groups and Lie algebras. Later, Van Est (1953-1955) would do it.

Also in 1948, A. A. Albert developed the concept of the Lie-*admissible* algebras, characterized by a product $X \cdot Y$, such that the associated commutator $[X,Y] = X \cdot Y - Y \cdot X$ is Lie. Albert demonstrates that the algebra associated with a Lie algebra, by means of the product commutator, is a Lie admissible algebra. In fact, he proves, using the product commutator, that every associative algebra is a Lie admissible algebra (see [2]).

2.1.4 First generalisations of Lie groups

One of the avenues of research arising from the theory of Lie groups is the study of *global* Lie groups. It was due to Hermann Weyl, who was inspired in turn by two theories developed independently: the theory of the *linear representations of complex semi-simple Lie algebras,* due to Cartan, and theory of the *linear representations of finite groups,* due to Frobenius and generalized to orthogonal groups by I. Schur in 1924, using the idea of Hurwitz (1897) of replacing the operator defined on a finite group by an integration relative to an invariant measure (see [34]). Schur used this procedure (see [167]) to show the complete reducibility of representations of the orthogonal group $\mathbf{O}(n)$ and the unitary group $\mathbf{U}(n)$, by constructing an invariant, positive, non-degenerate Hermitian form. He deduced, in addition, the complete reducibility of holomorphic representations of $\mathbf{O}(n, \mathbf{C})$ and $\mathbf{Sl}(n, \mathbf{C})$, establishing orthogonality relations for the characteristics of $\mathbf{O}(n)$ and $\mathbf{Sl}(n, \mathbf{C})$, and determining the characteristics of $\mathbf{O}(n)$. Weyl extended this method to semi-simple, complex Lie algebras in 1925 (see [183]). He showed that given a such algebra, it has a *real compact form,* i.e. it comes, by extension of scalars $\mathbf{R}$ to $\mathbf{C}$, from an algebra over $\mathbf{R}$ whose adjoint group is compact. He also showed that the fundamental group of the adjoint group is finite, since its covering is compact. He deduced the complete reducibility of representations of algebras of semi-simple, complex Lie algebras and determined all the characteristics of these representations.

After the works of Weyl, Cartan adopted a global perspective in his research on *symmetric spaces* and Lie groups, which would lay the foundations of his 1930 exposition (see vol. I, pp. 1165-1225 [20]) of the theory of *continuous and finite groups,* where he, in particular, found the first demonstration of the global variant of the third fundamental theorem of Lie (i.e., the existence of a Lie group over any given Lie algebra). Cartan also showed that any closed subgroup of a real Lie group is a Lie group, which generalizes a result of Von Neumann in 1927 on the closed subgroup of a linear group (see [82]). Neumann also

showed that any continuous representation of a complex semisimple group is real analytic.

Subsequently, Pontrjagin, in 1939, in his work on *topological groups* (see [91]), distinguished the local and global characters of the theory of Lie groups. In 1946, Chevalley developed a systematic discussion of the *analytic varieties* and the *exterior differential calculus* (see [23]). The *infinitesimal transformations* of Lie appeared as a vector field and the algebra of a Lie group was identified as the *space of fields of vectors invariant to the left* on that group.

Another generalization of Lie groups started in the year 1907 from the works of Hensel (see [32]), who developed the *p-adic* functions (as defined by the developments in the entire series) and the *p-adic Lie groups.* Hensel discovered a local isomorphism between the additive and multiplicative groups of $\mathbf{Q}_p$ (or more generally, of any complete ultrametric field of characteristic zero). A. Weil in 1936 (see [180]) and E. Lutz the following year (see [71]), starting from *p-adic elliptic curves,* deepened this subject. As an arithmetic function, it was the construction of a local isomorphism of commutative group with an additive group, based on the integration of an invariant differential form. This method also applies to the Abelian varieties, as Chabauty remarked shortly after in 1941 (see [22]), to demonstrate a particular case of the Mordell conjecture. R. Hooke, pupil of Chevalley, established in 1942, in his doctoral thesis, fundamental theorems about p-adic Lie groups and algebras (see [33]). Later, in 1965, this work would be developed in a more precise way by M. Lazard, in [60].

One last example of a generalization of Lie groups are the *Lie-Banach groups.* Lie groups are treated *of infinite dimension,* in which, from the local point of view, a neighborhood of zero is replaced within a Euclidean space by a neighborhood of zero in a Banach space. This was treated by Birkhoff in 1936 (see [13]), thus attaining to the notion of a *normed complete* Lie algebra. By 1950, Dynkin would complete the results of Birkhoff, although his results would remain local.

In addition to the previous works, a new study relates Lie algebras with Lie groups. The origin of this new work is in 1932, the year in

which P. Hall presents a study about a class of p-groups called *regular*, on which he develops the subject of *commutators* and builds the *descending central series* of a group (see [30]). Later, between the years 1935 and 1937, the work of Magnus [72] and Witt (see [186]) appears, in which, together with those of Hall, the *free* Lie algebras are defined, associated with free groups. Witt will show that the enveloping Lie algebra of a free Lie algebra L is a free associative algebra, deducing, then, the range of homogeneous components of L (*Witt formulas*). Finally, in 1950, M. Hall determines the basis of a free Lie algebra known as the *Hall's basis*, which already appeared implicitly in the works of P. Hall and Magnus.

2.2 The Lie-Santilli isotheory

In this section we discuss, first of all, those problems which gave rise to the emergence of this isotheory, followed by a treatment of the historical evolution of the same. Among the first we deal with the emerging application of Lie theory in physics and the concepts of an admissible algebra and universal enveloping Lie algebra.

2.2.1 Lie theory in physics

Lie theory, apart from its application in different branches of mathematics, also has numerous applications in the field of physics. Lie groups were introduced into physics even before the development of the theory of *quanta*, through representations by matrices of finite or infinite dimensions. They were useful for the description of (locally) symmetric and homogeneous pseudo-Riemann spaces, being used in particular in geometric theories of gravitation. However, it was the development of modern quantum theory, in the years 1925 and 1926, which facilitated the explicit introduction into physics of Lie groups.

In this theory, the physical observations appeared represented by Hermitian matrices, while the transformations were described by means of their representations by unitary or anti-unitary matrices. The operators used (with respect to a law similar to the one of commutators: $X \cdot Y - Y \cdot X$) pertained to a Lie algebra of finite dimension, while the transformations described by a finite number of continuous parameters belong to a Lie group.

Other reasons that led to the introduction of Lie theory into physics were the presence of exact kinematic symmetries or the use of dynamic models designed with a higher symmetry to that present in the real world. These exact kinematic symmetries appear by the use of a canonical formalism in classical mechanics and quantum theory. Thus, Lie theory finds applications not only in elementary particle or nuclear physics, but also in fields as diverse as: continuous mechanics, solid state physics, cosmology, control theory, statistical physics, astrophysics, superconductivity, computer modeling, and theoretical biophysics, among others.

However, as we will see in subsequent pages of this text, the theory of Lie seems unable to explain satisfactorily many other problems of physics, which will lead to the emergence of many generalizations of it that try to resolve these problems, one of the latter being, without doubt, the most important: the Lie-Santilli isotheory.

2.2.2 Origin of the isotopy

In 1958 R. H. Bruck ([16]) pointed out that the notion of *isotopy* already existed in the early stages of set theory, where two Latin squares are said to be *isotopically related* when their permutations coincide. The word *isotopic* comes from the Greek words "Ισοσ Τοποσ," which means *the same place*. This term intends to point out that the two figures have the same configuration, the same topology. Now, given that a Latin square could be considered as a multiplication table of a quasi-

group, its isotopies are propagated to these latter. Later it would be done for algebras and, more recently, for the majority of the branches of mathematics. As an example, K. Mc. Crimmon would study in 1965 (see [73]) the isotopy of the Jordan algebras, while already in 1967 R. M. Santilli himself would develop in [94] the isotopies of the associative universal enveloping algebra U for a fixed Lie algebra L, calling them *isoassociative enveloping algebras* ($\widehat{U}$). Other more recent works on isotopy can be seen in Tomber ([12]) in 1984 or in the monograph of Lõhmus, Paal, and Sorgsepp in 1994 (see [68]).

The term of isotopy appears also in other sciences. Thus, for example, in chemistry an *isotopy* is the condition of two or more simple bodies which represent the same atomic structure and have identical properties, although they disagree in atomic weight. Such elements are called *isotopes.*

2.2.3 Lie admissible algebras

In the same work of 1967 previously cited (see [94]), Santilli introduced and developed the new notion of a Lie admissible algebra, resulting from studies of his thesis in theoretical physics at the University of Turin. The first notion of *Lie admissibility* was due to the American mathematician A. A. Albert, who developed this concept in 1948 (see [2]), referring to a non-associative algebra U, with elements $\{a, b, c, \ldots\}$, and an (abstract) product $a \cdot b$, such that its associated commutative algebra U^- (which is the same vector space U, although equipped with the product commutator $[a, b]_U = a \cdot b - b \cdot a$) is a Lie algebra. As such, the algebra U does not necessarily contain a Lie algebra in its classification, thus being inapplicable for the construction of the mathematical and physical results of Lie theory. In fact, Albert began imposing that U should contain Jordan algebras as special cases, directing his studies toward quasi-associative algebras of product $(a, b) = \lambda \bullet a \cdot b - (1 - \lambda) \bullet b \cdot a$, where λ is a non-zero scalar

distinct from 1, taking into account that for $\lambda = \frac{1}{2}$ and the product $a \cdot b$ being associative, one obtains a Jordan commutative algebra, but which does not admit a Lie algebra under any finite value of λ.

Santilli then proposes that the algebra U can admit Lie algebras in its classification, i.e., that the product $a \cdot b$ admits as a particular case the Lie product bracket: $[a, b] = ab - ba$. This new definition was presented as a generalization of the *flexible* admissible Lie algebras, of product $(a, b) = \lambda \bullet a \cdot b - \mu \bullet b \cdot a$, with $\lambda, \mu, \lambda + \mu$ non-zero scalars under conditions in which $[a, b]_U = (a, b) - (b, a) = (\lambda + \mu) \bullet (a \cdot b - b \cdot a)$ is the Lie product bracket, such that the product (a, b) admits the Lie product as a particular case. The last conditions amount to $\lambda = \mu$ and that the product $a \cdot b$ is associative. Santilli also thus began the study of the so-called *q-deformations* (studied later in the 1980s by a large number of authors), which consist of considering the product $(a, b) = a \cdot b - q \bullet b \cdot a$, so, under the previous notations, $\lambda = 1$ and $\mu = q$.

In 1969, Santilli would identify the first Lie admissible structure in the classical dynamics of dissipative systems, which illustrated the physical need for his new concept (see [95]).

Subsequently, in 1978, Santilli introduced in [97] and [98] the generalization of Lie admissible algebras by means of the product $(a, b) = a \times R \times b - b \times S \times a$, where $a \times R$, $R \times b$, etc., are associative, $R, S, R + S$ being arbitrary, non-singular operators able to admit the scalars λ and μ as particular cases. He then discovered that the associated commutative algebra U^- was not a conventional Lie algebra with the commutator product $a \times b - b \times a$, but that it was characterized by the product $[a, b]_U = (a, b) - (b, a) = a \times T \times b - b \times T \times a$, where $T = R + S$. He called this product *Lie isotopic.* All this led to the third definition of Lie admissibility, today called *Albert-Santilli Lie admissibility,* which refers to the non-associative algebras U admitting Lie-Santilli isoalgebras both in the associated commutator algebra U^- as in algebras that are in its classification. In these same works, Santilli identified a classical representation and an operator of the general Lie admissible algebras, establishing the foundations of an admissible generalization of Lie in analytical mechanics and quantum mechanics, as well as in their

corresponding applications. The particular isotopic case that appears when one considers $R = S = T = T^\dagger \neq 0$ deserves special attention.

This notion of Lie admissibility of Albert-Santilli can be considered as the birth of Lie-Santilli isotheory, and can be found in section 3 (particularly in subsection 3.7) of [98] and in section 4 (particularly in the section 4.14) of [97]. In fact, Santilli recognized that the parentheses $(a, b)_U$ associated with the non-associative algebra U of product $(a, b) = a \times R \times b - b \times S \times a$, can be identically rewritten as the commutator product associated with the associative algebra $\widehat{A}$, of product $a \times R \times b$, $[a, b]_U = [a, b]_{\widehat{A}}$. This last identity indicated the transition of the studies within the context of non-associative algebras, given by Santilli until 1978, to the genuine study of the generalization of Lie theory given by Santilli since 1978, which is based on the lifting of the associative enveloping algebras, by lifting the conventional product $a \times b$ to the isotopic product $a \widehat{\times} b = a \times T \times b$.

2.2.4 Universal enveloping Lie algebra

A Lie algebra L being fixed, its associative universal enveloping algebra is conventionally defined as a pair (U, T), where U is an associative algebra and T is a homomorphism of L in the commutative algebra U^- associated with U, satisfying that if U' is another associative algebra and T' is a homomorphism of L in U'^-, then there is a unique homomorphism γ of U^- in U'^- such that $T' \equiv \gamma \circ T$, the following diagram being commutative:

$$\begin{array}{c} L \xrightarrow{T} U^- \\ \searrow^{T'} \# \downarrow^{\gamma} \\ U'^- \end{array}$$

In physics, the concept of universal enveloping algebra has played a very important role. It is used, for example, in the calculation of the

magnitude of angular momentum or, as an algebraic structure, to represent time evolution.

Santilli proved in 1978 (see [99]) that the enveloping algebra of the time evolution of Hamiltonian mechanics is not associative, so it was not then directly compatible with Lie theory. In fact, Santilli shows that Poisson brackets on an algebra of differentiable operators, L,

$$(X,Y) = \sum_{i=1}^{n} \left(\frac{\partial X}{\partial r_i} \cdot \frac{\partial Y}{\partial p_i} - \frac{\partial Y}{\partial r_i} \cdot \frac{\partial X}{\partial p_i} \right),$$

coincide with the commutator product associated with the enveloping algebra of product

$$X \cdot Y = \sum_{i=1}^{n} \frac{\partial X}{\partial r_i} \cdot \frac{\partial Y}{\partial p_i},$$

which is not associative. That is, the vector space of elements X_i (and associated polynomials) over the field $\mathbf{R}$ of real numbers, endowed with the previous product, is an algebra, since it satisfies the distributive laws to the left and right, and the scalar law. In addition, it is not associative, as in general $(X \cdot Y) \cdot Z \neq X \cdot (Y \cdot Z)$. Then, since associative and non-associative algebras are different, without a known function that connects them, Santilli argues that the non-associative character of the enveloping algebra does not permit the formulation of Hamiltonian mechanics according to the product given by the Poisson bracket. So he looks for a dual generalization of Lie theory according to the isotopic generalization (still based on associative enveloping algebras and formulated by means of the most general associative product possible) and the (genotopic) generalization by means of the theory of Lie admissible algebras (based on enveloping Lie admissible algebras that are developed by means of the most general non-associative product $X \cdot Y$ possible, whose associated commutator product, $X \cdot Y - Y \cdot X$, is Lie).

2.2.5 Other issues not resolved by Lie theory

The generalizations that we just saw were also demanded by other theoretical developments. For example, when in *simplectic and contact geometry* one achieves a transformation of the structure, the basis of the theory is still able to be used, although there is a loss in the formulation of the Lie theory. Thus, notions, properties, and theorems for the conventional structure are not necessarily applicable to the most general established structure. We just noted when the first of the generalizations, then, takes priority. This seeks to recover the compatibility of the formulation with the simplectic and contact geometry, i.e., it seeks to attain algebraic notions, properties, and theorems that are directly applicable to the most general representations possible. This generalization also has an associative nature, as it passes from the usual product $X \cdot Y$ to the more general associative product $X * Y$ possible. So, when we can pass from a certain non-associative structure to one that has a general associative character, then we can apply the theory that Santilli was looking for.

Another new problem that leads to the search for a generalization of Lie-Santilli theory appears when he passes directly to study the theory of algebras and Lie groups, with a view to its immediate applications in physics. This theory, in its usual formulation, is established from a *linear, local-differential, and canonical (Hamiltonian potential)* point of view. It is here where Santilli truly encounters the problem of applying this theory in various physical situations. An example of this is when one tries to pass from *questions of external dynamics in the vacuum to questions of internal dynamics in a physical medium.* In the former case, the particles move in empty space (which is homogeneous and isotropic), with interactions at a distance that can be considered as points (such as a spaceship in a stationary orbit around the Earth), with its consequent local-differential and canonical-potential equations of motion. In the second case, the particles have extension and are deformable, moving in an anisotropic and inhomogeneous physical medium with contact interactions and interactions at a dis-

tance (like a spaceship entering the Earth's atmosphere). Their consistent equations of motion then include ordinary differential terms for the trajectory of the center of mass and more surface or volume integral terms that represent the correction, due to the shape and size of the bodies, of the previous characterization. Thus, such equations of motion are of non-linear, non-integral, and non-canonical-potential type.

The non-equivalence between these two types of problems therefore lies essentially in the: (a) topological matter, given that conventional topologies are not applicable to non-local conditions, (b) analytic matter, given the loss of the Lagrangian of the first order, going from a variationally self-adjoint system to another that is non-self-adjoint, and (c) geometric matter, given the inability of conventional geometries to characterize, for example, local changes in the speed of light. These differences were already noticed by K. Schwarzschild in 1916, in his two works [168] and [169].

From a geometrical point of view, gravitational collapse and other interior gravitational problems cannot be considered as if they were caused by specific point bodies, but the presence of a large number of hyper-dense particles is necessary (such as protons, neutrons, and other particles) in conditions of mutual total mixture, as well as the compression of a large number of them into a small region of space. This implies the need for a new structure that is arbitrarily non-linear (in coordinates and velocities), non local-integral (in various quantities), and non-Hamiltonian (variationally non-self-adjoint). Therefore, Lie theory is not applicable or valid in general for internal dynamic problems. And although these problems sometimes tend to simplify when treating them as exterior dynamical problems, it should be noted that this is mathematically impossible; while the former are local-differential and variationally self-adjoint, the latter are non-local-differential and variationally non-self-adjoint.

This possible new generalized structure would also solve other problems of great importance in fields as diverse as astrophysics, superconductivity, theoretical biology, etc. So, a generalization of conven-

tional Lie theory is sought, one which would also be directly applicable to non-linear, non-integro-differential, and variationally non-self-adjoint equations that characterize matter; i.e., one not based on approximations or transformations of the original theory. Also, a further extension of this generalization using a consistent anti-automorphic characterization above the classical-astrophysical level and the level of the elementary constituents of matter would address the formulation of the Lie theory, as well as the geometries and the underlying mechanics, for the characterization of antimatter.

A final point to note is that the theory of Lie depends mainly on the basic n-dimensional unit $I = \text{diag}(1, 1, ..., 1)$, in all its aspects, such as the enveloping algebras, Lie algebras, Lie groups, representation theory, etc. Therefore, the generalization sought should begin with a generalization of the aforementioned basic unit.

2.2.6 Origin of Lie-Santilli theory

To try to solve the problems that have just been indicated, Santilli presented in 1978, at Harvard University, a report (see [98]) in which he first proposed a step-by-step generalization of the conventional formulation of Lie theory, designed specifically for non-linear, non-integro-differential, and non-canonical systems, giving rise to the so-called *Lie-Santilli isotheory*. The main feature of this theory, which differentiates it from other generalizations of Lie theory, was its isotopic character, in the sense of that at all times it preserves the original axioms of the theory of Lie, through a series of isotopic liftings, known today as *Santilli isotopies*. More specifically, the Santilli isotopies are today associated with functions that take any linear, local, and canonical structure into its non-linear, non-local, and non-canonical broader possible forms, which are able to reconstruct linearity, locality, and canonicity in certain generalized spaces within the coordinates fixed by an inertial observer.

In fact, this theory was nothing more than a special case of the more general theory, today known as *Lie-Santilli admissible theory* or *Lie-Santilli genotopic theory*, where the term *genotopic* (of Greek origin, meaning *that which induces a configuration*) was introduced to denote the characterization of the covering axioms in the Lie admissible theory. In the genotopic theory, Santilli studied non-associative algebras that contained Jordan algebras or Lie algebras, thus continuing the studies that he began in 1967 regarding the new notion of the aforementioned Lie admissible algebras.

Passing from the study of non-associative algebras to associative ones, on which Santilli founded his generalization of Lie theory, he based it on the lifting of the associative enveloping algebras and generalizing the product $X \times Y$ to the isotopic product $X \hat{\times} Y = X \times T \times Y$, where T would have inverse element $\hat{I} = T^{-I}$ (I being unit of the conventional theory), which would be called the *isounit*. This is where the Santilli isotopy differed from others existing in the scientific literature, that is, in being based on the (axioms-preserving) generalization of the conventional unit I.

In this way, the Lie-Santilli isotheory was initially conceived as the image of the conventional theory under the isotopic lifting of the usual trivial unit I, to a new arbitrary unit $\hat{I}$, under the condition that the isounit retain the original topological properties of I, to achieve the conservation of the axioms of primitive theory. It would thus be the basis for a further generalization of the usual concepts of conventional Lie theory, such as universal enveloping algebras, the three fundamental theorems of Lie, the usual notion of Lie group, etc., in ways compatible with a generalized unit $\hat{I}$, which would cease being linear, local, and canonical (as is the usual unit of primitive Lie theory) to being non-local, non-linear, and non-canonical.

In fact, this generalization of the starting unit was also in the aforementioned genotopic theory. The construction of the Santilli *genotopies* (which were a generalization of the Santilli isotopies) were based on the new unit $\hat{I}$ not necessarily being symmetric ($\hat{I} \neq \hat{I}^t$), a topological property that trivially possesses the usual unit I; thus two different

quantities resulted, depending on whether one considered the generalized unit on the left ($^{<}\widehat{I}$) or on the right ($\widehat{I}^{>}$), in the following manner:

$$^{<}\widehat{I} = \widehat{I}, \quad \widehat{I}^{>} = \widehat{I}^{t}, \quad ^{<}\widehat{I} = (\widehat{I}^{>})^{t}.$$

Finally, the quoted text [98] also included the property of the Lie-Santilli isoalgebras (isotopic lifting of Lie algebras) to unify the compact and non-compact simple Lie algebras of the same dimension, using a conservation of the basis of all of them, varying from each other only in the element T that characterizes each isotopic lifting.

2.2.7 Further development of the Lie-Santilli isotheory

Since its inception in 1978, the Lie-Santilli isotheory has evolved in many aspects, both in its foundations and its subsequent applications. We will then see what some of its fundamental advances have been so far from that year until today.

Once Santilli fixed the fundamental idea of his new generalization of Lie theory, he must begin to conduct such a generalization in all aspects of conventional Lie theory. In fact, he must begin with the study of the basic elements on which his theory was based. Thus, during a presentation delivered at the *Differential Geometric Methods in Mathematical Physics* Congress that took place in 1980, in Clausthal (Germany), Santilli first presented the new numbers that appear in developing the isotopic generalization of the product $a \times b$ to the isotopic product $a \widehat{\times} b$. He treated *isotopic numbers*, which he also called *isonumbers*, whose construction was based on the lifting $a \to \widehat{a} = a * \widehat{I}$, where $\widehat{I}$ was the new isounit established in the isotopy in question. Santilli also presented at that conference the isotopic lifting of the usual fields $K(a, +, \times)$ called *isofields*.

In that same year, 1980, Santilli studied in collaboration with C. N. Ktorides and H. C. Myung (see [62]) the non-associative Lie admissi-

ble generalization of universal enveloping algebras, which he had begun to study already in 1978 (see [98]). Santilli himself would already apply this generalization in simplectic geometry, in 1982 (see [109]).

In 1981 Santilli studied in [106] the isotopic lifting of an associative algebra starting from the *Santilli isotopy*, characterized by the product $X \cdot Y = W * X * W * Y * W$, where $W^2 = W \neq 0$, W being fixed.

In these first years of the 1980s, Santilli also recognized that conventional, isotopic, and Lie admissible formulations could be applicable to matter but not antimatter, due to the aforementioned anti-automorphic character of the latter. Santilli then reexamined his isotopies and discovered in the works [114] and [115] (written in 1983, but not published until 1985, due to editorial problems) that, once one abandons the element unit I, accepting as a new unit element $\widehat{I}$, this latter unit permits a natural way of acquiring negative values. This is achieved by means of the function $\widehat{I} > 0 \rightarrow \widehat{I}^d = -\widehat{I} < 0$, which was an anti-automorphism he defined in [111], which he called an *isodual function*, in the sense of being a dual form that necessarily requires the isotopic generalization of the unit I. This function gave an anti-automorphic image to any function based on $\widehat{I}$, verifying that:

$$\left(\widehat{I}^d\right)^d = -\left(\widehat{I}^d\right) = -\left(-\widehat{I}\right) = \widehat{I}.$$

Thus given, the *isodual Lie-Santilli isotheory* began, with which new ways to solve problems related to antimatter seemed to be opened. Thus new notions of space-time and internal symmetries to solve problems related to matter and its isodual, antimatter, would later appear. Among them, for example, the isorotational symmetries (which emerged as the earliest examples that illustrated the Lie-Santilli isotheory and that would be already studied in detail in 1993 [127] and 1994 [134], respectively). These isosymmetries have attained very important implications, such as the first numerical representations of the magnetic moment of the deuteron and the synthesis of the neutron within new stars from protons and electrons only; the prediction of

a new clean nuclear energy called *Hadronic energy*; the definition of isodual ellipsoids with negative semiaxes; etc.

However, Santilli refrained from indicating in his works of 1983 the applicability of the isodual theory in the characterization of antimatter, given its profound implications in various fields, such as the possibility of stepping back in time or the prediction of the antigravity of antiparticles. However, the emergence of the isodual isotheory entailed a parallel to the isotheory study which appeared in 1978, which on the other hand was strengthened. In fact, in [111], where the isodual application appeared, Santilli also introduced the isotopic lifting of vector and metric spaces, giving rise to the so-called *vector isospaces* and *metric isospaces.*

As for the isodual isotheory, Santilli started defining the *isotopic isodual product* or *isodual isoproduct* equal to what he already did for the isotheory with the isoproduct $\widehat{\times}$. Thus, he defined the lifting $a \times b \to a \widehat{\times}^d b = a \times T^d \times b = a \times (-T) \times b = -(a \times T \times b) = -(a \widehat{\times} b)$, where the element T was what determined a preset isotopy of isounit $\widehat{I} = T^{-I}$, where I would be the conventional starting unit.

He defined the concept of an *isodual number* by lifting $a \to a^d = -a$, in addition to the concept of an *isodual isonumber*, using the isodual isoproduct, $\widehat{\times}^d$, beginning with lifting $a \to \widehat{a}^d = \overline{a} * \widehat{I}^d = \overline{a} * (-\widehat{I}) = -(\overline{a} * \widehat{I}) = -\widehat{\overline{a}}$, where $\overline{a}$ denotes the conjugate element a in the field K to which it belongs. Thereby Santilli obtained in particular that if one takes in the real numbers (of conventional unit $I = 1$) an isotopy starting from the isounit $\widehat{I} = I = 1$, then the corresponding isodual isotopic lifting was achieved be taking $\widehat{I}^d = -I = -1$. In this way, the actual numbers corresponding to the isounit $\widehat{I}^d$ are obtained by lifting $a \to a^d = -a$ (coinciding with the isoreal isodual isonumbers, corresponding to the same isounit as $\overline{a} = a, \forall a \in \mathbf{R}$). Thus, the *negative isodual numbers* referred to the negative unit -1 are obtained, which made them different from negative numbers referred to positive unit $+1$, in the sense that given a real negative number $-a$, we will have that, referred to the isodual isounit, $\widehat{I}^d = -1$ is completely analogous to its corresponding opposite positive real number, a, referred to the con-

ventional positive unit $I = +1$. In this way, the change in sign under isoduality would occur only in the projection of the isodual numbers on the original usual field. With all of this, isodual numbers would later reach an important application in the study of anti-particles in physics, from research in the Dirac equations, which gave negative energy solutions.

The next step is the study of the isotopic isodual liftings of fields $K(a, +, \times)$, giving rise to isodual isofields with isounit $\hat{I}^d = -\hat{I}$. Later, the isotopic isodual liftings of the rest of the mathematical structures would appear at the same time that the corresponding isotopic liftings of these were appearing. The latter gave rise to the formation of the so-called *mathematical isostructures*, while the former gave rise to the so-called *isodual mathematical isostructures*, the construction of which was sought at all times to achieve the anti-automorphic character with respect to the corresponding mathematical isostructures.

In 1988, in [116] and [117], Santilli presents a solution to the question of the relationship between the dynamic interior and exterior problems, through *isogeometries* built from the different degrees of freedom granted to the conventional unit with which he is working, beginning from an isotopy of the usual product. The problem for solving these issues was that the different constructed geometries encountered until then did not arise from a general, non-linear, and non-local-differential point of view over anisotropic and non-homogeneous systems. Santilli propounds his theory as encompassing the former general point of view. He would later develop it in [127] and [134], until 1996 when he applies the concept of isodifferential calculus to the construction of the isogeometries, thus allowing a greater transparency of the abstract unit of the latter, allowing at the same time a unified treatment of interior and exterior dynamic problems.

Also in 1988 Santilli established in [118] the irreducibility of the internal dynamic problems to external dynamic problems, through the so-called *no-reduction theorems*, which forbade the reduction of a macroscopic system with a monotonically decreasing angular momentum to a finite collection of elementary particles, each with a con-

stant angular momentum. Subsequently, in 1994, Santilli would treat these problems in the monograph [134]. As a result of these findings and in view of the progress achieved, Santilli received some awards in 1989. These include the nomination by the Academy of Sciences of Estonia, as one of the best applied mathematicians of all times, thus joining prestigious mathematicians such as Gauss, Hamilton, Cayley, Lie, Frobenius, Poincaré, Cartan, and Riemann, among others. Santilli was also the only member of Italian origin in this list.

Two years later, Santilli managed in 1991 to solve the problems that had already been raised in 1983, on the implementation of his isodual isotheory for the characterization of antimatter. This was done in the monographs [122] and [123]. The equivalence between isoduality and charge conjugation would be tested in 1994 (see [135]), while important implications due to the isoduality would also be developed in the same year (see [132] and [136]), in which he also published his first monograph on isoduality (see [134]).

In 1992 J. V. Kadeisvili classified in [46] the isotopic lifting into five classes, depending on the type of isounit used. He also studies for the first time the notion of *isocontinuity*, proving that this concept can reduce to conventional continuity, given that the isomodules $\widehat{|\widehat{f}(\widehat{X})|}$ of a function $\widehat{f}$ on a fixed isospace would be given by the usual modulus multiplied by the isounit $\widehat{I}$, which is well-behaved: $\widehat{|\widehat{f}(\widehat{X})|} = |\widehat{f}(\widehat{X})| \times \widehat{I}$. Kadeisvili follows with an isotopic development of functional analysis, under the name of *functional isoanalysis*.

In 1993 Santilli develops in detail in [129] a study on the isotopy and its isodualities of Poincaré symmetry (which had already been studied in 1983 (see [110])). He is also treats the *isospinor covering*.

On the other hand, he develops a study on the isotopic lifting of number theory (see [128]). In it he recalls the construction of the isonumbers and their isoduals. He also treats the construction of *pseudoisostructures* through pseudoisotopies (which are based on the lifting of the operation $+$ to $\widehat{+}$, which was already treated by Santilli in 1989 (see [120])). These pseudoisotopies are liftings that retain all the axioms of the departing structure except for those relating to dis-

tributivity, so it cannot be considered an authentic isotopy (and hence the name Santilli adopted). He develops this study for real and complex numbers and the *quaternions* and *octonions*. Santilli performs his representation of the isotopic lifting of the latter (isoquaternions and isooctonions, respectively) for the first time in terms of the isotopies and isodualities of the Pauli matrices. He also studies the emergence of new numbers of dimensions 3, 5, 6, and 7 for developing the classification of the isonormalized isoalgebras, and he indicates the generalization of the theory of numbers by using the *genotopic lifting*.

Finally, Santilli developed in that year, 1993, a monograph [127] on the isotopic liftings of vector and metric spaces. He treats *isoeuclidean spaces*, the *isoeuclidean metric*, and *isoeuclidean* and *isominkowskian geometry*. He also shows that the first of these isogeometries depends on the class to which the isotopy pertains, under which the starting geometry is lifted. Class I isoeuclidean geometry allows one to treat all the conventional geometry of the same dimension and signature, in addition to all its possible isotopies. Those of Class III allow the unified treatment of all previous geometries, regardless of their signature.

Also, Santilli defined the concept of *geometric propulsion*, by which a point particle of mass moves from one point to another without the application of force, but through the alteration of the underlying geometry. He achieved this by altering the units of space with which he works, to show that any such space leaves the product (length)$\times$(unit) invariant. In this way, through a proper isotopy (i.e., by a suitable change of unit), one can mathematically transform very large distances into very small distances and *vice versa*.

Another subject matter of his study during that year consists of developing, as Kadeisvili already had, the notions of isotopies of continuity, limits, series, etc. Santilli would prove that series which are conventionally divergent can be transformed by means of an isotopy into convergent series. This property has important applications in the reconstruction of perturbed convergent series for strong interactions, which are conventionally divergent.

In that same year, 1993, the Greek mathematicians D. Sourlas and G. Tsagas wrote the first mathematical book on isotheory (see [175]). In it these authors address the different existing mathematical isostructures and their immediate applications in physics. They also define and develop the isostructure of *isomanifolds*, the isotopic generalization of differentiable manifolds of differential geometry, which would be identified for the first time in 1995, in [176]. Finally, they introduced for the first time the concept of a *topological isospace* starting from an *isotopology*. G. Tsagas would carry out a study in 1994 (see [173]) of *isoaffine connections* and *isoriemannian metrics* over an isomanifold.

In 1994, Santilli develops a new representation of *antimatter* (see [134]) starting at the classical level and reaching the operator level, based on the isoduality function. He obtains that this representation is equivalent to the conjugate of the charges. In particular, all characteristics which are conventionally positive for matter come to be defined negative for antimatter, including energy, time, curvature, etc. However, as he had shown in [128], he had to take into account that the positive characteristics referred to a positive unit are equivalent to the negative characteristics referred to a negative unit. This elemental property would have important implications, such as the prediction of antigravity for elementary antiparticles.

Also in [134], Santilli develops his study of the theory of *isorepresentation*, which maintains the non-linear, non-local, and non-canonical representations of Lie groups. He also detailed the inequivalence existing between the interior and exterior dynamical problems.

In that same year of 1994, Santilli carried out a study (see [133]) on experimental verifications of the isotopies in nuclear physics, at the same time as he carried out a development of the isotopies of the *Dirac equations*, previously introduced by M. Nishioka in [83]. Santilli also studied that year (see [154]) the *deforming Q-operators of the isospinor symmetry* $\widehat{SU_Q}(2)$, an isotopic lifting spinor symmetry $SU(2)$, testing the isomorphism $\widehat{SU_Q}(2) \approx SU(2)$ and building and classifying the isorepresentations of $\widehat{SU_Q}(2)$. This work would be not yet published until 1998, although the conventional q-deformations of the as-

sociative algebras had already been studied previously, in 1993 (see [70]). Basically, the conventional q-deformations of associative algebras, $AB \to qAB$, were reformulated in isotopic terms as $qAB = A\widehat{\times}B$, considering the isounit for $\widehat{I} = q^{-1}$, which allowed its generalization and axiomatization into the most general integro-differential operator T possible (which is therefore sometimes denoted by Q).

In 1995, Santilli develops together with A. O. E. Animalu a study of the experimental verification of the isotopy in superconductivity, based on models due to non-linear, non-local, and non-Hamiltonian interactions, showing its ability to represent and to predict such models (see [4]).

In 1996, Tsagas carried out a study on the classification of the Lie-Santilli algebras (see [174]). This paper examines the fundamental concepts of the Lie algebras and explores the relationship between a Lie algebra and a type of isoalgebra called the *Lie-Santilli isoalgebra*, which consists of the isotopic generalization of a Lie algebra.

In that same year, Santilli introduces for the first time the concept of a *hyperstructure* beginning with a unit of multiple values (see [143]). The concept of a hyperstructure in general had already been introduced by T. Vougiouklis in 1994 (see [179]). In that year of 1996, Santilli studied, in collaboration with Vougiouklis himself, the *hyperstructures with units of singular values*. This new concept of a hyperstructure with a unit of multiple values came to be a generalization of the genotopic structure (arising from a genotopy), which happened to be a special case.

In hyperstructures the new units on the left and on the right are given by a finite, ordered set as follows: $\{<\widehat{I}\} = \{<\widehat{I}_1, <\widehat{I}_2, <\widehat{I}_3, \ldots\}$, $\{\widehat{I}>\} = \{\widehat{I}_1>, \widehat{I}_2>, \widehat{I}_3>, \ldots\}$, $\{<\widehat{I}\} = \{\widehat{I}>\}^t$, where the last operation refers to each element of the ordered sets.

In [145] Santilli examines the isotopic, axiom-preserving generalization of ordinary differential calculus, called *isodifferential calculus*, which he already posed previously, implicitly in his works [127] and [134] in 1993 and 1994, respectively. In fact, he gave the official presentation on it in December 1994 at the International Congress *Work-*

shop on Differential Geometry and Lie algebras, held in the Department of mathematics of the University of Aristotle in Thessaloniki (Greece).

This isodifferential calculus is based on the generalization of the basic unit with compatible generalizations of differential varieties, vector spaces, and fields. With this new type of calculation, applied to the isotopic lifting of the Newton equations, Santilli opens up new possibilities such as the representation of non-spherical and deformable particles, the admission of non-local-integral forces, and the ability to transform non-Hamiltonian Newtonian systems in the given space into systems that are Hamiltonian in the corresponding isospaces.

Also with isodifferential calculus Santilli proves that analytical and quantum mechanics can be isotopically lifted, and he constructs new isotopies of simplectic and Riemannian geometries, being non-linear (in coordinates and velocities), integro-differential, and not the first order of Lagrange. In this way, these new geometries are useful for internal dynamic problems such as the geometrization of a locally varying speed of light. Santilli also presents the existence of the corresponding generalizations by genotopic and hyperstructural methods, as well as their analogous isoduals.

In [145] Santilli also developed for first time the isotopies of the Newtonian equations (calling them *Newtonian isoequations*) and the rest of the fundamental equations of classical mechanics. In fact, he studied the isotopies and anti-automorphic images under isoduality of all the basic equations of Newtonian, quantum, and analytical mechanics. These new equations allow the representation of antimatter at the Newtonian level for the first time in an anti-automorphic sense. With all of this he develops *isohamiltonian* and *isolangragian* mechanics. In fact, he also develops them from genotopic and hyperstructural methods.

Finally, he treats the concept of *isogeometries* starting from the isodifferential calculus. Thus he reformulated the isosimplectic geometry (achieving direct universality for interior systems, i.e., the representation of all the interior systems, directly in the inertial frame fixed by the observer). This isosimplectic geometry emerges as the underlying iso-

geometry in isohamiltonian mechanics and Lie-Santilli isotheory. He also develops the *isoriemannian* geometry, achieving applications in the study of the local variation of the speed of light, gravitational theory, geodesic theory, etc. It also results that the isoriemannian geometry is a particular case with zero curvature of Riemannian geometry. He finally mentions *isodual isogeometries, genogeometries* (genotopic lifting of the geometries), and the *hypergeometries* (hyperstructural lifting of the geometries).

In 1997 a work by Santilli appeared [152] which shows that all non-uniform deformations (including the q-, k-, quantum, Lie isotopic, Lie admissible, and other deformations), although mathematically correct, have a series of problematic aspects of a physical character when formulated on given conventional spaces over conventional fields. These problems include the loss of the invariability of the basic units of spacetime, a loss of invariant numerical predictions, a loss of the observed Hermiticity in time, etc. In this way he also shows that the contemporary formulation of gravity is subject to similar problems, since Riemannian spaces are in fact non-canonical deformations of Minkowskian spaces, thus having non-invariant spacetime units. Santilli then constructed—based on the isotopies, genotopies, hyperstructures, and their isoduals—a non-unitary one known as *relativistic hadronic mechanics*, which saves the axiomatic inconsistencies of relativistic quantum mechanics, retains the abstract axioms of special relativity, and is a complement to the conventional mechanics apropos the argument of Einstein-Podolski-Rosen. This theory is mainly given by the following non-unitary transformations:

$$I \to \hat{I} \quad = UIU^\dagger \qquad\qquad A \to A' = UAU^\dagger$$

$$B \to B' = UBU^\dagger \qquad\qquad AB \to UABU^\dagger = A'\hat{T}B'$$

where:

$$UU^\dagger = \hat{I} \neq I \qquad \hat{T} = (UU^\dagger)^{-I}$$

$$\hat{I} = \hat{T}^{-I} \qquad \hat{I} = \hat{I}^\dagger$$

$$\hat{T} = \hat{T}^\dagger \qquad U(AB - BA)U^\dagger = A'\hat{T}B' - B'\hat{T}A'$$

With this new theory, Santilli opens new possibilities for study, due even today to develop many related aspects, in the sense that all concepts relating to isotopies, genotopies, and hyperstructures should be generalized into a non-unitary form. Santilli himself has continued investigating this topic, developing these physical inconsistencies into different quantum deformations (see [156] and [150]), as in other generalizing theories (as in [160]). On the other hand, these inconsistencies have come to be of such importance to the isotheory that Santilli himself has denoted them as *catastrophic inconsistencies* of the Lie-Santilli isotheory.

Let us note to complete this historical introduction of the Lie-Santilli theory that in recent years Santilli has directed his research to further develop the various implications that this isotheory has in fields as varied as: the use of isominkowskian geometry for the gravitational treatment of matter and antimatter (see [149], [151], [153] and [154]), together with an experimental verification in particle physics (see [5]), an application of hadronic mechanics in the prediction and experimental verification of new chemical species, called *magnecules* (distinguishing them from conventional molecules), which permitted the industrial application of his theory for new hadronic reactors, recycling liquid waste to produce a combustible, clean, and inexpensive gas (see [157]), an application of the isodual theory antimatter in the prediction of antigravity (see [159]), and many other applications in other fields that tested the strength of this theory and its definitive implementation in modern science.

Chapter 3
LIE-SANTILLI ISOTHEORY: ISOTOPIC STRUCTURES (I)

In this chapter and in the following, we will carry out the study of the *Lie-Santilli isotheory*. It is convenient to indicate that most of the concepts and properties that appear in them have been introduced by Santilli himself and by some other authors who studied his work. Our personal contribution to this text focuses on incorporating a large number of examples to systematize all the knowledge related to the same isostructure (with the need to unify the notation, having obtained this knowledge from several authors on some occasions) and to provide new demonstrations of some of them, which, in our opinion, contribute to shortening and improving—from the modern mathematical point of view, above all—what existed previously. In some cases these demonstrations did not even exist, and the facts were given by assumptions.

Then we begin this study with the definition of the concept of *isotopy*. As this new concept has a meaning too general for what is intended, however, we restrict ourselves to the case of the *Santilli isotopy*, which will be a basic tool for the development of this generalization of the theory of Lie, known as *isotheory* or *Lie-Santilli theory*. Later, after introducing a few preliminary notions, we will achieve an isotopic lifting of the basic mathematical structures, thus forming new *isotopic structures* or *isostructures*.

3.1 Isotopies

Definition 3.1.1 *Given any mathematical structure, an* isotopy *or* isotopic lifting *of the same is defined as any lifting of it which could result in a new mathematical structure, such that the same basic axioms which characterize the primitive structure are satisfied. This new structure will be given the name of* isotopic structure *or* isostructure.

Note that this notion of isotopy encompasses a wide range of possible liftings. So, for example, we would have first the basic isotopy of identity, which gives as resulting structure the starting structure. So, if we have, for example, a field $K(a, +, \times)$ of elements $\{a, b, c, \ldots\}$ and operators of the usual sum and product, $+$ and $\times$, the same field $K(a, +, \times)$ would be an isotopy of it, as trivially a mathematical structure that verifies the axioms which characterize the first structure is obtained. In this way, the isotopic theory becomes a covering of the usual theory, in the sense of being made of structurally more general foundations that admit the conventional formulation as a trivial special case.

The previous case is the simplest possible. We are interested in studying other cases and seeing how we can obtain them. We will do it, in the first place, from a theoretical point of view, and later we will see examples of it.

We observe first that in any mathematical structure (be it a group, field, vector space, etc.) there appear two mathematical objects that constitute it: the *elements* which form it and the *operations* or *laws* which relate these elements. Therefore, since any isotopy is a lifting of a mathematical structure, we will be able to form a classification of these according to the object in question which is raised. Thus, we have three types of non-trivial isotopies:

1. Isotopies of TYPE I. Only the elements of the mathematical structure are lifted.
2. Isotopies of TYPE II. Only the operations associated with the mathematical structure are lifted, leaving its set of elements invari-

ant (although each element itself can become another in the said set).

3. Isotopies of TYPE III. Both the elements and laws associated with the mathematical structure are lifted.

In all three cases one needs to take into account that to really have an isotopy it is necessary that the axioms that determine the primitive structure be preserved. This ensures that the type of structure is the same, i.e., the isotopic lifting of a field is a field, a ring is a ring, a group is a group, etc. Now, in these conditions there are naturally some questions, such as: What progresses the implementation of an isotopy? or What advantages does a new mathematical structure of the same type that we already had attain through an isotopy? To answer these and other similar questions, it must be observed that although an isotopy preserves the type of mathematical structure, it generally need not preserve the concepts, properties, and theorems that the conventional structure developed. In this way, a isotopy becomes a generalization of the concepts of homomorphisms and homeomorphisms, in the sense that we no longer need to restrict ourselves to the conditions that they had to satisfy these, but we can use the most general possible liftings.

We still have to resolve the question of how to obtain isotopies. As seen, each isotopy will depend on the type of initial structure that we have. Thus, in principle we cannot give a basic model that serves for any given structure. Therefore, each isotopy must be "built" for each specific case. This is also due to the definition of isotopy, since having to keep axioms no longer serves for any lifting known of the given structure. It is here where the problem of the construction of isotopies lies; therefore, if we set the initial structure and know a lifting of it, we have to check if it verifies *all* the axioms of the base structure. If not, we will have to modify the lifting for axiomatic preservation. This last procedure might be very costly because you have to do it according to each of the axioms, carefully, so as not to alter any of them. With all of this, once we have lifted the elements and laws associated with them,

so that they will satisfy the axioms of this structure, we will have managed to construct an isotopy. It also follows that an isotopy of a mathematical structure is not unique, since every "construction" is subject to modifications that allow the obtaining of new, useful liftings.

However, the above ideas are still very general. When we impose conditions on the isotopies, the true utility of these isotopies appears. Santilli, in 1978 (see [98]), obtained one of the possible constraints that can be made; when attending to physical problems and looking for a generalization of Lie theory, he began to use a type of isotopy, later called the *Santilli isotopy*, which satisfied the following:

Definition 3.1.2 *The isotopies of a linear, local, and canonical structure are called* Santilli isotopies *which will result in an isostructure in the most general possible non-linear, non-local, and non-canonical forms and that are able to reconstruct linearity, locality, and canonicity in certain generalized spaces within the coordinates set by an inertial observer.*

In that same work of 1978, Santilli gives a possible model for this type of isotopy. This is in addition to a model that, with appropriate modifications, will be valid for different types of mathematical structures. Fundamentally, these isotopies are based on a generalization of the conventional unit and its usual properties. This generalization of the unit, which Santilli will give the name of *isounit*, will be key to the lifting of the elements of the conventional structure and in the lifting of the associated laws. Obtaining such isounit is given in the following:

Definition 3.1.3 *Let E be any linear, local, and canonical mathematical structure defined over a set of elements C. Let $V \supseteq C$ be a set equipped with an operator $*$, with unit element I. The set V will be called the* general set of the isotopy. *Let $\widehat{I} \in V$ be such that its inverse $T = \widehat{I}^{-I}$ with respect to the operation $*$ exists (where the superscript indicates the unit under which the inverse is calculated, i.e., such that $\widehat{I} * T = T * \widehat{I} = I$). $\widehat{I}$ will then be called the* isotopic unit *or* isounit, *and it will be the basic unit of the lifting following from the structure E. The element T is called* isotopic element. *Finally, the pair of elements $\widehat{I}$ and $*$ constitute the* elements of the isotopy.

In practice, given that the essential element of a Santilli isotopy is the isounit $\widehat{I}$, it is not usually indicated what the set V of the definition is, but rather, to establish the isotopy, it suffices to indicate the operation $*$ and isounit $\widehat{I}$ (or isotopic element T). We also see that the operator $*$ need not appear in the structure E.

In the most general case possible, $\widehat{I}$ can possess a non-linear and non-local dependence on time t of the coordinates x and their derivatives of arbitrary order $\dot{x}, \ddot{x}, \ldots$, i.e., $\widehat{I} = \widehat{I}(t, x, \dot{x}, \ddot{x}, \ldots)$. It could also occur that $\widehat{I}$ depends on other local variables such as temperature, density of the medium, etc. In any case, given that the fundamental characteristic of any isotopy is the preservation of axioms and given that we want to establish an isotopy by means of the isounit $\widehat{I}$, we will have to impose that $\widehat{I}$ satisfies the topological properties of the usual unit I. Thus, assuming that I is n-dimensional, $\widehat{I}$, apart from having the same dimension n, should be like I, not unique at all points, invertible (in the region where the local values are considered), and Hermitian (i.e., symmetric and real-valued).

Note finally that in the definition we can establish that $\widehat{I}$ be an isounit both to the right and left, depending on how we multiply. Two similar theories could be established as well. However, given that we are interested in $\widehat{I}$ satisfying the topological properties of the usual unit, we will admit that $\widehat{I}$ is an isounit to the left and right.

Before seeing how we get the isostructures from the isounit $\widehat{I}$, we observe that in the case of Santilli isotopies, we are not interested in isotopies of type I seen above, since the non-linear, non-local, and non-canonical character sought is given in the operations. Thus, if these do not vary we will not, achieve this characteristic. However, given that Santilli isotopies are based on the isounit $\widehat{I}$, J. V. Kadeisvili, in 1992 (see [46]), already carried out the following classification of these Santilli isotopies:

1. SANTILLI ISOTOPIES of CLASS I. The isounits are sufficiently differentiable, bounded, non-degenerate at all points, Hermitian,

and positive-definite. They are the Santilli isotopies properly so-called.

2. SANTILLI ISOTOPIES of CLASS II. They are similar to class I isotopies, except that $\widehat{I}$ will be negative-definite.
3. SANTILLI ISOTOPIES of CLASS III. The isotopies are the union of the two previous classes.
4. SANTILLI ISOTOPIES of CLASS IV. The isotopies are the union of those of class III with those that have singular isounits.
5. SANTILLI ISOTOPIES of CLASS V. The isotopies are the union of those of class IV with those that have isounits without any restriction, which may depend on discontinuous functions, distributions, etc.

It is important to note that this classification not only serves for isotopies, but for all the structures generated by them, i.e., for the isostructures. In our study, we will develop those of class I and II, sometimes joining to them those of class III.

Next, it remains to be seen how to construct the Santilli isotopy beginning with the isounit $\widehat{I}$. For this we must see, for each structure in particular, how to perform a lifting, both of the set of elements of the structure and of the associated laws. This particularization will be reflected also in the initial conditions that the isounit $\widehat{I}$ and the operation $*$ must obey, conditions that will vary according to the structure that we are studying.

In general, if C is the set of elements of any fixed structure, E, the lifting what is carried out is, for each element $X \in C$, to associate $\widehat{X} = X * \widehat{I}$, where $*$ would correspond to the set V seen previously. We again note that this operation need not be a law of the structure E. However, for reasons of simplification and when there is no room for confusion, we will not write it, resulting in the notation $\widehat{X} = X\widehat{I}$. The set then obtained, $\widehat{C}_{\widehat{I}} = \{\widehat{X} = X\widehat{I} : X \in C\}$, will be called an *isotopic set* associated with C by means of the isounit $\widehat{I}$. Also, when there are no problems of interpretation, it will be notated simply by $\widehat{C}$.

Let us also underline that, by convention, symbols with a hat ^ will indicate that what is represented corresponds to the plane of the isotopic lifting carried out. So we will see, for example, that the majority of the usual operations can be isotopically raised in one way or another. To prevent complications in notation, the sign ^ will differentiate the operations of the starting structure from those of the isostructure.

However, it is important to note that this way of attaining the isotopic lifting of the set C, although correct, is not the most appropriate, because it leads to a series of mathematical inconsistencies identified by Santilli himself at the beginning of his studies (see [99]) and recently (see [145] and [161]). However, to begin the study of the Santilli isotheory, one should start handling this special lifting.

Finally, although we have yet to study how to perform the lifting associated with the structure E, this aspect should be seen for each structure in particular. For this reason, let us go see in the following sections in this and in the following chapters, separately, the set of isotopic liftings of the most important mathematical structures that will lead, respectively, to the following isostructures: isogroups, isorings, isofields, vector isospaces, isomodules, metric vector isospaces, and isoalgebras. As will be seen, the names of all of the above isostructures are formed by adding the prefix *iso-* to the names of the usual structures. With this procedure (common in other fields of the isotheory), it is to be noted that we have seen the new structures are the same type as the previous ones, because they retain the axioms that define them.

Finally, we point out that the definitions of the isostructures serve for any isotopy in general, although we will apply them directly to the Santilli isotopy. Therefore, given that there is no room for confusion, we will designate by isotopy simply these latter, always remembering that they are a restriction of the set of general isotopies.

3.2 Isonumbers

Before beginning to study the first of the isostructures, we will dedicate a section to the formation of the isotopic set $\widehat{C}$, noted above, in the case of the structure of fields. This will help us to begin to familiarize ourselves with the new concepts, because it is the simplest case possible.

Suppose then that we have any field $K = K(a, +, \times)$ of elements $\{a, b, c, \ldots\}$, with the usual associative operations $+$ and $\times$, with additive unit 0 and multiplicative unit 1. Given that conventionally we call the elements of a field numbers, we call the elements of the isostructure that we want to construct *isonumbers*. With the notation of the previous section, we would have in this case: $E = K$, $C = \{a, b, c, ...\}$, and the associated operations would be $+$ and $\times$. A more comprehensive development, as well as the historical genesis of isonumbers, can see in [128].

The isounit $\widehat{I}$, which may or may not belong to K (remember that $\widehat{I} \in V$, where $V \supseteq K$, although we said that in practice it need not indicate said set) and an operation $*$ having been fixed, we know, by the model seen above, that the isotopic set associated with C by means of the isounit $\widehat{I}$ is: $\widehat{C} = \{\widehat{a} = a * \widehat{I} = a\widehat{I} \mid a \in C\}$. The isotopic set is the set of the isonumbers which corresponds to the isotopic lifting by $\widehat{I}$ and $*$ of the field K.

Let us see below two examples of isonumbers originating from real numbers, i.e., we will work with the structure of $\mathbf{R}(+, \times)$, with the usual sum and product. In the first example, we will take $\widehat{I} \in \mathbf{R}$, while in the second $\widehat{I} \notin \mathbf{R}$.

Example 3.2.1 *With the previous notations, suppose that*

$$V = C = \mathbf{R}, \qquad * \equiv \times, \qquad \widehat{I} = 2.$$

We will have in this case that the isotopic set associated with $\mathbf{R}$ *by means of* $\widehat{I}$ *and* $*$ *will be the set* $\widehat{\mathbf{R}}_2 = \{\widehat{a} = a \times 2 = 2a : a \in \mathbf{R}\}$. *So, it turns out that*

$\widehat{\mathbf{R}}_2 = \mathbf{R}$, *thus the isotopic lifting of the set of initial elements rising gives us the same set.* ◁

Example 3.2.2 *Suppose now that*

$$V = \mathbf{C}, \qquad * \equiv \bullet \text{ (the product in } \mathbf{C}), \qquad \widehat{I} = i.$$

We will then get as the isotopic set $\widehat{\mathbf{R}}_i = \{\widehat{a} = a \bullet i = ai : a \in \mathbf{R}\}$. *We will arrive at* $\widehat{\mathbf{R}}_i = Im(\mathbf{C})$.

We note that in the second example it was enough to designate the isounit $\widehat{I} = i$ and indicate what the operation $*$ to obtain the isotopic set is, since it has not been necessary at any time to know what the whole V was.

In general, as the operation $*$ is internal, when we want the isotopy we seek not to change the set of all elements of the starting structure, we assume that $V = C$. Otherwise, we will take $V \supset C$. That is why, when we designate an isotopy just by an isounit $\widehat{I} \in C$ and operation $*$, we assume that $V = C$. If $\widehat{I} \notin C$, then $V \supset C$.

We see finally that, for the construction of the isotopic set, we made no distinction regarding the characteristic of the field with which we worked. Let us note also, as already noted previously, that to be able make an isotopy, apart from the conditions imposed on the isounit $\widehat{I}$, the isounit can also depend on time, the coordinates and their derivatives, etc. This is important from the physical and analytical point of view. However, to fix ideas, we will first start by assuming that $\widehat{I}$ is a constant element.

3.3 Isogroups

We will start by giving the definition of an isogroup (it can be seen more comprehensively in [98]) then indicating the method of constructing an isogroup from an isounit and fixed operation $*$. After finally showing some examples, the possibility arises of an isotopic lift-

ing of some substructures related to groups and of functions that exist between isogroups.

Definition 3.3.1 *Let $(G, \circ)$ be a group, $\circ$ being an internal associative operator with unit element e. An* isogroup $\widehat{G}$ *is an isotopy of G endowed with a new internal operator $\widehat{\circ}$ such that the pair $(\widehat{G}, \widehat{\circ})$ satisfies the properties of a group, i.e., that $\forall \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma} \in \widehat{G}$, the following are satisfied:*

1. Associativity: $(\widehat{\alpha}\widehat{\circ}\widehat{\beta})\widehat{\circ}\widehat{\gamma} = \widehat{\alpha}\widehat{\circ}(\widehat{\beta}\widehat{\circ}\widehat{\gamma})$.
2. Unit element (isounit): $\exists \widehat{I} \in \widehat{G}$ *such that* $\widehat{\alpha}\widehat{\circ}\widehat{I} = \widehat{I}\widehat{\circ}\widehat{\alpha} = \widehat{\alpha}$.
3. Inverse element (isoinverse): *Given* $\widehat{\alpha} \in \widehat{G}$, *there exists* $\widehat{\alpha}^{-\widehat{I}} \in \widehat{G}$ *such that* $\widehat{\alpha}\widehat{\circ}\widehat{\alpha}^{-\widehat{I}} = \widehat{\alpha}^{-\widehat{I}}\widehat{\circ}\widehat{\alpha} = \widehat{I}$.

If, in addition, $\widehat{\alpha}\widehat{\circ}\widehat{\beta} = \widehat{\beta}\widehat{\circ}\widehat{\alpha}$ for all $\widehat{\alpha}, \widehat{\beta} \in \widehat{G}$, then $\widehat{G}$ is called an isoabelian isogroup *or* isocommutative isogroup.

Let us first observe that this notion of isogroup, so defined, is general; the unit element associated with $\widehat{\circ}$, which is called the isounit, does not always correspond with the isounit which we made reference to in the construction of a Santilli isotopy. However, when we carry out such a construction, we seek to make it so that these two elements coincide. We note that writing $\widehat{I}$ and not $\widehat{e}$ is not coincidental. This is due that, if we follow the notations used so far, $\widehat{e}$ is the isotopic lifting of the element e, but in general $\widehat{e}$ need not be a unit element of the operation $\widehat{\circ}$. This is one of the fundamental reasons why concepts, properties, and theorems developed by the starting structure are not necessarily applicable to the new, more general structure.

Let us look at this last fact in the case of the construction of a Santilli isotopy from a fixed isounit and operator $*$.

To do so, once we have the starting group $(G, \circ)$, we consider the isounit $\widehat{I}$ (not necessarily belonging to G) and define the operation $*$ with which we want to work. We already know the way to construct the isotopic set associated with G by means of the isounit $\widehat{I}$ and operation $*$, as it has already been said that the method of construction seen in the previous sections applies to any structure. $\widehat{G} = \{\widehat{\alpha} = \alpha * \widehat{I} = \alpha\widehat{I} \mid \alpha \in G\}$ then results.

We will see next, for the first time in this text, how to lift operations associated with the starting structure (in our case we only need to lift the operator $\circ$). Depending on the case in which the isounit $\widehat{I}$ in question is (that is, $\widehat{I} \in G$ or $\widehat{I} \notin G$), we can make use of as many operator liftings as we can construct. However, we will start with a lifting that will always work and will be fundamental in the theory, not only of isogroups, but of the whole of isostructures in general.

This lifting is, in our particular case, in defining $\widehat{\alpha}\widehat{\circ}\widehat{\beta} = \widehat{\alpha} * T * \widehat{\beta}$, $\forall \widehat{\alpha}, \widehat{\beta} \in \widehat{G}$, where $T = \widehat{I}^{-I}$ is the given isotopic element in Definition 3.1.3. Taking into account then the definition of the isotopic set, it turns out that $\widehat{\alpha} = \alpha * \widehat{I}$ and $\widehat{\beta} = \beta * \widehat{I}$ and therefore $\widehat{\alpha}\widehat{\circ}\widehat{\beta} = (\alpha * \widehat{I}) * T * (\beta * \widehat{I})$.

Observing this last expression we realize the importance that the condition of associativity would impose on the operation $*$, a condition which is not required for the formation of Santilli isotopies in general. However, when the operation that we want to lift by this procedure has the property of associativity (as in the case which concerns us), yes, we will have to impose that $*$ be associative, as we will see later. Therefore, we assume now that $*$ is an associative law.

Then, assuming the condition of associativity of $*$, we finally get that

$$\widehat{\alpha}\widehat{\circ}\widehat{\beta} = \widehat{\alpha} * T * \widehat{\beta} = (\alpha * \widehat{I}) * T * (\beta * \widehat{I}) = \alpha * (\widehat{I} * T) * \beta * \widehat{I} = \alpha * I * \beta * \widehat{I},$$

where I is the unit element associated with $*$. Hence, $\widehat{\alpha}\widehat{\circ}\widehat{\beta} = (\alpha * \beta) * \widehat{I}$. The product resulting from this construction is called the *isoproduct*.

We already said that this model of constructing the isotopic lifting of the operation $\circ$ associated with the group G is not proper to isogroups, and indeed we will continue to use it in the construction of the rest of the isostructures. This is because the use of the isoproduct allows, after appropriate restrictions on the isounit $\widehat{I}$ and the operation $*$, the correct lifting of the structure of the corresponding starting isostructure. In the case of isogroups, the condition that we are going to impose is that,

with the usual notations, $(G, *)$ be a group with unit element $I \in G$ (where I is the unit element with respect to $*$ in the general set V).

We can demonstrate, then, that $(\widehat{G}, \widehat{\circ})$, as defined, is an isogroup, seeing that $\widehat{\circ}$ is an internal operation and that it also verifies the three conditions of Definition 3.3.1.

In fact, $\forall \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma} \in \widehat{G}$, we see that

1. $\widehat{\circ}$ is an internal operation for $\widehat{G}$, since $\widehat{\alpha} \widehat{\circ} \widehat{\beta} = (\alpha * \beta) * \widehat{I} \in \widehat{G}$, with $\alpha * \beta \in G$, $*$ being an internal operation on G by hypothesis (remember that $(G, *)$ is a group).
2. $(\widehat{\alpha} \widehat{\circ} \widehat{\beta}) \widehat{\circ} \widehat{\gamma} = (\widehat{\alpha} * T * \widehat{\beta}) * T * \widehat{\gamma} = \widehat{\alpha} * T * (\widehat{\beta} * T * \widehat{\gamma}) = \widehat{\alpha} \widehat{\circ} (\widehat{\beta} \widehat{\circ} \widehat{\gamma})$. ((Note that it is essential that $*$ is associative so $\widehat{\circ}$ also is.)
3. $I * \widehat{I} = \widehat{I} \in \widehat{G}$, given that $I \in G$. Also, $\widehat{I}$ is the isounit we seek, as $\widehat{\alpha} \widehat{\circ} \widehat{I} = \widehat{\alpha} * T * \widehat{I} = \widehat{\alpha} * (T * \widehat{I}) = \widehat{\alpha} = \widehat{I} \widehat{\circ} \widehat{\alpha}$.
4. Given $\widehat{\alpha} \in \widehat{G}$, $\widehat{\alpha} = \alpha * \widehat{I}$ will be satisfied, with $\alpha \in G$. So, for $(G, *)$ being a group with unit element I, $\alpha^{-I} \in G$ exists, such that $\alpha * \alpha^{-I} = \alpha^{-I} * \alpha = I$. Thus, it suffices to take the element $\widehat{\alpha^{-I}} = \alpha^{-I} * \widehat{I}$ as the isoinverse of $\widehat{\alpha}$ with respect to $\widehat{\circ}$, so then we obtain that $\widehat{\alpha} \widehat{\circ} \widehat{\alpha^{-I}} = \widehat{\alpha} * T * \widehat{\alpha^{-I}} = (\alpha * \alpha^{-I}) * \widehat{I} = I * \widehat{I} = \widehat{I} = \widehat{\alpha^{-I}} \widehat{\circ} \widehat{\alpha}$. We also observe that if $\alpha * \widehat{I} = \beta * \widehat{I}$, then $\alpha = \alpha * \widehat{I} * T = \beta * \widehat{I} * T = \beta$. Therefore, the isoinverse is well-defined.
5. In addition, if $*$ is commutative, $(\widehat{G}, \widehat{\circ})$ will be commutative, so $\widehat{\alpha} \widehat{\circ} \widehat{\beta} = \widehat{\alpha} * T * \widehat{\beta} = (\alpha * \beta) * \widehat{I} = (\beta * \alpha) * \widehat{I} = \widehat{\beta} \widehat{\circ} \widehat{\alpha}$

Therefore, we have proved the following:

Proposition 3.3.2 *Let $(G, \circ)$ be an associative group and let $\widehat{I}$ and $*$ be two isotopic elements in the conditions of Definition 3.1.3. If $(G, *)$ has an associative group structure with unit element $I \in G$ (I being the unit element of $*$ in the corresponding general set V), then the isotopic lifting $(\widehat{G}, \widehat{\circ})$ achieved by the procedure of the isoproduct has an isogroup structure. If $(G, *)$ is also commutative, then $(\widehat{G}, \widehat{\circ})$ is a commutative isogroup.* □

Let us next see some examples of isogroups:

Example 3.3.3 *Let us consider the group $(\mathbf{R}, +)$ of real numbers with a binary operation given by the usual sum. A trivial isotopic lifting would be given from the isounit $\widehat{I} = 0$ and the operation $* \equiv +$ (obviously $(\mathbf{R}, *) = (\mathbf{R}, +)$ would be a group with unit element $0 \in \mathbf{R}$), with which we get the pair $(\widehat{\mathbf{R}}_0, \widehat{+})$, where $\widehat{\mathbf{R}}_0 = \{\widehat{a} = a * 0 = a + 0 = a \mid a \in \mathbf{R}\} = \mathbf{R}$. On the other hand, with $* \equiv +$, the unit element with respect to $*$ in $\mathbf{R}$ will be $I = 0$ and so $T = \widehat{I}^{-I} = \widehat{I}^{-0} = 0^{-0} = 0$. In this way, the isoproduct would be defined according to*

$$\widehat{a}\widehat{+}\widehat{b} = \widehat{a}*0*\widehat{b} = (a*0)*0*(b*0) = (a+0)+0+(b+0) = (a+b)+0 = \widehat{a+b} = a+b.$$

So, we would have $\widehat{+} \equiv * \equiv +$. We should indicate, however, that although we should properly speak of an isosum, *we will keep the term isoproduct for reasons that we will see in the following sections.*

Therefore, the isotopy of $(\mathbf{R}, +)$, given by the isounit 0 and operation $* \equiv +$, coincides with the trivial isotopy, i.e. the identity. This shows that the construction that we are carrying out of a Santilli isotopy is correct because if we do not vary the starting unit or the operation associated with the group, it remains invariant after the isotopy. $\triangleleft$

Example 3.3.4 *Let us consider now—for the group $(\mathbf{R}^*, \times)$, $\mathbf{R}^*$ being the set all real numbers except zero—the isotopy that emerges considering $\widehat{I} = i$ as the isounit and using the product of complex numbers $* \equiv \bullet$ as the operator.*

As seen above, we arrive at the isotopic set being $\widehat{\mathbf{R}^*}_i = \mathrm{Im}(\mathbf{C}) \setminus \{0\}$. We will now study the isotopic lifting of the product $\times$. For this, as $* \equiv \bullet$, the unit element with respect to $*$ will be $I = 1$. So, $\widehat{I}^{-I} = \widehat{I}^{-1} = i^{-1} = -i$, since $i \bullet (-i) = (-i) \bullet i = 1$.

Finally, the isoproduct is defined in the following way: $\widehat{a}\widehat{\times}\widehat{b} = \widehat{a}*(-i)*\widehat{b} = (a*i)*(-i)*(b*i) = (a \bullet i) \bullet (-i) \bullet (b \bullet i) = (a \bullet b) \bullet i = \widehat{a \bullet b} = \widehat{a \times b}$, for all $a, b \in \mathbf{R}$. $\triangleleft$

We also observe that the isogroups obtained in the two examples above are isocommutative, as the corresponding starting groups are also, Proposition 3.3.2 then being applicable.

Now, as a prelude to our development and in order to continue with our intention of distinguishing the isotopic concepts from the usual ones, let us give the following:

Definition 3.3.5 *Given the isotopy $(\widehat{G}, \widehat{\circ})$ of a group $(G, \circ)$, if there exists a minimal positive p such that $\overbrace{\widehat{a}\widehat{\circ}\ldots\widehat{\circ}\widehat{a}}^{p \text{ times}} = \widehat{I}$ ($\widehat{I}$ being the isounit of the isogroup in question), we will say that the isogroup $\widehat{G}$ has* isocharacteristic *p. Otherwise, we will say that it has* isocharacteristic zero.

Next, we will study possible liftings of substructures related to groups, i.e., subgroups. To follow the construction that we have done, we need to define an isosubgroup as the isotopy of a subgroup H of a fixed group G. The problem arises from the moment in which we want any isotopic lifting of a given structure to be a structure of the same type. Thus, any isotopy H should have a subgroup structure and, therefore, the isotopic lifting from H might not be independent of the lifting of G. With all of this, the definition of isosubgroup would be as follows:

Definition 3.3.6 *Let $(G, \circ)$ be an associative group and $(\widehat{G}, \widehat{\circ})$ be an associated isogroup. Let H be a subgroup of G. $\widehat{H}$ is called an* isosubgroup *of $\widehat{G}$ if, being an isotopy of H, the pair $(\widehat{H}, \widehat{\circ})$ is a subgroup of $\widehat{G}$, i.e. if $\widehat{H} \subseteq \widehat{G}$, $\widehat{\circ}$ is a binary operation for $\widehat{H}$ and $(\widehat{H}, \widehat{\circ})$ has the structure of a group.*

We will now apply this previous definition to the method of construction that we have been making, through an isounit and operation $*$. We suppose then that we have the associative group $(G, \circ)$ and isogroup $(\widehat{G}, \widehat{\circ})$, obtained by means of an isounit $\widehat{I}$ and a fixed operation $*$. Let H be a subgroup of G. Given that we wish that in the future isosubgroup $\widehat{H}$ the associated law be $\widehat{\circ}$ itself, if we follow the given construction of the isoproduct, both the operation and the isounit under which we make the isotopy have to be, respectively, $*$ and $\widehat{I}$, since otherwise we would not get the same operation $\widehat{\circ}$ in general. Here is an example of this:

Example 3.3.7 *We will consider the group $(\mathbf{Z}, +)$ of the whole numbers with the usual sum. Let us take, with the usual notations, $* \equiv +$, $\widehat{I} = 2$. As $(\mathbf{Z}, *) = (\mathbf{Z}, +)$ is a group with unit element $0 \in \mathbf{Z}$, we can achieve the isotopy of the elements $*$ and $\widehat{I}$. Then, $\widehat{\mathbf{Z}}_2 = \{\widehat{a} = a*2 = a+2 \mid a \in \mathbf{Z}\} = \mathbf{Z}$. Therefore, as $* \equiv +$, $I = 0$ will be the unit element with respect to $*$. So, $\widehat{I}^{-I} = \widehat{I}^{-0} = 2^{-0} = -2$ and we thus arrive at the isoproduct given by $\widehat{a}\widehat{+}\widehat{b} = \widehat{a} * (-2) * \widehat{b} = (a+2) * (-2) * (b+2) = a + 2 + (-2) + b + 2 = a + b + 2 = (a+b) * 2 = \widehat{a+b}$, for all $a, b \in \mathbf{Z}$. We have thus obtained the isogroup $(\widehat{\mathbf{Z}}_2, \widehat{+})$ arising from the additive group $(\mathbf{Z}, +)$.*

We consider now the subgroup $(\mathbf{P}, +)$ of the even integers and zero. If we make the isotopy relative to the same previous elements (which again can be done, $(\mathbf{P}, *) = (\mathbf{P}, +)$ being a group with unit element $0 \in \mathbf{P}$), we get on the one hand the isotopic set $\widehat{\mathbf{P}}_2 = \{\widehat{m} = m * 2 = m + 2 \mid m \in \mathbf{P}\} = \mathbf{P}$, and on the other hand, we would arrive at the same isoproduct $\widehat{+}$.

We now show that $(\widehat{\mathbf{P}}_2, \widehat{+})$ is an isosubgroup of $(\widehat{\mathbf{Z}}_2, \widehat{+})$, bearing in mind that, of course, $\widehat{\mathbf{P}}_2$ is an isotopy of $\mathbf{P} \subseteq \mathbf{Z}$. We observe that

1. *$\widehat{\mathbf{P}}_2 \subseteq \widehat{\mathbf{Z}}_2$, as $\widehat{\mathbf{P}} = \mathbf{P}$, $\widehat{\mathbf{Z}} = \mathbf{Z}$ and $\mathbf{P} \subseteq \mathbf{Z}$.*
2. *For all $m, n \in \mathbf{P}$, $\widehat{m}\widehat{+}\widehat{n} = m+n+2 \in \mathbf{P}$ is satisfied. So, $\widehat{+}$ is an internal operation in $\widehat{\mathbf{P}}_2$.*
3. *The conditions of the group are also satisfied:*
   a. *It inherits the associativity from $(\widehat{G}, \widehat{\circ})$.*
   b. *The isounit $\widehat{I} = 2$ (which is the unit element with respect to $\widehat{+}$) belongs to $\widehat{\mathbf{P}}_2 = \mathbf{P}$. Thus, $\widehat{I} = \widehat{0} = 0 + 2$.*
   c. *$\forall m \in \mathbf{P}$, $m^{-\widehat{I}} = -m \in \widehat{\mathbf{P}}_2$, since $m\widehat{+}(-m) = (m + (-m)) + 2 = 0 + 2 = 2 = \widehat{I} = (-m)\widehat{+}m$.*

Therefore, it is demonstrated that $(\widehat{\mathbf{P}}_2, \widehat{+})$ is an isosubgroup of $(\widehat{\mathbf{Z}}_2, \widehat{+})$. $\triangleleft$

Note that some parts of the previous example can be suppressed. For example, given that the operation $*$ and the isounit used are the same in both isotopies, we will have that if H is a subset of a fixed group $(G, \circ)$, then $\widehat{H} \subseteq \widehat{G}$, with the usual notation, $\widehat{h} = h * \widehat{I}$ with $h \in H \subseteq G \Rightarrow h \in G \Rightarrow \widehat{h} \in \widehat{G}$. On the other hand, once it is

proved that we can carry out the corresponding isotopy with the elements that we used to construct $\widehat{G}$, to construct $\widehat{H}$, we will avoid some more proofs. Recall that, according to Proposition 3.3.2, one of the conditions that must be satisfied in order to construct the isotopy is that the pair $(H, *)$ be a group with the same unit element which V had with respect to $*$, I, which in turn should coincide with the unit element of $(G, *)$ (since we already showed, of course, that one can achieve the isotopy of the group G). Then—similarly to how we found that $(\widehat{G}, \widehat{\circ})$ had group structure, by means of the isoproduct $\widehat{\circ}$ formed from $*$—we have the fact that $\widehat{+}$ is a binary operation in $\widehat{\mathbf{P}}_2$ and conditions (2) and (3) are held by construction. Finally, the associativity condition (1) obviously holds, as $\widehat{\circ}$ is associative in $(\widehat{G}, \widehat{\circ})$, $*$ being associative by hypothesis.

All this is proved as follows:

Proposition 3.3.8 *Let $(G, \circ)$ be an associative group and let $(\widehat{G}, \widehat{\circ})$ be the associated isogroup corresponding to the isotopy of elements $\widehat{I}$ and $*$. Let H be a subgroup of G. Then if $(H, *)$ has a group structure, with the unit element the same as that of $(G, *)$, the isotopic lifting $(\widehat{H}, \widehat{\circ})$ corresponding to the isotopy of elements $\widehat{I}$ and $*$ is an isosubgroup of $\widehat{G}$.* □

Indeed, since the method of constructing isogroups already notes this condition, we could simply indicate that, in the case of achieving the corresponding the isotopy corresponding to $\widehat{I}$ and $*$, the isotopic lifting $(\widehat{H}, \widehat{\circ})$ is already an isosubgroup of $\widehat{G}$. Therefore, the only problem that arises now is that this isotopy cannot be performed due to a lack of initial conditions. Let us look at this situation in the following:

Example 3.3.9 *Let us consider the group $(\mathbf{Z}/\mathbf{Z_2}, +)$ of the quotient set $\mathbf{Z}/\mathbf{Z_2}$ with the usual sum. We will now consider, with the usual notation, $\widehat{I} = 1 + \mathbf{Z_2}$ and the operation $*$ defined according to*

$$(1 + \mathbf{Z_2}) * (1 + \mathbf{Z_2}) = 1 + \mathbf{Z_2} = (0 + \mathbf{Z_2}) * (0 + \mathbf{Z_2}),$$

$$(1 + \mathbf{Z_2}) * (0 + \mathbf{Z_2}) = (0 + \mathbf{Z_2}) * (1 + \mathbf{Z_2}) = 0 + \mathbf{Z_2}.$$

Let us first prove that $*$, so defined, is an associative operation. For it,

$((1+\mathbf{Z_2})*(1+\mathbf{Z_2}))*(1+\mathbf{Z_2}) = (1+\mathbf{Z_2})*(1+\mathbf{Z_2}) = 1+\mathbf{Z_2} = (1+\mathbf{Z_2})*(1+\mathbf{Z_2}) = (1+\mathbf{Z_2})*((1+\mathbf{Z_2})*(1+\mathbf{Z_2}))$,

$((1+\mathbf{Z_2})*(1+\mathbf{Z_2}))*(0+\mathbf{Z_2}) = (1+\mathbf{Z_2})*(0+\mathbf{Z_2}) = 0+\mathbf{Z_2} = (1+\mathbf{Z_2})*(0+\mathbf{Z_2}) = (1+\mathbf{Z_2})*((1+\mathbf{Z_2})*(0+\mathbf{Z_2}))$,

$((1+\mathbf{Z_2})*(0+\mathbf{Z_2}))*(0+\mathbf{Z_2}) = (0+\mathbf{Z_2})*(0+\mathbf{Z_2}) = 1+\mathbf{Z_2} = (1+\mathbf{Z_2})*(1+\mathbf{Z_2}) = (1+\mathbf{Z_2})*((0+\mathbf{Z_2})*(0+\mathbf{Z_2}))$,

$((0+\mathbf{Z_2})*(0+\mathbf{Z_2}))*(0+\mathbf{Z_2}) = (1+\mathbf{Z_2})*(0+\mathbf{Z_2}) = 0+\mathbf{Z_2} = (0+\mathbf{Z_2})*(1+\mathbf{Z_2}) = (0+\mathbf{Z_2})*((0+\mathbf{Z_2})*(0+\mathbf{Z_2}))$,

and the other possible cases would hold by commutativity. It results that $(\mathbf{Z}/\mathbf{Z}_2,*)$ *has group structure with unit element unit* $I = \widehat{I} = 1+\mathbf{Z_2} \in \mathbf{Z}/\mathbf{Z_2}$.

In addition, achieving now the corresponding isotopy for $\widehat{I}$ *and* $*$, *we obtain as the isotopic set for* $\widehat{\mathbf{Z}/\mathbf{Z}_{\mathbf{2}1+Z_2}} = \{0+Z_2, 1+Z_2\} = \mathbf{Z}/\mathbf{Z_2}$.

In turn, as $\widehat{I}^{-I} = (1+\mathbf{Z}_2)^{-(1+\mathbf{Z_2})} = 1+\mathbf{Z}_2$, *the corresponding isoproduct* $\widehat{+}$ *will be given by*

$$(0+\mathbf{Z_2})\widehat{+}(0+\mathbf{Z_2}) = ((0+\mathbf{Z_2})*(1+\mathbf{Z_2}))*(1+\mathbf{Z_2})*((0+\mathbf{Z_2})*(1+\mathbf{Z_2})) =$$

$$((0+\mathbf{Z_2})*(0+\mathbf{Z_2}))*(1+\mathbf{Z_2}) = (1+\mathbf{Z_2})*(1+\mathbf{Z_2}) = 1\widehat{+}\mathbf{Z_2} = 1+\mathbf{Z_2}$$

$$(0+\mathbf{Z_2})\widehat{+}(1+\mathbf{Z_2}) = ((0+\mathbf{Z_2})\widehat{*}(1+\mathbf{Z_2})) = 0\widehat{+}\mathbf{Z_2} = 0+\mathbf{Z_2}$$

$$(1+\mathbf{Z_2})\widehat{+}(1+\mathbf{Z_2}) = ((1+\mathbf{Z_2})\widehat{*}(1+\mathbf{Z_2})) = 1\widehat{+}\mathbf{Z_2} = 1+\mathbf{Z_2}$$

So, $\widehat{+} \equiv *$ *and therefore* $(\widehat{\mathbf{Z}/\mathbf{Z}_{\mathbf{2}1+\mathbf{Z_2}}}, \widehat{+}) = (\mathbf{Z}/\mathbf{Z_2}, *)$ *is a new isogroup.*

On the other hand, let us consider the subgroup $(\{0+\mathbf{Z_2}\},+)$ *of* $(\mathbf{Z}/\mathbf{Z_2},+)$. *We see that* $(\{0+\mathbf{Z_2}\},*)$ *does not, however, have a group structure, as* $*$ *is not a binary operation for* $\{0+\mathbf{Z_2}\}$, *since* $(0+\mathbf{Z_2})*(0+\mathbf{Z_2}) = 1+\mathbf{Z_2} \notin \{0+\mathbf{Z_2}\}$. *Therefore, the conditions of Proposition 3.3.8 are not verified for an isosubgroup by applying the isotopy of elements* $\widehat{I} = 1+\mathbf{Z_2}$ *and* $*$, *because in the case where we constructed the isotopic set and the corresponding isoproduct, the result we obtained would not have a group structure. We should have, in fact, that* $\{\widehat{0+\mathbf{Z}_2}\}_{1+\mathbf{Z}_2} = \{(0+\mathbf{Z}_2)*(1+\mathbf{Z}_2)\} = \{0+\mathbf{Z}_2\}$, *with* $(0+\mathbf{Z}_2)\widehat{+}(0+\mathbf{Z}_2) = 1\widehat{+}\mathbf{Z}_2 = 1+\mathbf{Z}_2 \notin \{\widehat{0+\mathbf{Z}_2}\}_{1+\mathbf{Z}_2}$. $\triangleleft$

It might be good now to pose a new question. Let us first give the necessary conditions to pose it: Let $(G, \circ)$ be an associative group of unit element I and $(\widehat{G}, \widehat{\circ})$ the isogroup associated with the isotopy of elements $\widehat{I}$ and $*$. We know that any isosubgroup $\widehat{G}$ has subgroup structure. We have also seen examples where subgroups of G may not give rise to isosubgroups of $\widehat{G}$, using as isotopic elements both $\widehat{I}$ and $*$. We finally ask if any subset of $\widehat{G}$ has isosubgroup structure $(\widehat{G}, \widehat{\circ})$, that is, if it comes from the isotopic lifting of a subgroup of G. It has certainly been clear that as a subgroup$(\widehat{H}, \widehat{\circ})$ de $(\widehat{G}, \widehat{\circ})$, the operation $\widehat{\circ}$ must be the same in both pairs, the elements of the corresponding isotopy must coincide. That is, if $\widehat{H}$ is an isosubgroup, it must come from an isotopy with the same elements as that which constructs $\widehat{G}$. Therefore, the only possible subset of G that would give rise to the possible isosubgroup would be $H = \{a \in G : \widehat{a} \in \widehat{H}\} \subseteq G$. However, the pair $(H, \circ)$ does need not be a subset of $(G, \circ)$ in general, as we can see in the following:

Example 3.3.10 *Let us consider* $(\mathbf{Z}/\mathbf{Z_2}, +)$ *and the isogroup* $(\widehat{\mathbf{Z}/\mathbf{Z}}_{2_{1+\mathbf{Z_2}}}, *)$ *given in the previous example. As unique subgroups of both, we have the pairs* $(\{0 + \mathbf{Z_2}\}, +)$ *and* $(\{1 + \mathbf{Z_2}\}, *)$, *respectively.*

As seen, if with the previous notations we take $\widehat{H} = (\{1 + \mathbf{Z_2}\}, *)$ *as subgroup of* $(\widehat{\mathbf{Z}/\mathbf{Z}}_{2_{1+\mathbf{Z_2}}}, *)$, *the only possible subset of* $\mathbf{Z}/\mathbf{Z_2}$ *which could give* $\widehat{H}$ *an isosubgroup structure would be* $H = \{1 + \mathbf{Z_2}\}$, *then* $(1 + \mathbf{Z_2}) * (1+\mathbf{Z_2}) = 1+\mathbf{Z_2}$, $\widehat{I} = 1+\mathbf{Z_2}$ *being the isounit we use in such an example to construct the isotopy. However,* $(\{1 + \mathbf{Z_2}\}, +)$ *is not a subset of* $(\mathbf{Z}/\mathbf{Z_2}, +)$, *as, for example,* $+$ *is not a binary operation for* $\{1 + \mathbf{Z}_2\}$, *where* $(1 + \mathbf{Z_2}) + (1 + \mathbf{Z_2}) = 0 + \mathbf{Z_2}$. $\triangleleft$

We therefore see that with this example the question that we raised is answered negatively, the problem of the relationship between groups and isogroups finally being resolved.

We end this section giving definitions of the possible functions that are established among isogroups:

Definition 3.3.11 *Let $(\widehat{G}, \widehat{\circ})$ and $(\widehat{G'}, \widehat{\bullet})$ be two isogroups. A function $f : \widehat{G} \to \widehat{G'}$ is called an isogroup* homomorphism *if $f(\widehat{\alpha}\widehat{\circ}\widehat{\beta}) = f(\widehat{\alpha})\widehat{\bullet}f(\widehat{\beta})$ is satisfied for all $\widehat{\alpha}, \widehat{\beta} \in \widehat{G}$. If f is bijective, it is then called an isogroup* isomorphism. *If $\widehat{G'} = \widehat{G}$, f is called an* endomorphism, *and if, in addition, f is isomorphic, then it is called an* automorphism.

3.4 Isorings

For the study of this new isostructure, we will follow the same procedure as in the case of isogroups. In the first subsection, we will study *isorings* and *isosubrings* themselves. In the following two subsections we will discuss *isoideals* and *quotient isorings*.

3.4.1 Isorings and Isosubrings

Definition 3.4.1 *Let $(A, \circ, \bullet)$ be a ring with unit element e. Any isotopy A equipped with two binary operations $\widehat{\circ}$ and $\widehat{\bullet}$, the second with a unit element $\widehat{I} \in \widehat{A}$ (isounit) not necessarily belonging to A, verifying the ring axioms, is called an* isoring *$\widehat{A}$. I.e., such that for every $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma} \in \widehat{A}$, it satisfies:*

1. *$(\widehat{A}, \widehat{\circ})$ is an Abelian group.*
2. *Associativity of $\widehat{\bullet}$: $(\widehat{\alpha}\widehat{\bullet}\widehat{\beta})\widehat{\bullet}\widehat{\gamma} = \widehat{\alpha}\widehat{\bullet}(\widehat{\beta}\widehat{\bullet}\widehat{\gamma})$.*
3. *Distributivity:*
   a. $\widehat{\alpha}\widehat{\bullet}(\widehat{\beta}\widehat{\circ}\widehat{\gamma}) = (\widehat{\alpha}\widehat{\bullet}\widehat{\beta})\widehat{\circ}(\widehat{\alpha}\widehat{\bullet}\widehat{\gamma})$
   b. $(\widehat{\alpha}\widehat{\circ}\widehat{\beta})\widehat{\bullet}\widehat{\gamma} = (\widehat{\alpha}\widehat{\bullet}\widehat{\gamma})\widehat{\circ}(\widehat{\beta}\widehat{\bullet}\widehat{\gamma})$.

If in addition $\widehat{\alpha}\widehat{\bullet}\widehat{\beta} = \widehat{\beta}\widehat{\bullet}\widehat{\alpha}$ is satisfied for all $\widehat{\alpha}, \widehat{\beta} \in \widehat{A}$, $\widehat{A}$ is called isocommutative.

We note that in the case of isorings there must exist two isounits: one with respect to the operation $\widehat{\circ}$, which we designate by $\widehat{S}$, and the

other with respect to $\widehat{\bullet}$, which we have already denoted by $\widehat{I}$. If we focus on the case of Santilli isotopies, we have already seen that each of these is determined by an isounit and operation $*$. Moreover, the construction we made for isogroups would favor that the isounit of the isotopy coincide with the isounit of the isogroup. However, here are two isounits in the isostructure. Would the use of two different isotopies for the construction of an isoring be required, then? To answer this question, we will set up in the first place an associative operation $*$ and an isounit which we call $\widehat{I}$, since we will seek that, by construction, it matches the isounit with respect to $\widehat{\circ}$, already cited above. Regarding this isotopy, we already know explicitly to construct the isotopic set associated with A, which shall be given by $\widehat{A} = \{\widehat{a} = a * \widehat{I} \mid a \in A\}$.

Once the set of elements of the starting structure has been raised, we must make the lifting of the associated operations, $\circ$ and $\bullet$. Therefore, we will begin isotopically lifting the second operation, $\bullet$, by means of the procedure for constructing the isoproduct, already seen in the previous section. That is, if $*$ has as a unit for I and has $T = \widehat{I}^{-I}$, we will have as the isoproduct $\widehat{\bullet}$ for the operation defined as

$$\widehat{a}\widehat{\bullet}\widehat{b} = \widehat{a} * T * \widehat{b} = (a * b) * \widehat{I}, \ \forall \widehat{a}, \widehat{b} \in \widehat{A}.$$

In this way, just as we did for isogroups, we get that the operation $\widehat{\bullet}$ is associative for $*$ being associative. Moreover, imposing $I \in A$ we have that $\widehat{I} \in \widehat{A}$, the isounit being indicated in the above definition with respect to $\widehat{\bullet}$.

Lifting the operation $\circ$ would still be lacking. If we seek an analogous procedure as the method for an isoproduct, we would need an isounit $\widehat{S}$ and operation $\star$, similar to $\widehat{I}$ and $*$. Now, given that the isotopic set of the future isoring is already constructed, the operation $\star$ should be such that the isotopic set formed starting from it matches that which it already had. In addition, given that $(\widehat{A}, \widehat{\circ})$ should be an isogroup, we would have to check $\widehat{S} \in \widehat{A}$, so that $\widehat{S}$ should be of the form $\widehat{S} = s * \widehat{I}$, with $s \in A$. For this reason, following the notation of Definition 3.1.3, we will consider the general set V associated with the

lifting of the operation $\circ$ the same as that used for the lifting of A and $\bullet$. On the other hand, if $\star$ was an operation with unit element S, we know that as conditions for an isotopy of a group it is necessary to impose that $(A, \star)$ be a group with $S \in A$. Also we will impose that $\star$ be associative and along with $*$ that it satisfies the distributive property for $\widehat{A}$, i.e., that $\forall a, b, c \in A$, it satisfies

1. $(a \star b) \star c = a \star (b \star c)$.
2. $a * (b \star c) = (a * b) \star (a * c)$; $(a \star b) * c = (a * c) \star (b * c)$.

As a result, supposing that $\widehat{S}^{-S} = \widehat{R} = r * \widehat{I} \in \widehat{A}$ (with $r \in A$), we would construct the isoproduct $\widehat{\circ}$ defining for all $\widehat{a}, \widehat{b} \in \widehat{A}$ according to $\widehat{a}\widehat{\circ}\widehat{b} = \widehat{a} \star \widehat{R} \star \widehat{b} = (\widehat{a} \star \widehat{R}) \star \widehat{b} = ((a * \widehat{I}) \star (r * \widehat{I})) \star \widehat{b} = ((a \star r) * \widehat{I}) \star (b * \widehat{I}) = (a \star r \star b) * \widehat{I}$ that is trivially in $\widehat{A}$, since $a \star r \star b \in A$, $(A, \star)$ being a group.

Then, in a way analogous to the general case we saw, we would arrive at, under these conditions, that $(\widehat{A}, \widehat{\circ})$ is an isogroup, thus verifying the condition (1) of the above definition, so the only condition that remains to check would be distributivity. However, as it will be seen below, in general this condition is not met. Indeed, let there be $\widehat{a}, \widehat{b}, \widehat{c} \in \widehat{A}$. Then: $\widehat{a}\widehat{\bullet}(\widehat{b}\widehat{\circ}\widehat{c}) = \widehat{a}\widehat{\bullet}(\widehat{b} \star \widehat{R} \star \widehat{c}) = \widehat{a}\widehat{\bullet}((b \star r \star c) * \widehat{I}) = (a * (b \star r \star c)) * \widehat{I} = ((a * b) \star (a * r) \star (a * c)) * \widehat{I} \neq ((a * b) \star r \star (a * c)) * \widehat{I} = (\widehat{a}\widehat{\bullet}\widehat{b})\widehat{\circ}(\widehat{a}\widehat{\bullet}\widehat{c})$ and similarly, nor is the distributive operation to the left in general satisfied. It would be to the right when $a * r = r, \forall a \in A$ (to the left when $r * a = r$, respectively), $\forall a \in A$. In the case of $a * r = r = r * a$ being satisfied, $\forall a \in A$, then, it would satisfy distributivity and therefore we would have already constructed the isoring, which finally would come from the isotopy of the principal elements $\widehat{I}$ and $*$ and from the secondary elements $\widehat{S}$ and $\star$.

We note that in the event that the isotopy could be constructed, as $*$ (the principal element of it) has been used to lift the operation $\bullet$, the constructed isoring received the name of *isoring with respect to multiplication*, given that in practice $\circ \equiv +$ and $\bullet \equiv \times$. Making an analogous procedure, using $*$ to lift $\circ$, we arrive at an *isoring with respect to the sum*. In fact, according to this criterion, from now on we will denote

the isotopic lifting of the first operation $\circ$ the *isosum*, while the lifting of the second we will continue calling the isoproduct.

On the other hand, although the final condition $a * r = r, \forall a \in A$, might not be fulfilled (and therefore will not satisfy distributivity), as all other conditions are met, we will denote the lifting obtained by the previous procedure *pseudoisotopy* or *pseudoisotopic lifting*. So we would obtain in this way a new type of mathematical structure, generally called a *pseudoisostructure* (see [128]). In this particular case, we would obtain a *pseudoisoring*.

We also noticed that the procedure used can be simplified, taking into account that the second binary operation of a ring need not verify the condition of an inverse element, so we could also simplify this condition in the case of $*$, if it is an isoring with respect to multiplication, or in the case of $\star$, if it is an isoring with respect to the sum. The general procedure will work for more particular structures as, for example, the isofields that we will see in the next section.

All of the above suggests the following:

Proposition 3.4.2 *Let $(A, \circ, \bullet)$ be a ring and let $\widehat{I}, \widehat{S}, *$, and $\star$ be elements of an isotopy under the conditions of Definition 3.1.3, I and S being the respective units of $*$ and $\star$. In these conditions, if $(A, \star, *)$ has a ring structure with respective units $S, I \in A$, then the isotopic lifting $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$ obtained by the procedure of the isoproduct corresponding to the isotopy of the principal elements $\widehat{I}$ and $*$ and secondary elements $\widehat{S}$ and $\star$ has an isoring structure with respect to multiplication if $a * r = r = r * a, \forall a \in A$, $r \in A$ being such that $\widehat{R} = \widehat{S}^{-S} = r * \widehat{I}$. Analogously, if $(A, *, \star)$ has a ring structure with respective units $I, S \in A$, then the isotopic lifting by the construction of the isoproduct, $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$, corresponding to the same previous isotopy, has an isoring structure with respect to the sum, if $a \star t = t = t \star a, \forall a \in A$ is met, $t \in A$ being such that $t * \widehat{I} = T = \widehat{I}^{-I}$.* □

Agreeing then with this definition, the answer to the question of whether two isotopies were necessary for an isoring is negative, since only one isotopy is necessary but with two principal elements $\widehat{I}$ and $*$,

and two secondary elements $\widehat{S}$ and $\star$, which will have to be explicitly indicated anyways.

Let us see below some examples of isorings:

Example 3.4.3 *Consider the ring $(\mathbf{Z}, +, \times)$ of the integers with the usual sum and product. Analogously to Example 3.3.3 we prove that the isotopy of elements $\widehat{I} = 0$ and $* \equiv +$ (of the unit element $I = 0 = \widehat{I}$, and therefore $T = \widehat{I}^{-I} = 0^{-0} = 0 = 0 + 0 = 0 * 0$) isotopically raises the group $(\mathbf{Z}, +)$ into $(\widehat{\mathbf{Z}}, \widehat{+}) = (\mathbf{Z}, +)$, such that this isotopy is equal to the identity.*

Similarly, if we add the secondary isotopic elements $\widehat{S} = 1$ and $\star \equiv \times$ (although in this case they could also be considered as primary elements, given that there would be no difference in the end result), we would get for the isotopic lifting of the ring $(\mathbf{Z}, +, \times)$ this same ring.

To prove this assertion, we observe firstly that the isotopic set constructed by $\star$ and $\widehat{S}$ would be $\widehat{\mathbf{Z}}_1 = \{\widehat{a} = a \times 1 = a \mid a \in \mathbf{Z}\} = \mathbf{Z}$, which coincides with that constructed from the previous elements $\widehat{I} = 0$ and $* \equiv +$. Now, it would suffice to define the isoproduct originated by $\star$ for the previous elements $\widehat{I} = 0$ and $* \equiv +$. Now, as the element unit with respect to $\star$ is $S = 1$ and therefore $\widehat{S}^{-S} = 1^{-1} = 1$, we deduce that $\widehat{a}\widehat{\times}\widehat{b} = \widehat{a} \star 1 \star \widehat{b} = \widehat{a} \times 1 \times \widehat{b} = (a+0)\times(1+0)\times(b+0) = (a\times 1\times b)+0 = \widehat{a \times 1 \times b} = \widehat{a \times b} = a\times b$, for all $\widehat{a}, \widehat{b} \in \widehat{\mathbf{Z}}_0$. Then $\widehat{\times} \equiv \times$, thus finishing the demonstration, since $(\mathbf{Z}, *, \star) = (\mathbf{Z}, +, \times)$ being a ring with respective units $0, 1 \in \mathbf{Z}$ and satisfying $a \star 0 = a \times 0 = 0 = 0 \times a = 0 \star a$, $\forall a \in \mathbf{Z}$, Proposition 3.4.2 ensures that the isotopic lifting of the principal elements $\widehat{I}$ and $*$ and secondary elements $\widehat{S}$ and $\star$ of the ring $(\mathbf{Z}, +, \times)$ is an isoring. Thus we get a trivial isotopy of the starting ring, which corroborates the fact that if any isotopic lifting varies neither the starting operations nor the corresponding unit elements, the departing structure does not change. $\triangleleft$

Example 3.4.4 *Let us consider again the ring $(\mathbf{Z}, +, \times)$. Let us take the operation $* \equiv \times$ and isounit $\widehat{I} = -1$ ($T = \widehat{I}^{-I} = (-1)^{-1} = -1 = \widehat{I}$, given that the element unit with respect to $*$ is $I = 1$) and seek to construct an isoring with respect to multiplication. As an isotopic set, it would then be $\widehat{\mathbf{Z}}_{-1} = \{\widehat{a} = a \times (-1) = -a : a \in \mathbf{Z}\} = \mathbf{Z}$.*

In turn, the isoproduct is defined as $\widehat{a}\widehat{\times}\widehat{b} = \widehat{a} * (-1) * \widehat{b} = (a * b) * (-1) = (a \times b) * (-1) = \widehat{a \times b} = -(a \times b)$, *for all* $\widehat{a}, \widehat{b} \in \widehat{\mathbf{Z}}_{-1}$.

In general, we call the isotopy that arises from considering as isounit for $\widehat{I} = -I$ *the* isodual *isotopy. Santilli himself introduced this isotopy in* **[110]**, **[111]**, **[114]** *and* **[115]**.

We now seek to raise the operation $+$, *leaving it invariant. For this, it would suffice to take* $\widehat{S} = 0$ *as the secondary isounit and* $\star$ *as the operation for* $+$ *itself, such as it has been done in previous examples. So, given that the unit element of* $\star$ *would be* $S = 0$, *we would have that* $\widehat{R} = \widehat{S}^{-S} = 0^{-0} = 0 = 0 * 1$, *with* $a * 0 = a \times 0 = 0 = 0 \times a = 0 * a$, $\forall a \in \mathbf{Z}$. *In this way,* $(\mathbf{Z}, \star, *) = (\mathbf{Z}, +, \times)$ *is a ring verifying the conditions of Proposition 3.4.2; thus, the isotopic lifting* $(\mathbf{Z}, +, \widehat{\times})$ *turns out to be an isoring with respect to multiplication.* $\triangleleft$

Example 3.4.5 *Continuing with the* $(\mathbf{Z}, +, \times)$ *ring, we can study a case not seen thus far. Here is an example in which the isounit* $\widehat{I}$ *is in the resulting isotopic set, but the isotopic element* $T = \widehat{I}^{-I}$ *is not. In our case, it suffices to consider, for example,* $* \equiv \times$ *and* $\widehat{I} = 2$. *The isotopic set* $\widehat{\mathbf{Z}}_2 = \{\widehat{a} = a \times 2 \mid a \in \mathbf{Z}\} = \mathbf{P} = \mathbf{Z_2}$ *would result. On the other hand, given that the unit element with respect to* $*$ *is* $I = 1$, *then* $\widehat{I}^{-I} = 2^{-1} = \frac{1}{2} \notin \mathbf{P}$, *the isoproduct being defined by:* $\widehat{a}\widehat{\times}\widehat{b} = \widehat{a} * \frac{1}{2} * \widehat{b} = (a \times 2) \times \frac{1}{2} \times (b \times 2) = (a \times b) \times 2 = \widehat{a \times b}$, *for all* $\widehat{a}, \widehat{b} \in \widehat{\mathbf{Z}}_2$.

In any case, if as in the previous example we want operation $+$ *to remain invariant, we could only take as the secondary isounit* $\widehat{S} = 0 = 0 * 2$, *as* $r = 0$ *is the unique element of* $\mathbf{Z}$ *such that* $a * r = r = r * a$ *for all* $a \in \mathbf{Z}$, *which is a necessary condition to construct the isoring according to Proposition 3.4.2. Therefore, completing our isotopy with the secondary elements* $\widehat{S} = 0$ *and* $\star \equiv +$, *we would get that* $(\mathbf{P}, +, \widehat{\times})$ *is an isoring with respect to multiplication.* $\triangleleft$

We finally note that if we want to obtain an isoring (with respect to the sum) of the ring $(\mathbf{Z}, +, \times)$, starting from $* \equiv +$ and $\star \equiv \times$, the main isounit $\widehat{I}$ would necessarily be $\widehat{I} = 0$, to ensure the designated condition that $a \star t = a \times t = t = t \times a = t \star a, \forall a \in \mathbf{Z}, t \in \mathbf{Z}$ being such

that $T = \widehat{I}^{-I} = t * \widehat{I}$. In this way, the isoring $(\mathbf{P}, +, \widehat{\times})$ above can also be considered an isoring with respect to the sum.

We will then study the isotopies of substructures associated with rings, i.e., the subrings. We give, in the first place, the definition of an isosubring.

Definition 3.4.6 *Let $(A, \circ, \bullet)$ be an isoring and $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$ an associated isoring with unit element $\widehat{I}$ with respect to $\widehat{\bullet}$. Le B be a subring of A. $\widehat{B}$ is called an* isosubring *of $\widehat{A}$ if, being an isotopy of B, $(\widehat{B}, \widehat{\circ}, \widehat{\bullet})$ is a subring of $\widehat{A}$, i.e., it satisfies:*

1. *$(\widehat{B}, \widehat{\circ}, \widehat{\bullet})$ is closed, satisfying the conditions of associativity and distributivity.*
2. *$(\widehat{B}, \widehat{\circ})$ is an isosubgroup of $(\widehat{A}, \widehat{\circ})$.*
3. *$\widehat{I} \in \widehat{B}$.*

We will then see if it is possible to apply the above definition to the method of constructing the isotopy that we are carrying out. To do this, we suppose we have the ring $(A, \circ, \bullet)$ and isoring $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$ obtained by the isotopy of the principal elements $\widehat{I}$ and $*$ (of unit element I) and secondary elements $\widehat{S}$ and $\star$ (of the unit element S), in the conditions of Proposition 3.4.2. We will see it in the case of isorings with respect to multiplication, being analogous in the case of the sum, with the appropriate modifications.

As was the case for isosubgroups, since we want that in the future isosubring $\widehat{B}$ the associated laws be the same as those of the ring $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$, if we follow the construction given by the isoproduct, the principal and secondary isotopic elements we should use for the lifting of the subring B have to be exactly the same as those used for lifting the ring A. Thus we obtain in particular that $\widehat{B} \subseteq \widehat{A}$, a necessary condition for the structure of a subring. If in addition to imposing that $(B, \star, *)$ have a ring structure, we will obtain by construction the condition (1) of Definition 3.4.6 (the distributivity condition is met, given that we are in the conditions of Proposition 3.4.2, $(B, \star, *)$ inherits from $(A, \star, *)$ the fact that $a * r = r = r * a$, $\forall a \in B$, where r is the element

indicated in this proposition). On the other hand, imposing that $I \in B$, we will get that $\widehat{I} = I * \widehat{I} \in \widehat{B}$, obtaining condition (3). Finally, given that that $(B, \star)$ already has a group structure, $(B, \star, *)$ being a ring, if we impose in addition that $S \in B$, we will have $\widehat{S} = S \star \widehat{S} \in \widehat{B}$ (let us recall that the secondary elements of an isotopy act in the same way as the primary ones, except that they should produce the same isotopic set the primary ones produce), thus Proposition 3.3.8 would ensure that the condition (2) is also verified.

It would also be equivalent to impose that $s \in B$, in the case of $\widehat{S} = s * \widehat{I}$, because in the same way we would arrive at $\widehat{S} \in \widehat{B}$ and could again apply Proposition 3.3.8. In fact, we observe that $s \in B \Leftrightarrow S \in B$, then the following are satisfied:

1. $s \in B \Rightarrow \widehat{S} = s * \widehat{I} \in \widehat{B} \Rightarrow \exists a \in B$ such that $a \star \widehat{S} = \widehat{S}$. Now, if $\widehat{R} = \widehat{S}^{-S}$, then $a \star \widehat{S} \star \widehat{R} = \widehat{S} \star \widehat{R} \Rightarrow a \star S = S \Rightarrow a = S \Rightarrow S \in B$.
2. $S \in B \Rightarrow \widehat{S} = S \star \widehat{S} \in \widehat{B} \Rightarrow \exists a \in B$ such that $a * \widehat{I} = \widehat{S}$. However, we have that $s * \widehat{I} = \widehat{S}$. Then, if $T = \widehat{I}^{-I}$, we have $a * \widehat{I} * T = \widehat{S} * T = s * \widehat{I} * T \Rightarrow a * I = s * I \Rightarrow a = s \Rightarrow s \in B$.

Note that this previous development in fact also applies to isorings. With this we also arrive at that if we have that $s \in B$ and $S \notin B$, or *vice versa*, the isotopy of elements $\widehat{I}, \widehat{S}, *$, and $\star$ is not possible, for the operations $*$ and $\star$ would not be compatible for the formation of the same isotopic set. From all of this, the following results:

Proposition 3.4.7 *Let $(A, \circ, \bullet)$ be a ring and $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$ the associated isoring corresponding to the isotopy with principal elements $\widehat{I}$ and $*$ (of unit element I), secondary elements $\widehat{S} = s * \widehat{I}$ and $\star$ (of unit element S), in the conditions of Proposition 3.4.2. Let B be a subring of A. In these conditions, if $(B, \star, *)$ is a subring of $(A, \star, *)$ with $I \in B$ and $S \in B$ (or, $s \in B$), then the isotopic lifting $(\widehat{B}, \widehat{\circ}, \widehat{\bullet})$, corresponding to the isotopy of the previous elements themselves, is an isosubring of $\widehat{A}$.* □

Here below is an example of an isosubring:

Example 3.4.8 *Let us consider the ring* $(\mathbf{Q}, +, \times)$ *of the rational numbers with the usual sum and product. Let us take the isounit* $\widehat{I} = 2$ *and the operation* $* \equiv \times$, *the isotopic set* $\widehat{\mathbf{Q}}_2 = \{\widehat{a} = a \times 2 \mid a \in \mathbf{Q}\} = \mathbf{Q}$ *resulting. As* $* \equiv \times$, *the respective unit element* $*$ *has to be* $I = 1 \in \mathbf{Q}$, *thus* $\widehat{I}^{-I} = 2^{-1} = \frac{1}{2}$, *then obtaining the isoproduct defined according to* $\widehat{a}\widehat{\times}\widehat{b} = \widehat{a} * \frac{1}{2} * \widehat{b} = \widehat{a * b} = \widehat{a \times b} = (a \times b) \times 2$, *for all* $\widehat{a}, \widehat{b} \in \widehat{\mathbf{Q}}_2$.

If on the other hand we consider the elements of the secondary isotopies $\widehat{S} = 0$ *and* $\star \equiv +$ *(with which* $s = S = 0$*), we get, in an analogous way to how it was done in Example 3.4.5, that* $(\widehat{\mathbf{Q}}_2, +, \widehat{\times}) = (\mathbf{Q}, +, \widehat{\times})$ *is an isoring.*

We now consider the subring $(\mathbf{Z}, +, \times)$ *of* $(\mathbf{Q}, +, \times)$, *of the integers, and try to realize the isotopy of this subring, of the same elements as those used in the construction of the isoring* $(\mathbf{Q}, +, \widehat{\times})$.

The isotopic set $\widehat{\mathbf{Z}}_2 = \{\widehat{a} = a \times 2 \mid a \in \mathbf{Z}\} = \mathbf{P}$ *would then result. Therefore, as* $(\mathbf{Z}, \star, *) = (\mathbf{Z}, +, \times)$ *has ring structure* $(\mathbf{Q}, \star, *) = (\mathbf{Q}, +, \times)$, *with the unit element with respect to* $*$, $I = 1 \in \mathbf{Z}$ *(the same as for* $(\mathbf{Q}, \star, *)$*) and* $s = S = 0 \in \mathbf{Z}$, *we arrive by Proposition 3.4.7 at* $(\widehat{\mathbf{Z}}_2, +, \widehat{\times}) = (\mathbf{P}, +, \widehat{\times})$ *being an isosubring of* $\widehat{\mathbf{Q}}_2$.

In this way we observe, if we take into account Example 3.4.5, that $(\mathbf{P}, +, \widehat{\times})$ *can give itself both an isoring and isosubring structure with respect to the same designated isotopy.* $\triangleleft$

As was the case with the isogroups, we can also ask ourselves if any subring gives rise to an isosubring under a particular isotopy or if any subring of a given isoring has an isosubring structure. In the same way as in the isosubgroups, the possible counter-examples should be in those cases in which the conditions of Proposition 3.4.7 are not met. However, except that we keep in mind some examples of more complicated liftings than we have been using so far, to find such counterexamples is quite complicated, because if, as we have almost always done, we leave the starting operations practically invariant, their properties will remain conserved at all times. However, theoretically it is possible to find counterexamples that give a negative answer to the two questions posed. It would suffice, for example, in the conditions

of Proposition 3.4.7, that we have a subring B of A such that $S \notin A$ or $I \notin B$.

Along the same lines, if we had a subring $\widehat{B}$ of the isoring $\widehat{A}$, with $\widehat{S} \in \widehat{B}$, such that $S \notin C$ for all C, subring of A;s then we could not provide an isosubring structure for $\widehat{B}$, since we could not find any subring in A that gives $\widehat{B}$ itself as a result, after the corresponding isotopic lifting. In this way, we can guess that in general not every subring can be isotopically lifted to an isosubring using a fixed isotopy, nor can any subring of an isoring have the structure of an isosubring by means of the isotopy corresponding to said isoring.

Concerning the various functions that exist between isoring, we have the following:

Definition 3.4.9 *Let $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$ and $(\widehat{A'}, \widehat{+}, \widehat{\times})$ be two isorings with isounits and operations $\{\widehat{I}, \widehat{\bullet}\}$ and $\{\widehat{I'}, \widehat{\times}\}$, respectively. A function $f : \widehat{A} \to \widehat{A'}$ is called an* isoring homomorphism *if for all $\widehat{a}, \widehat{b} \in \widehat{A}$, the following are satisfied:*

1. $f(\widehat{a} \widehat{\circ} \widehat{b}) = f(\widehat{a}) \widehat{+} f(\widehat{b})$,
2. $f(\widehat{a} \widehat{\bullet} \widehat{b}) = f(\widehat{a}) \widehat{\times} f(\widehat{b})$,
3. $f(\widehat{I}) = \widehat{I'}$.

If f is bijective, it is called an isomorphism, *and if $\widehat{A} = \widehat{A'}$, an* endomorphism. *In the latter case, if f is also bijective, it is called an* automorphism.

In the next two subsections we will study, respectively, the basic notions of two new isostructures related to isorings: isoideals and quotient isorings.

3.4.2 Isoideals

Definition 3.4.10 *Let $(A, \circ, \bullet)$ be a ring and $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$ an associated isoring. Let $\Im \subseteq A$ be an ideal of A. $\widehat{\Im}$ is called an* isoideal *of $\widehat{A}$ if, being an isotopy of $\Im$, $\widehat{\Im}$ has the structure of an ideal with respect to $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$, i.e., if the two following conditions are satisfied:*

1. $(\widehat{\Im}, \widehat{\circ})$ *is an isosubgroup of* $\widehat{A}$.
2. $\widehat{\Im}\widehat{\bullet}\widehat{A} \subseteq \widehat{\Im}$, $\widehat{A}\widehat{\bullet}\widehat{\Im} \subseteq \widehat{\Im}$, *i.e.,* $\widehat{x}\widehat{\bullet}\widehat{a}$ *and* $\widehat{a}\widehat{\bullet}\widehat{x} \in \widehat{\Im}$, *for all* $\widehat{a} \in \widehat{A}$ *and* $\widehat{x} \in \widehat{\Im}$.

We now consider the model of the isotopy that we are carrying out. Suppose then that we have a ring $(A, \circ, \bullet)$ and isoring $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$ obtained by the isotopy of principal elements $\widehat{I}$ and $*$ and secondary elements $\widehat{S}$ and $\star$. As we already did previously, we will prove only the case of isorings with respect to multiplication, understanding that with regard to the sum we would proceed similarly.

As in previous cases, if we have an ideal $\Im$ of A, we wish that the future isoideal have the associated laws of $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$. To do so, if we follow the construction of the isoproduct, we will take as primary and secondary isotopic elements exactly those that were needed for the construction of the isoring $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$. In this way we will have that $\widehat{\Im} \subseteq \widehat{A}$, since $\Im \subseteq A$.

If, in addition, we impose that $\Im$ be an ideal of the ring $(A, \star, *)$, we see that the two conditions of the above definition are verified, since:

1. condition (1) holds without needing to apply Proposition 3.3.8.
2. if T is the associated isotopic element for the isotopy of isounit $\widehat{I}$ and operation $*$, then $\widehat{x}\widehat{\bullet}\widehat{a} = \widehat{x} * T * \widehat{a} = (x * a) * \widehat{I} \in \widehat{\Im}$, for all $\widehat{x} \in \widehat{\Im}$ and $\widehat{a} \in \widehat{A}$, since for $\Im$ being the ideal of $(A, \star, *)$, $x * a \in \Im$.

Therefore, we finally arrive at that $\widehat{\Im}$ is an isoideal of $\widehat{A}$, thus the following has been proved:

Proposition 3.4.11 *Let* $(A, \circ, \bullet)$ *be a ring and* $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$ *the associated isoring corresponding to the isotopy of principal elements* $\widehat{I}$ *and* $*$ *and secondary elements* $\widehat{S}$ *and* $\star$, *under the conditions of Proposition 3.4.2. Let* $\Im$ *be an ideal of* $(A, \circ, \bullet)$. *If* $\Im$ *is an ideal of* $(A, \star, *)$, $(\Im, \star)$ *being a subset of* $(A, \star)$, *with the unit element being that of this latter, then the isotopic lifting* $(\widehat{\Im}, \widehat{\circ}, \widehat{\bullet})$ *corresponding to the isotopy of the items listed above is an isoideal of* $\widehat{A}$. □

Here below are some examples of isoideals:

Example 3.4.12 *Let us consider the ring* $(\mathbf{Z}, +, \times)$ *and the isoring* $(\mathbf{Z}, +, \widehat{\times})$ *associated with it from Example 3.4.4. Let us now take* $\mathbf{P} = \mathbf{Z_2}$ *as the ideal of* $(\mathbf{Z}, +, \times)$*. Then, with the notation from the cited example,* $(\mathbf{P}, +)$ *is a subgroup of* $(\mathbf{Z}, +)$*, with unit element* $0 \in \mathbf{P}$*,* $(\mathbf{P}, \star, *) = (\mathbf{P}, +, \times)$ *being the ideal of* $(\mathbf{Z}, +, \times)$*. Proposition 3.4.11 then tells us that* $(\widehat{\mathbf{P}}_{-1}, +, \widehat{\times})$ *is an isoideal of* $(\mathbf{Z}, +, \widehat{\times})$*. However, as* $\widehat{\mathbf{P}}_1 = \{\widehat{a} = a * (-1) = a \times (-1) = -a : a \in \mathbf{P}\} = \mathbf{P}$*, we have that* $(\widehat{\mathbf{P}}_{-1}, +, \widehat{\times}) = (\mathbf{P}, +, \widehat{\times})$ *is the designated ideal.* $\triangleleft$

Example 3.4.13 *Let us now consider the ring* $(\mathbf{Z}, +, \times)$ *and the associated isoring* $(\mathbf{P}, +, \widehat{\times})$ *given in Example 3.4.5. Taking again the ideal* $(\mathbf{P}, +, \times)$ *of* $(\mathbf{Z}, +, \times)$*, we would have, with the notations of Example 3.4.5, that* $(\mathbf{P}, \star, *) = (\mathbf{P}, +, \times)$*, which is, in turn, an ideal of* $(\mathbf{Z}, \star, *) = (\mathbf{Z}, +, \times)$*,* $(\mathbf{P}, \star)$ *being a subgroup of* $(\mathbf{Z}, \star)$ *with the same unit element (in this case,* $I = 0 \in \mathbf{P} \cap \mathbf{Z}$*). In this way, Proposition 3.4.11 assures us that* $(\widehat{\mathbf{P}}_2, +, \widehat{\times})$ *is an isoideal of* $\mathbf{Z}$*. Now,* $\widehat{\mathbf{P}}_2 = \{\widehat{a} = a * 2 = a \times 2 \mid a \in \mathbf{P}\} = \mathbf{Z_4}$ *and therefore* $(\widehat{\mathbf{P}}_2, +, \widehat{\times}) = (\mathbf{Z_4}, +, \widehat{\times})$ *is the designated isoideal.* $\triangleleft$

We now define the concept of isosubideal.

Definition 3.4.14 *Let* $(A, \circ, \bullet)$ *be a ring,* $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$ *an associated isoring, and* $\Im$ *an ideal of* A*, such that the corresponding isotopic lifting* $\widehat{\Im}$ *is an isoideal. Let* J *be a subideal of* $\Im$*.* $\widehat{J}$ *is called an* isosubideal *of* $\widehat{\Im}$ *if, being an isotopy of* J*,* $(\widehat{J}, \widehat{\circ}, \widehat{\bullet})$ *is a subideal of* $\widehat{\Im}$ *with respect to* $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$*, i.e., if it is satisfied that*

1. $(\widehat{J}, \widehat{\circ})$ *is an isosubgroup of* $\widehat{A}$*.*
2. $\widehat{J} \widehat{\bullet} \widehat{A} \subseteq \widehat{J} \subseteq \widehat{\Im}$*.*

It is easily proved that with the usual construction of an isotopy by means of an isounit and the method of construction of the isoproduct we arrive, similarly to the case of ideals, at the following result:

Proposition 3.4.15 *Let* $(A, \circ, \bullet)$ *be a ring and* $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$ *the associated isoring corresponding to the isounit of the principal elements* $\widehat{I}$ *and* $*$ *and secondary elements* $\widehat{S}$ *and* $\star$*, in the conditions of Proposition 3.4.2. Let* $\Im$ *be an ideal of* A *such that the corresponding isotopic lifting,* $(\widehat{\Im}, \widehat{\circ}, \widehat{\bullet})$*, be an*

isoideal of $\widehat{A}$. *Finally, let* J *be a subideal* $\Im$. *If* $(J, \star, *)$ *is a subideal of* $(\Im, \star, *)$, *with the unit element of the group* $(\Im, \star)$ *the same as that of* $(\Im, *)$, *then the corresponding isotopic lifting* $(\widehat{J}, \widehat{\circ}, \widehat{\bullet})$ *is an isosubideal of* $\widehat{\Im}$. □

We observe that the usual construction allows that, given that $J \subseteq \Im$ (being its subgroup) and that the isotopic elements used are the same as for the construction of $\widehat{\Im}$, $\widehat{J} \subseteq \widehat{\Im}$.

As an example of a subideal, we have the following:

Example 3.4.16 *In Example 3.4.12 let us consider the subideal* $(\mathbf{Z_6}, +, \times)$ *of* $(\mathbf{P}, +, \times)$. *With the notation of said example, we have that* $(\mathbf{Z_6}, \star, *) = (\mathbf{Z_6}, +, \times)$ *is a subideal of* $(\mathbf{P}, \star, *) = (\mathbf{P}, +, \times)$, *with the unit element of* $(\mathbf{Z_6}, \star)$ *the same as that of* $(\mathbf{P}, \star)$ *(in this case,* $S = 0 \in \mathbf{Z_6} \cap \mathbf{P}$*). Therefore, applying Proposition 3.4.15 we have that* $(\mathbf{Z_6}, +, \widehat{\times})$ *is an isosubideal of* $(\mathbf{P}, +, \widehat{\times})$, *with* $\widehat{\mathbf{Z_6}}_{-1} = \{\widehat{a} = a * (-1) = a \times (-1) = -a \mid a \in \mathbf{Z_6}\} = \mathbf{Z_6}$. *Therefore,* $(\mathbf{Z_6}, +, \widehat{\times})$ *is the designated isosubideal.* ◁

Just as in the previous cases and given the particularity of the conditions imposed in Proposition 3.4.15, we can guess that not all subideals of a given ring can be isotopically lifted to an isosubideal using a fixed isotopy, nor must all subideals of an isoring have an isosubideal structure. To answer these conjectures affirmatively would require the use of more complicated liftings of the starting operations than those we have been using. These, however, do not provide an important characteristic for the development that we are realizing to give a basis of the Lie-Santilli isotheory.

We end this section with a final subsection in which we will carry out the study of quotient isorings. We will give the definition of this new isostructure and the problems that arise will be raised in its construction.

3.4.3 Quotient isorings

Definition 3.4.17 *Let $(A, \circ, \bullet)$ be a ring, $\Im$ and ideal of A, and $A/\Im$ the quotient ring with usual structure $(A/\Im, +, \times)$. $\widehat{A/\Im}$ is called a* quotient isoring *if, being an isotopy of $A/\Im$, $(\widehat{A/\Im}, \widehat{+}, \widehat{\times})$ has the structure of a quotient ring, i.e., if a ring $(B, \Box, \Diamond)$ and an ideal J of B exist, such that $\widehat{A/\Im} = B/J$, $\widehat{+}$ and $\widehat{\times}$ being the usual operations of quotient rings arising from $\Box$ and $\Diamond$.*

Observe that the definition allows that in general the ring B and its ideal J need not be isotopic liftings of the ring A and its ideal $\Im$, allowing one to differentiate the concept of quotient isoring from the quotient ring built from an isoring and its isoideal. In fact, theoretically we can give the case that either only B, or else just J, are isotopic liftings of A or $\Im$, respectively. In this way, although all had quotient ring structure, we would have to distinguish between the possible sets B/J, $\widehat{A}/J$, $B/\widehat{\Im}$, and $\widehat{A}/\widehat{\Im}$.

Note also that if we study the method of constructing isotopies that we are carrying out, such a distinction is more evident. Since we already studied isorings and isoideals, in the case of being able to construct the quotient ring $\widehat{A}/\widehat{\Im}$, we know that the isotopy used to obtain $\widehat{A}$ and $\widehat{\Im}$ must have the same principal and secondary isotopic elements. Moreover, given that the isotopic lifting $\widehat{A/\Im}$ of the ring quotient $A/\Im$ would be done according to the model we already saw of liftings of rings, the isotopy used will also consist of two principal and two secondary elements. Certainly, given distinct characteristics of the rings A and $A/\Im$, the isotopic elements will not be the same in general, because, in particular, the operations would be defined on different sets. However, we could give the case in which $\widehat{A/\Im} = \widehat{A}/\widehat{\Im}$, although in general they will not be equal. Here is an example of this:

Example 3.4.18 *Let us consider the ring $(\mathbf{Z}, +, \times)$ and its ideal $(\mathbf{Z_3}, +, \times)$, with the usual sum and product. Realizing the isotopy of Example 3.4.5, of principal elements $\widehat{I} = 2$ and $* \equiv \times$ and secondary elements $\widehat{S} = 0$ and $\star \equiv +$, we obtain the isoring $(\mathbf{P}, +, \widehat{\times})$ and its isoideal $(\mathbf{Z_6}, +, \widehat{\times})$*

(keeping in mind that $\widehat{\mathbf{Z}_{32}} = \{\widehat{a} = a \times 2 \mid a \in \mathbf{Z_3}\} = \mathbf{Z_6}$*). We would thus construct the quotient ring* $\mathbf{P}/\mathbf{Z}_6 = \{0+\mathbf{Z_6}, 2+\mathbf{Z_6}, 4+\mathbf{Z_6}\}$*, with the usual sum and product operations in quotient rings, arising from* $+$ *and* $\widehat{\times}$*. Denoting these also by* $+$ *and* $\widehat{\times}$*, we may indicate the second explicitly, which is given by:*

1. $(0+\mathbf{Z_6})\widehat{\times}(a+\mathbf{Z_6}) = (0\widehat{\times}a)+\mathbf{Z_6} = ((0\times a)\times 2)+\mathbf{Z_6} = 0+\mathbf{Z_6} = (a+\mathbf{Z_6})\widehat{\times}(0+\mathbf{Z_6})$*, for all* $a \in \{0,2,4\}$.
2. $(2+\mathbf{Z_6})\widehat{\times}(2+\mathbf{Z_6}) = (2\widehat{\times}2)+\mathbf{Z_6} = (2\times 2\times 2)+\mathbf{Z_6} = 8+\mathbf{Z_6} = 2+\mathbf{Z_6}$
3. $(2+\mathbf{Z_6})\widehat{\times}(4+\mathbf{Z_6}) = (2\times 4\times 2)+\mathbf{Z_6} = 16+\mathbf{Z_6} = 4+\mathbf{Z_6} = (4+\mathbf{Z_6})\widehat{\times}(2+\mathbf{Z_6})$.
4. $(4+\mathbf{Z_6})\widehat{\times}(4+\mathbf{Z_6}) = (4\times 4\times 2)+\mathbf{Z}_6 = 32+\mathbf{Z_6} = 2+\mathbf{Z_6}$.

So we arrive at, in particular, that the unit element of $\mathbf{P}/\mathbf{Z_6} = \mathbf{Z_2}/\mathbf{Z_6} = \widehat{\mathbf{Z}}_2/\widehat{\mathbf{Z_{32}}}$ *is* $2+\mathbf{Z_6}$.

Let us consider, on the other hand, the quotient ring $\mathbf{Z}/\mathbf{Z_3}$*, with the usual operations in quotient rings, arising from the operations* $+$ *and* $\times$ *of the ring* $(\mathbf{Z},+,\times)$*, which we will denote* $\circ$ *and* $\bullet$*. We now realize a lifting* resembling *the previous one, with principal isotopic elements* $\widehat{I} = 2+\mathbf{Z_3}$ *and* $* \equiv \bullet$*, and secondary elements* $\widehat{S} = 0+\mathbf{Z_3}$ *and* $\star \equiv \circ$*. Analogously to the examples we have already seen, the final result of such an isotopy is the isoring* $(\mathbf{Z}/\mathbf{Z_3}, \circ, \widehat{\bullet})$*, where the isoproduct* $\widehat{\bullet}$ *is defined by:*

1. $(0+\mathbf{Z_3})\widehat{\bullet}(a+\mathbf{Z_3}) = ((0+\mathbf{Z_3})\times(a+\mathbf{Z_3}))\times(2+\mathbf{Z_3}) = 0+\mathbf{Z}_3 = (a+\mathbf{Z_3})\widehat{\bullet}(0+\mathbf{Z_3})$*, for all* $a \in \{0,1,2\}$.
2. $(1+\mathbf{Z_3})\widehat{\bullet}(1+\mathbf{Z_3}) = ((1+\mathbf{Z_3})\times(1+\mathbf{Z_3}))\times(2+\mathbf{Z_3}) = 2+\mathbf{Z_3}$.
3. $(1+\mathbf{Z_3})\widehat{\bullet}(2+\mathbf{Z_3}) = ((1+\mathbf{Z_3})\times(2+\mathbf{Z_3}))\times(2+\mathbf{Z_3}) = 1+\mathbf{Z_3} = (2+\mathbf{Z_3})\widehat{\bullet}(1+\mathbf{Z_3})$.
4. $(2+\mathbf{Z_3})\widehat{\bullet}(2+\mathbf{Z_3}) = ((2+\mathbf{Z_3})\times(2+\mathbf{Z_3}))\times(2+\mathbf{Z_3}) = 2+\mathbf{Z_3}$

In this way, the isoring obtained is a quotient isoring, as it was constructed from the ring $(\mathbf{Z},+,\widehat{\times})$ *and its ideal* $(\mathbf{Z_3},+,\widehat{\times})$*, where* $+$ *and* $\widehat{\times}$ *are the operations cited from Example 3.4.5.*

Finally, we note that despite the resemblance *of the isotopy we used, we arrived at that* $\widehat{\mathbf{Z}/\mathbf{Z_3}} \neq \widehat{\mathbf{Z}}/\widehat{\mathbf{Z_3}}$ *(where we do not intentionally indicate the isounits to which both sets are referred, to glimpse the difference between*

them), which shows the care that must be taken to distinguish between a quotient isoring and a quotient isoring coming from an isoring and its isoideal. $\triangleleft$

3.5 Isofields

The next isostructure that we will study in this chapter comes from the isotopic lifting of fields (see [121]). As we have done with the previous isostructures, we will first give the definition of an *isofield*, followed by the method of constructing them by means of the isoproduct and several examples. Also, we indicate the most common operations between the elements of the isofields, the *isonumbers*.

Definition 3.5.1 *Let $K = K(a, +, \times)$ be a field of elements $\{a, b, \ldots\}$ with associated product $\times$ (or alternate product, respectively, i.e., such that $a \times (b \times b) = (a \times b) \times b$ and $(a \times a) \times b = a \times (a \times b)$, for all $a, b \in K$). An* isofield *$\widehat{K}$ is an isotopy K equipped with two new operations $\widehat{+}$ and $\widehat{\times}$, satisfying the field axioms, i.e.,*

1. *Axioms of addition:*
   a. *$(\widehat{K}, \widehat{+})$ is closed, i.e., $\widehat{a}\widehat{+}\widehat{b} \in \widehat{K}, \forall \widehat{a}, \widehat{b} \in \widehat{K}$.*
   b. *Commutativity: $\widehat{a}\widehat{+}\widehat{b} = \widehat{b}\widehat{+}\widehat{a}, \forall \widehat{a}, \widehat{b} \in \widehat{K}$.*
   c. *Associativity: $(\widehat{a}\widehat{+}\widehat{b})\widehat{+}\widehat{c} = \widehat{a}\widehat{+}(\widehat{b}\widehat{+}\widehat{c}), \forall \widehat{a}, \widehat{b}, \widehat{c} \in \widehat{K}$.*
   d. *Neutral element: $\exists \widehat{S} \in \widehat{K}$ such that $\widehat{a}\widehat{+}\widehat{S} = \widehat{S}\widehat{+}\widehat{a} = \widehat{a}, \forall \widehat{a} \in \widehat{K}$.*
   e. *Inverse element: Given $\widehat{a} \in \widehat{K}$, $\widehat{a}^{-\widehat{S}} \in \widehat{K}$ exists such that $\widehat{a}\widehat{+}\widehat{a}^{-\widehat{S}} = \widehat{a}^{-\widehat{S}}\widehat{+}\widehat{a} = \widehat{S}$.*
2. *Axioms of multiplication:*
   a. *$(\widehat{K}, \widehat{\times})$ is closed, i.e., $\widehat{a}\widehat{\times}\widehat{b} \in \widehat{K}, \forall \widehat{a}, \widehat{b} \in \widehat{K}$.*
   b. *Isocommutativity: $\widehat{a}\widehat{\times}\widehat{b} = \widehat{b}\widehat{\times}\widehat{a}, \forall \widehat{a}, \widehat{b} \in \widehat{K}$.*
   c. *Isoassociativity: $(\widehat{a}\widehat{\times}\widehat{b})\widehat{\times}\widehat{c} = \widehat{a}\widehat{\times}(\widehat{b}\widehat{\times}\widehat{c}), \forall \widehat{a}, \widehat{b}, \widehat{c} \in \widehat{K}$.*
   d. *(Isoalternancy, respectively): $\widehat{a}\widehat{\times}(\widehat{b}\widehat{\times}\widehat{b}) = (\widehat{a}\widehat{\times}\widehat{b})\widehat{\times}\widehat{b}$, $(\widehat{a}\widehat{\times}\widehat{a})\widehat{\times}\widehat{b} = \widehat{a}\widehat{\times}(\widehat{a}\widehat{\times}\widehat{b}), \forall \widehat{a}, \widehat{b} \in \widehat{K}$.*

e. Isounit: $\exists \hat{I} \in \hat{K}$ *such that* $\hat{a}\hat{\times}\hat{I} = \hat{I}\hat{\times}\hat{a} = \hat{a}, \forall \hat{a} \in \hat{K}$.

f. Isoinverse: Given $\hat{a} \in \hat{K}$, $\hat{a}^{-\hat{I}} \in \hat{K}$ *exists such that* $\hat{a}\hat{\times}\hat{a}^{-\hat{I}} = \hat{a}^{-\hat{I}}\hat{\times}\hat{a} = \hat{I}$.

3. *Axioms of addition and multiplication:*

   a. $(\hat{K}, \hat{+}, \hat{\times})$ *is closed:* $\hat{a}\hat{\times}(\hat{b}\hat{+}\hat{c}), (\hat{a}\hat{+}\hat{b})\hat{\times}\hat{c} \in \hat{K}, \forall \hat{a}, \hat{b}, \hat{c} \in \hat{K}$.

   b. Isodistributivity: $\hat{a}\hat{\times}(\hat{b}\hat{+}\hat{c}) = (\hat{a}\hat{\times}\hat{b})\hat{+}(\hat{a}\hat{\times}\hat{c})$, $(\hat{a}\hat{+}\hat{b})\hat{\times}\hat{c} = (\hat{a}\hat{\times}\hat{c})\hat{+}(\hat{b}\hat{\times}\hat{c}), \forall \hat{a}, \hat{b}, \hat{c} \in \hat{K}$.

We finally note that the elements of the isofield $\hat{K}$ *are usually called* isonumbers.

We will then apply the method of constructing isotopies by means of an isounit and isoproduct to obtain isofields from a given field. Let us therefore fix a field $K = K(a, +, \times)$ in the conditions of Definition 3.5.1. We must bear in mind that two isounits and two operations should appear in this construction, in the conditions of Definition 3.1.3 (since, by the definition of an isofield, a neutral element $\hat{S}$ for the lifting $\hat{+}$ and an isounit $\hat{I}$ for $\hat{\times}$).

In addition, we must not lose sight that any field is no more than a ring that satisfies the condition of the inverse element with respect to the second operation, so the construction that we have of isorings must be very similar to what we are seeking for isofields. In fact, if we restrict ourselves to the conditions of Proposition 3.4.2, we would get all the axioms mentioned in the above definition, except that of the isoinverse.

On the other hand, if we are working with the alternancy axiom instead of the associativity axiom, it would suffice to restrict, in turn, the associativity of the operation $*$ (or $\star$, if we search for an isoring with respect to the sum) to the degree of alternancy. Therefore from now on we will assume that we are always working with the axiom of associativity, given the analogous handling of alternancy. We will also assume that in the use of Proposition 3.4.2 we will be managing isorings with respect to multiplication, being analogous with respect to the sum. Indeed, observing the analogy between the construction

of isorings and isofields, we already noted that any isotopy of a field, constructed using the model of the isoproduct, will consist, as in the case of the isorings, of two principal isotopic elements, $\widehat{I}$ and $*$, and two secondary ones, $\widehat{S}$ and $\star$. Also, depending on whether $* \equiv \times$ or if $* \equiv +$, we will denote the isotopic lifting of the starting field the *isofield with respect to multiplication* or *with respect to the sum*, respectively. Due to expansion, we will carry out our construction seeking to obtain an isofield only with respect to multiplication, given that the procedure would be analogous for the sum.

To solve the problem raised by the isoinverse, it would suffice to impose on Proposition 3.4.2 that the ring $(A, \star, *)$ (in our case it would be $A = K$) have the property of the inverse element for the second operation, i.e., it would suffice to impose that $(A, \star, *)$ have a field structure.

With the notation of Proposition 3.4.2, to see that the isoinverse condition is effectively verified, let us set an element $\widehat{a} \in \widehat{A}$ such that $\widehat{a} = a * \widehat{I}$, with $a \in A$. Then, as $(A, \star, *)$ would have a field structure, $a^{-I} \in A$, the inverse element of a with respect to $*$ would exist, thus it would suffice to take $\widehat{a}^{-\widehat{I}} = \widehat{a^{-I}}$, since then $\widehat{a}\widehat{\times}\widehat{a^{-I}} = (a * a^{-I}) * \widehat{I} = I * \widehat{I} = \widehat{I} = \widehat{a^{-I}}\widehat{\times}\widehat{a}$. In this way, the existence of the isoinverse would be verified, which was the missing axiom for obtaining the isofield with respect to multiplication.

As a result we have demonstrated the following:

Proposition 3.5.2 *Let $K = K(a, +, \times)$ be an associative field. Let $\widehat{I}, \widehat{S}, *$ and $\star$ be isotopic elements in the conditions of Definition 3.1.3, I and S being the respective unit elements of $*$ and $\star$. In these conditions, if $K(a, \star, *)$ has a field structure with respective field elements $S, I \in K$, then the isotopic lifting $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$, realized by the previous procedure, corresponding to the isotopy of principal elements $\widehat{I}$ and $*$ and secondary elements $\widehat{S}$ and $\star$, has an isogroup structure with respect to multiplication, provided that $a * r = r = r * a$, $\forall a \in K$, with $r \in K$ such that $\widehat{R} = \widehat{S}^{-S} = r * \widehat{I}$.*

Similarly, if $K(a, *, \star)$ has a field structure with respective unit elements $I, S \in K$, then the isotopic lifting by the construction of the isoproduct $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$, corresponding to the isotopy of the listed items above, has isofield

structure with respect to the sum if $a \star t = t = t \star a, \forall a \in K$, *is satisfied,* $t \in K$ *being such that* $t * \widehat{I} = T = \widehat{I}^{-I}$. □

Again, as in the case of isorings, if the final condition of $a * r = r = r * a$ or $a \star t = t = t \star a, \forall a \in K$ is not fulfilled, then it does not satisfy the distributivity axiom. However, as the rest of axioms are satisfied, the resulting structure is given the name of *pseudoisofield* (see [128]).

Before seeing some examples of isofields, let us point out that the conservation of the axioms of K allow that the isounit $\widehat{I}$ concerning the operation $\widehat{\times}$ continues to satisfy the usual unit axioms of the operation $\times$ of the initial field (see [145]). Thus, for example, with the usual notation, we have:

1. $\widehat{I}^2 = \widehat{I}\widehat{\times}\widehat{I} = (I * I) * \widehat{I} = I * \widehat{I} = \widehat{I}$.
2. $\widehat{I}^n = \overbrace{\widehat{I}\widehat{\times} \ldots \widehat{\times}\widehat{I}}^{\text{n times}} = \overbrace{(I * \ldots * I)}^{\text{n times}} * \widehat{I} = I * \widehat{I} = \widehat{I}$.
3. $\widehat{I}\widehat{\times}\widehat{I}^{-I} = \widehat{I}$, since $\widehat{I}\widehat{\times}\widehat{I} = \widehat{I}$.

In addition, the isotopic lifting we are carrying out does not imply a change in the numbers used in a determinate theory, since if we multiply an isonumber $\widehat{n}$ by an amount Q, we have that $\widehat{n}\widehat{\times}Q = (n * \widehat{I}) * T * Q = n * Q$ (see [175]). On the other hand, all the usual operations dependent on multiplication over K are generalized to $\widehat{K}$ in a unique way through the corresponding lifting (see [145]). Thus, we have the following:

Definition 3.5.3 *With the usual notation, for all* $\widehat{a}, \widehat{b} \in \widehat{K}$, *the following* isooperations *are defined:*

1. Isoquotient: $\widehat{a}\widehat{/}\widehat{b} = \widehat{a}\widehat{\times}\widehat{b}^{-\widehat{I}} = (a * b^{-I}) * \widehat{I} \in \widehat{K}$.
2. Isosquare: $\widehat{a}^{\widehat{2}} = \widehat{a}\widehat{\times}\widehat{a} = (a * a) * \widehat{I} \in \widehat{K}$.
3. Isosquareroot: $\widehat{a}^{\widehat{\frac{1}{2}}} = \widehat{b} \in \widehat{K} \Leftrightarrow \widehat{b}^{\widehat{2}} = \widehat{a}$.
4. Isonorm: $\widehat{|\widehat{a}|} = (\widehat{\overline{a} * a})^{\widehat{\frac{1}{2}}} \in \widehat{K}$, *where* $\overline{a}$ *represents the conjugate of* $a \in K$ *with respect to* $*$.

However, in practice it is normal that $* \equiv \times$. In these cases, the lifting of the previous operations looks even clearer in the following:

Definition 3.5.4 *They are in practice defined as:*

1. Isoquotient: $\widehat{a}\widehat{/}\widehat{b} = (a * b^{-I}) * \widehat{I} = (a \times b^{-I}) * \widehat{I} = (a/b) * \widehat{I} = \widehat{a/b} \in \widehat{K}$.
2. Isosquare: $\widehat{a}^{\widehat{2}} = (a * a) * \widehat{I} = (a \times a) * \widehat{I} = a^2 * \widehat{I} = \widehat{a^2} \in \widehat{K}$.
3. Isosquareroot: $\widehat{a}^{\widehat{\frac{1}{2}}} = a^{\frac{1}{2}} * \widehat{I} = \widehat{a^{\frac{1}{2}}} \in \widehat{K}$, *since* $\widehat{a^{\frac{1}{2}}}^{\widehat{2}} = (a^{\frac{1}{2}})^2 * \widehat{I} = a * \widehat{I} = \widehat{a}$.
4. Isonorm: $\widehat{|\widehat{a}|} = (\widehat{\overline{a} * a})^{\widehat{\frac{1}{2}}} = (\overline{a} \times a)^{\frac{1}{2}} * \widehat{I} = |a| * \widehat{I} = \widehat{|a|} \in \widehat{K}$.

Note that it is clear, in addition, that all defined isooperations are closed in $\widehat{K}$. Finally we will define the concept of isocharacteristic of an isofield:

Definition 3.5.5 *An isofield* $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$ *is said to be of* isocharacteristic *p if a minimal positive number p exists such that* $\overbrace{\widehat{a}\widehat{\times} \ldots \widehat{\times}\widehat{a}}^{p \text{ times}} = \widehat{I}$ *(*$\widehat{I}$ *being the isounit of the isofield in question with respect to the operation* $\widehat{\times}$*). Otherwise it is said to be of* zero isocharacteristic.

Here below are some examples of isofields:

Example 3.5.6 *In an analogous way to Example 3.4.3 we prove, by applying Proposition 3.5.2, that if we consider the field* $(\mathbf{R}, +, \times)$ *of the real numbers, with the usual sum and product, and take as principle elements of the isotopy* $\widehat{I} = 1$ *and* $* \equiv \times$ *(with unit element* $I = 1 = \widehat{I} \in \mathbf{R}$*) and as secondary elements* $\widehat{S} = 0 = 0 * 1$ *and* $\star \equiv +$ *(with the unit element* $S = 0 = \widehat{S} \in \mathbf{R}$*), then the isotopic lifting of* $(\mathbf{R}, +, \times)$*, corresponding to the isotopy of the aforementioned elements, is equivalent to the identity, leaving invariant both the operations and the elements of the starting field. So,* $(\widehat{\mathbf{R}}, \widehat{+}, \widehat{\times}) = (\mathbf{R}, +, \times)$. $\triangleleft$

Note that this example confirms again that if in an isotopic lifting of a structure neither the starting operations nor its corresponding units varies, the resulting isostructure coincides with the original structure.

Example 3.5.7 *Let us consider the field* $(\mathbf{R}, +, \times)$ *again. Let us now take the principal isotopic elements* $\widehat{I} = i$ *and* $* \equiv \bullet$ *(the product of complex numbers) and the secondary elements* $\widehat{S} = 0$ *and* $\star \equiv +$ *(the sum of complex*

numbers). Then we have that $(\mathbf{R}, \star, *) = (\mathbf{R}, +, \bullet) = (\mathbf{R}, +, \times)$ *is a field with respective unit elements* $S = 0$ *and* $I = 1$*, both in* $\mathbf{R}$*. As, in addition,* $\widehat{S}^{-S} = 0^{-0} = 0 = 0 * 1$ *and* $a * 0 = 0 = 0 * a$*,* $\forall a \in \mathbf{R}$*, Proposition 3.5.2 then assures us that the isotopic lifting corresponding to the isotopy of the aforementioned elements is an isofield with respect to multiplication. The resulting isotopic set would be* $\widehat{\mathbf{R}}_i = \{\widehat{a} = a * i = a \bullet i \mid a \in \mathbf{R}\} = \mathrm{Im}(\mathbf{C})$*. On the other hand, the isoproduct* $\widehat{\times}$ *would be defined according to:* $\widehat{a}\widehat{\times}\widehat{b} = (a * b) * i = (a \bullet b) \bullet i = \widehat{a \bullet b} = \widehat{a \times b}$*, for all* $\widehat{a}, \widehat{b} \in \widehat{\mathbf{R}}_i$*. Therefore,* $(\mathrm{Im}(\mathbf{C}), +, \widehat{\times})$ *would be the indicated isofield.* $\triangleleft$

It is important to note that using this isotopy we have given $\mathrm{Im}(\mathbf{C})$ a field structure, a structure that does not have the complex product $\bullet$, since, given $z = z_0 \bullet i$, $\omega = \omega_0 \bullet i \in \mathrm{Im}(\mathbf{C})$, we have that $z \bullet \omega = z_0 \bullet i \bullet \omega_0 \bullet i = z_0 \bullet \omega_0 \bullet (-1) = -(z_0 \bullet \omega_0) \notin \mathrm{Im}(\mathbf{C})$. However, the new isoproduct does give a field structure to $\mathrm{Im}(\mathbf{C})$, since $\widehat{\times}$ is an internal operation due to, with the previous notation, our having that $z\widehat{\times}\omega = (z_0 \bullet i)\widehat{\times}(\omega_0 \bullet i) = (z_0 \bullet \omega_0) \bullet i \in \mathrm{Im}(\mathbf{C})$.

We end this section with a remark not yet considered. It is that given a structure and an associated isostructure, if we consider that we are working at the abstract level of the axioms (i.e., where we only take into account the axioms which satisfy a particular mathematical object), the former structure and the isostructure can be considered equivalent, being associated with the same axioms (see [175]). This will serve for any structure in general and can therefore can be considered equivalent in the abstract level of the axioms, groups and isogroups, rings and isorings, ideals and isoideals, etc. Also, in the case of the fields and isofields, we can consider them equivalent in that abstract axiomatic level.

The latter suggests that whole field can be obtained starting from from one given, by means of a determinate isotopy or, at least, through a series of successive isotopies. In fact, Santilli showed in 1991 that the whole field of zero characteristic is the isotopic lifting of the field of real numbers corresponding to a particular isotopy (see [121]). What happens, however, is that this isotopic lifting need not follow the model

analyzed so far. Let us see, to finish this section, an example of what was mentioned above for the particular case of the field $\mathbf{C}$ of the complex numbers:

Example 3.5.8 *Let us consider* $(\mathbf{R}, +, \times)$ *and the isofields* $(\widehat{\mathbf{R}}_1, +, \times)$ *of Example 3.5.6 and* $(\widehat{\mathbf{R}}_i, +, \widehat{\times})$ *of Example 3.5.7. It would suffice to reach an isotopy of* $\mathbf{R}$ *that would give as a result* $\mathbf{C} = \widehat{\mathbf{R}} = \widehat{\mathbf{R}}_1 \oplus \widehat{\mathbf{R}}_i$ *(where* $\oplus$ *denotes the usual direct sum), establishing as an isoproduct the usual product associated with the direct sum. That is, taking into account that the generic element of* $\widehat{\mathbf{R}}$ *would be of the form* $z = \alpha \oplus \beta$*, with* $\alpha \in \widehat{\mathbf{R}}_1$ *and* $\beta \in \widehat{\mathbf{R}}_i$ *(i.e.,* $\alpha \in \mathbf{R}$ *and* $\beta \in \mathrm{Im}(\mathbf{C})$*), then, given* $z = \alpha \oplus \beta$*,* $z' = \alpha' \oplus \beta' \in \widehat{\mathbf{R}}$*, we would consider the isoproduct* $\bullet : \widehat{\mathbf{R}} \times \widehat{\mathbf{R}} \to \widehat{\mathbf{R}}$*, such that* $(z, z') \to z \bullet z' = (\alpha \oplus \beta) \bullet (\alpha' \oplus \beta') = (\alpha \times \alpha') \oplus (\beta \widehat{\times} \beta') \in \widehat{\mathbf{R}}_1 \times \widehat{\mathbf{R}}_i$ *(and therefore* $\bullet$ *is well-defined). In this way, the isounit associated with* $\widehat{\mathbf{R}}$ *with respect to* $\bullet$ *would be* $\widehat{I} = 1 \oplus i$*, since if* $z = \alpha \oplus \beta$*, then* $z \bullet \widehat{I} = (\alpha \oplus \beta) \bullet (1 \oplus i) = (\alpha \times 1) \oplus (\beta \widehat{\times} i) = \alpha \oplus \beta = z = \widehat{I} \bullet z$*.*

On the other hand, such as lifting the operator $+$*, we would give the usual associated sum for the direct sum. So, for the previous elements* z *and* z'*, it would be* $\circ : \widehat{\mathbf{R}} \times \widehat{\mathbf{R}} \to \widehat{\mathbf{R}}$*, such that* $(z, z') \to z \circ z' = (\alpha \oplus \beta) \circ (\alpha' \oplus \beta') = (\alpha + \alpha') \oplus (\beta + \beta') \in \widehat{\mathbf{R}}_1 \times \widehat{\mathbf{R}}_i = \widehat{\mathbf{R}}$ *(and therefore* $\circ$ *is well-defined). The isounit of* $\widehat{\mathbf{R}}$ *with respect to* $\circ$ *would then be* $\widehat{S} = 0 \oplus 0$*, since then, if* $z = \alpha \oplus \beta$*, we would have* $z \circ \widehat{S} = (\alpha \oplus \beta) \circ (0 \oplus 0) = (\alpha + 0) \oplus (\beta + 0) = \alpha \oplus \beta = z = \widehat{S} \circ z$*.*

Therefore, we arrive at being able to consider the field of complex numbers as an isotopy of $(\mathbf{R}, +, \times)$*, taking* $\mathbf{C} = \widehat{\mathbf{R}}_{1 \oplus i}$ *and the operations* $\circ$ *and* $\bullet$ *defined before.* $\triangleleft$

Chapter 4
LIE-SANTILLI ISOTHEORY: ISOTOPIC STRUCTURES (II)

Chapter 4 continues the study of the Lie-Santilli isotheory, performing the isotopic lifting of more complex algebraic structures than those seen before. Thus, *vector isospaces* and *metric vector isospaces* are studied, followed by *isomodules*. In addition, considering it of great interest, by the important consequences that are derived from them, we also felt it appropriate to include a section dedicated to the study of *isotransformations*.

4.1 Vector isospaces

In the first subsection of this section, we look at the vector isospaces and isosubspaces (see [77]). In the second section, we deal with the *metric vector isospaces* (see [121] and [111]). In both sections, we follow the same procedure as for the isostructures we already saw.

4.1.1 Vector isospaces and isosubspaces

Definition 4.1.1 *Let $(U, \circ, \bullet)$ be a vector space defined over a field $K = K(a, +, \times)$. Let $\widehat{K} = \widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$ be an isofield associated with K. $\widehat{U}$ is called a* vector isospace *on $\widehat{K}$ if, being an isotopy of U equipped with two new operations $\widehat{\circ}$ and $\widehat{\bullet}$, $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$ has a vector space structure over $\widehat{K}$, i.e., if $\forall \widehat{a}, \widehat{b} \in \widehat{K}$ and $\forall \widehat{X}, \widehat{Y} \in \widehat{U}$, we verify that*

1. *$(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$ is closed, $(\widehat{U}, \widehat{\circ})$ being an isosubgroup;*
2. *the 4 axioms of the external operation:*

   a. $\widehat{a}\widehat{\bullet}(\widehat{b}\widehat{\bullet}\widehat{X}) = (\widehat{a}\widehat{\times}\widehat{b})\widehat{\bullet}\widehat{X}$.
   b. $\widehat{a}\widehat{\bullet}(\widehat{X}\widehat{\circ}\widehat{Y}) = (\widehat{a}\widehat{\bullet}\widehat{X})\widehat{\circ}(\widehat{a}\widehat{\bullet}\widehat{Y})$.
   c. $(\widehat{a}\widehat{+}\widehat{b})\widehat{\bullet}\widehat{X} = (\widehat{a}\widehat{\bullet}\widehat{X})\widehat{\circ}(\widehat{b}\widehat{\bullet}\widehat{X})$.
   d. $\widehat{I}\widehat{\bullet}\widehat{X} = \widehat{X}$,

   $\widehat{I}$ being the isounit associated with $\widehat{K}$ with respect to the operation $\widehat{\times}$.

The elements of the isospace $\widehat{U}$ are usually called isovectors.

Note that by the last axiom of the external operation, the element $\widehat{I}$, which is the isounit associated with the isofield $\widehat{K}$ with respect to $\widehat{\times}$, also becomes the isounit $\widehat{U}$ with respect to $\widehat{\bullet}$. In addition, it is important to note the presence of two distinct isotopies in the isotopic lifting of a vector space. On the one hand, we would have the isotopy for obtaining the isofield $\widehat{K}$, while on the other hand it would be the isotopy corresponding to the vector isospace $\widehat{U}$, properly said.

On the other hand, if we now turn to study the model of constructing isotopies that we are carrying out from an isounit and from the isoproduct, we will notice a number of differences that appear with respect to the cases already studied. To start, we cannot make a distinction between vector isospaces with respect to the sum or multiplication, since we have already imposed one of the isounits that we must use. Let us look at this more closely.

Suppose we have a vector space $(U, \circ, \bullet)$ defined over a field $K(a, +, \times)$ and an isofield with respect to multiplication (with regard to the sum, it would be analogous) associated with K, corresponding to the isotopy of principal elements $\widehat{I}$ and $*$ (of unit element I) and secondary elements $\widehat{S}$ and $\star$. We seek to construct a structure $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$

satisfying the conditions of the previous definition. To do this we must impose that the principal isounit be $\widehat{I}$ and that it must act in the lifting of the operation $\bullet$ to obtain $\widehat{\bullet}$. Moreover, given that we want to get two new operations $\widehat{\circ}$ and $\widehat{\bullet}$, we will complement the isounit $\widehat{I}$ with the elements of primary and secondary isotopies, which are still common in this type of lifting. Let us assume therefore that we have principal isotopic elements $\widehat{I}$ and $\Box$, and secondary ones $\widehat{S'}$ and $\diamond$, both pairs of the compatible elements for the construction being of the same isotopic set $\widehat{U}_{\widehat{I}} = \{\widehat{X} = X\Box\widehat{I} \mid X \in U\}$, in the sense already seen for isorings.

However, it should be noted that in a vector space it makes no sense to talk about the inverse element with respect to the second operation, the latter being external. For this reason, if in a way analogous to the cases already studied, it interested us impose that $(U, \diamond, \Box)$ have vector space structure on $K(a, \star, *)$ (which has a field structure according to the condition imposed on Proposition 3.5.2 to obtain an isofield), it would not make sense to talk about the inverse element of $\Box$, nor for both isotopic elements, in the sense of being the inverse of $\widehat{I}$. What it would indicate is that the general set V' associated with the isotopy we are building was such that $K \cup U \subseteq V'$, as it would be necessary to define $\Box$ as an external operation to obtain that $(U, \diamond, \Box)$ be a vector space over $K(a, \star, *)$. Thus, the model of construction of the isoproduct that has been done so far will not be valid to obtain the operation $\widehat{\bullet}$.

To construct this last operation, we would impose that $(U, \circ)$ were a group with the same unit element as that of $(V, \circ)$, V being the general set associated with the isotopy in question. We denote the unit element by S'. Thus, if if $\widehat{S'}^{-S'} = \widehat{R'} = R'\Box\widehat{I}$, we would finally have that $\widehat{\circ}$ would be defined as $\widehat{X}\widehat{\circ}\widehat{Y}=\widehat{X} \diamond \widehat{R'} \diamond \widehat{Y}$, for all $\widehat{X}, \widehat{Y} \in \widehat{U}$. In this way we would also have, by Proposition 3.3.2, that $(\widehat{U}, \widehat{\circ})$ would be an isogroup.

In terms of the operation $\widehat{\bullet}$, despite not being able to define it as we have been doing until now, we can, by construction, make it directly, saying that $\widehat{a}\widehat{\bullet}\widehat{X}=(a\Box X)\Box\widehat{I} = \widehat{a\Box X}$, for all $\widehat{a} \in \widehat{K}, \forall \widehat{X} \in \widehat{U}$.

We should then impose that $\square$ have as unit element for I (unit element of $*$), because then we would have $\widehat{I}\widehat{\bullet}\widehat{X} = (I\square X)\square\widehat{I} = X\square\widehat{I} = \widehat{X}$, for all $\widehat{X} \in \widehat{U}$, and it would already satisfy the last axiom (2.d) of the external operation of the definition.

In addition, imposing, as already noted earlier, that $(U, \diamond, \square)$ be a vector space over $K(a, \star, *)$, we will have that $\widehat{a}\widehat{\bullet}\widehat{X} = (a\square X)\square\widehat{I} \in \widehat{U}$, for all $\widehat{a} \in \widehat{K}$ and $\widehat{X} \in \widehat{V}$, as then $a\square X \in U$. In this way, we will have proved that $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$ is closed, and, as we have already seen, that $(\widehat{U}, \widehat{\circ})$ is an isogroup, we have also proved condition (1) of the definition.

In addition, the fact that $(U, \diamond, \square)$ is a vector space over $K(a, \star, *)$ implies that $\square$ has I as its unit element, whereby we would have already stated it before for obtaining $\widehat{I}$ as the isounit of $\widehat{\bullet}$.

Condition (2.b) already complies with all the above, satisfying that $\widehat{a}\widehat{\bullet}(\widehat{b}\widehat{\bullet}\widehat{X}) = \widehat{a}\widehat{\bullet}((b\square X)\square\widehat{I}) = (a\square(b\square X))\square\widehat{I} = ((a * b)\square X)\square\widehat{I} = \widehat{a * b}\widehat{\bullet}\widehat{X} = (\widehat{a}\widehat{\times}\widehat{b})\widehat{\bullet}\widehat{X}$, for all $\widehat{a}, \widehat{b} \in \widehat{K}$ and $\widehat{X} \in \widehat{U}$.

To prove condition (2.c), it suffices to impose that $\widehat{a}\widehat{\bullet}\widehat{R'} = \widehat{R'}$, for all $\widehat{a} \in \widehat{K}$, since then, given $\widehat{a} \in \widehat{K}$ and $\widehat{X}, \widehat{Y} \in \widehat{U}$, we would have $\widehat{a}\widehat{\bullet}(\widehat{X}\widehat{\circ}\widehat{Y}) = \widehat{a}\widehat{\bullet}(\widehat{X} \diamond \widehat{R'} \diamond \widehat{Y}) = \widehat{a}\widehat{\bullet}((X\square\widehat{I}) \diamond (R'\square\widehat{I}) \diamond (Y\square\widehat{I})) = \widehat{a}\widehat{\bullet}((X \diamond R' \diamond Y))\square\widehat{I}) = (a\square(X \diamond R' \diamond Y))\square\widehat{I} = ((a\square X) \diamond (a\square R') \diamond (a\square Y))\square\widehat{I} = (a\square X\square\widehat{I}) \diamond (a\square R'\square\widehat{I}) \diamond (a\square Y\square\widehat{I}) = (\widehat{a}\widehat{\bullet}\widehat{X}) \diamond (\widehat{a}\widehat{\bullet}\widehat{R'}) \diamond (\widehat{a}\widehat{\bullet}\widehat{Y}) = (\widehat{a}\widehat{\bullet}\widehat{X}) \diamond \widehat{R'} \diamond (\widehat{a}\widehat{\bullet}\widehat{Y}) = (\widehat{a}\widehat{\bullet}\widehat{X}) \diamond (\widehat{a}\widehat{\bullet}\widehat{Y})$.

Finally, so that the condition (2.d) is satisfied, we should impose that if $\widehat{S}^{-S} = \widehat{R} = R * \widehat{I}$, then $\widehat{R}\widehat{\bullet}\widehat{X} = \widehat{R'}, \forall\widehat{X} \in \widehat{U}$, since then, given $\widehat{a}, \widehat{b} \in \widehat{K}$ and $\widehat{X} \in \widehat{U}$, we would have that $(\widehat{a}\widehat{+}\widehat{b})\widehat{\bullet}\widehat{X} = (\widehat{a} \star \widehat{R} \star \widehat{b})\widehat{\bullet}\widehat{X} = (a \star \widehat{R \star} b)\widehat{\bullet}\widehat{X} = ((a \star R \star b)\square X)\square\widehat{I} = (a \star R \star b)\square(X\square\widehat{I}) = (a\square X\square\widehat{I}) \diamond (R\square X\square\widehat{I}) \diamond (b\square X\square\widehat{I}) = (\widehat{a}\widehat{\bullet}\widehat{X}) \diamond (\widehat{R}\widehat{\bullet}\widehat{X}) \diamond (\widehat{b}\widehat{\bullet}\widehat{X}) = (\widehat{a}\widehat{\bullet}\widehat{X}) \diamond \widehat{R'} \diamond (\widehat{b}\widehat{\bullet}\widehat{X}) = (\widehat{a}\widehat{\bullet}\widehat{X})\widehat{\circ}(\widehat{b}\widehat{\bullet}\widehat{X})$.

Therefore, as a result of the foregoing, the following is now proved:

Proposition 4.1.2 *Let $(U, \circ, \bullet)$ be a vector space defined over the field $K(a, +, \times)$. Let $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$ be the isofield with respect to multiplication associated with K, corresponding to the isotopy of principal elements $\widehat{I}$ and $*$ (of unit element I) and secondary elements $\widehat{S}$, and $\star$ (of unit element S, with $\widehat{S}^{-S} = \widehat{R} = R * \widehat{I}$), in the conditions of Proposition 3.5.2. Let $\square$ (of unit*

element I), $\widehat{S}'$, and $\diamond$ (of unit element S', with $\widehat{S}'^{-S'} = \widehat{R'} = R'\Box\widehat{I}$), be elements of the isotopy that, together with $\widehat{I}$, are in the conditions of Definition 3.1.3, the associated general set V' being such that $K \cup U \subseteq V'$. In these conditions, if $(U, \diamond, \Box)$ has a vector space structure over the field $K(a, \star, *)$, $(U, \diamond)$ being a group with unit element $S' \in U$, $\widehat{a}\widehat{\bullet}\widehat{R'} = \widehat{R'}$, for all $\widehat{a} \in \widehat{K}$ and $\widehat{R}\widehat{\bullet}\widehat{X} = \widehat{R'}$, for all $\widehat{X} \in \widehat{U}$, then the isotopic lifting $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$, corresponding to the isotopy of principal elements $\widehat{I}$ and $\Box$ and secondary elements $\widehat{S}$ and $\diamond$, by means of the isoproduct procedure, has an isovector space structure over $\widehat{K}$. $\Box$

Let us see below some examples of vector isospaces:

Example 4.1.3 *We will give an example of a vector isospace of the general type, which is widely used as a model in practice.*

Let $(U, \circ, \bullet)$ be a vector space over the field $K(a, +, \times)$ (of respective unit elements $0, I \in K$), of respective unit elements $\overrightarrow{0}, I \in U$, satisfying the usual properties (as $0 \bullet X = \overrightarrow{0}$, $\forall X \in U$, $a \bullet \overrightarrow{0} = \overrightarrow{0}$, $\forall a \in K$, 0^{-I} does not exist, etc). Let $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times}) = \widehat{K}(\widehat{a}, +, \widehat{\times})$ be the isofield with respect to multiplication associated with K, corresponding to the isotopy of principal elements $\widehat{I} \in K$ and $* \equiv \times$ (of unit element I) and secondary elements $\widehat{S} = 0$ and $\star \equiv +$ (of unit element $S = 0$). We are going to isotopically lift the vector space U utilizing as principal isotopic elements $\widehat{I}$ and $\Box \equiv \bullet$ (of unit element I) and as secondary isotopic elements $\diamond \equiv \circ$ and $\widehat{S}' = \overrightarrow{0} = \overrightarrow{0} \bullet \widehat{I}$.

As $\widehat{I} \in K$, we have that the associated isotopic set is U itself, i.e., $\widehat{U}_{\widehat{I}} = \{\widehat{X} = X\Box\widehat{I} = X \bullet \widehat{I} : X \in U\} = U$; then if there is an element $X \in U$ such that $X \notin \widehat{U}$, taking $X \bullet T \in U$, $T = \widehat{I}^{-I} \in K$ (since $\widehat{I} \in K$, $K(a, \star, *) = K(a, +, \times)$ being a field, as we saw in the construction made in Proposition 3.5.2), we would arrive at $(X \bullet T) \bullet \widehat{I} = X \in \widehat{U}$, which would be a contradiction. In this way, the secondary elements $\diamond$ and $\widehat{S}'$ are well-selected, that they also remain for U an associated isotopic set.

The operations $\widehat{\circ}$ and $\widehat{\bullet}$ would be defined as follows:

1. $\widehat{X}\widehat{\circ}\widehat{Y} = (X\Box\widehat{I}) \circ 0 \circ (Y\Box\widehat{I}) = (X\Box\widehat{I}) \circ (Y\Box\widehat{I}) = (X \bullet \widehat{I}) \circ (Y \bullet \widehat{I}) = (X \circ Y) \bullet \widehat{I} = (X \circ Y)\Box\widehat{I} = \widehat{X} \circ \widehat{Y} \Rightarrow \widehat{\circ} \equiv \circ$ *for all* $\widehat{X}, \widehat{Y} \in \widehat{U}$.
2. $\widehat{a}\widehat{\bullet}\widehat{X} = (a * \widehat{I})\widehat{\bullet}(X\Box\widehat{I}) = (a \bullet X)\Box\widehat{I}$, *for all* $\widehat{a} \in \widehat{K}$ *and for all* $\widehat{X} \in \widehat{U}$.

In addition we have that

3. $(U, \diamond, \square) = (U, \circ, \bullet)$ *has a vector space structure defined over the field* $K(a, \star, *) = K(a, +, \times)$, $(U, \diamond) = (U, \circ)$ *being a group with unit element* $S' = \overrightarrow{0} = \widehat{S'} \in U$.
4. *As* $\widehat{R'} = \widehat{S'}^{-S'} = \overrightarrow{0}^{-\overrightarrow{0}} = \overrightarrow{0} = \overrightarrow{0} \square \widehat{I}$, *for* $\widehat{a} \in \widehat{K}$, *we have that* $\widehat{a} \widehat{\bullet} \widehat{R'} = (a \square \overrightarrow{0}) \square \widehat{I} = (a \bullet \overrightarrow{0}) \square \widehat{I} = \overrightarrow{0} \square \widehat{I} = \overrightarrow{0} = \widehat{R'}$.
5. *Also, as* $\widehat{R} = \widehat{S}^{-S} = 0^{-0} = 0 = 0 \times \widehat{I} = \widehat{0}$, *too, we have, for all* $\widehat{X} \in \widehat{U}$, *that* $\widehat{0} \widehat{\bullet} \widehat{X} = (0 \square X) \square \widehat{I} = (0 \bullet X) \square \widehat{I} = \overrightarrow{0} \square \widehat{I} = \overrightarrow{0} = \widehat{R'}$.

That is why we are able to apply Proposition 4.1.2, thus resulting in $(\widehat{U}_{\widehat{I}}, \widehat{\circ}, \widehat{\bullet}) = (U, \circ, \widehat{\bullet})$ *being an isovector space over the field* $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times}) = \widehat{K}(\widehat{a}, +, \widehat{\times})$. $\triangleleft$

We will now see a concrete example of the previous model.

Example 4.1.4 *Let* $(M_{m \times n}(\mathbf{R}), +, \bullet)$ *be a vector space of real matrices of dimension* $m \times n$, *with the usual sum and product of matrices over the field* $(\mathbf{R}, +, \times)$ *of real numbers with the usual sum and product. Let us consider the isofield with respect to multiplication* $(\widehat{\mathbf{R}}_2, +, \widehat{\times})$ *associated with* $(\mathbf{R}, +, \times)$, *corresponding to the isotopy of principal elements* $\widehat{I} = 2$ *and* $* \equiv \times$ *and secondary elements* $\widehat{S} = 0$ *and* $\star \equiv +$. *Then,* $\widehat{\mathbf{R}}_2 = \{\widehat{a} = a \times 2 \mid a \in \mathbf{R}\} = \mathbf{R}$, *the isoproduct* $\widehat{\times}$ *remaining defined according to* $\widehat{a} \widehat{\times} \widehat{b} = (a \times b) \times 2 = \widehat{a \times b}$, *for all* $\widehat{a}, \widehat{b} \in \widehat{K}$. *It can be observed, analogously to the previous examples, that we can apply Proposition 3.5.2, then arriving at* $(\widehat{\mathbf{R}}_2, +, \widehat{\times}) = (\mathbf{R}, +, \widehat{\times})$ *effectively being an isofield.*

We now consider the principal isotopic elements $\widehat{I} = 2$ *and* $\square \equiv \bullet$ *and the secondary ones* $\widehat{S} = \mathbf{0}$ *(the null matrix) and* $\diamond \equiv +$. *Example 4.1.3 then assures that the isotopic lifting* $(\widehat{M_{m \times n}(\mathbf{R})}, \widehat{+}, \widehat{\bullet})$ *associated with* $(M_{m \times n}(\mathbf{R}), +, \bullet)$ *is a vector isospace over the field* $(\mathbf{R}, +, \widehat{\times})$. *Also, the following are satisfied:*

1. $\widehat{M_{m \times n}(\mathbf{R})}_2 = \{\widehat{A} = A \bullet 2 \mid A \in M_{m \times n}(\mathbf{R})\} = M_{m \times n}(\mathbf{R})$.
2. $\widehat{+} \equiv +$, *as we already saw in Example 4.1.3.*

Finally, the isoproduct $\widehat{\bullet}$ *would be defined as* $\widehat{a} \widehat{\bullet} \widehat{A} = (a \bullet A) \bullet 2$, *for all* $\widehat{a} \in \widehat{\mathbf{R}}_2$ *and for all* $\widehat{A} \in \widehat{M_{m \times n}(\mathbf{R})}_2$.

Therefore, $(M_{m\times n}(\mathbf{R}), +, \widehat{\bullet})$ *is the sought isovector space.* $\triangleleft$

We will then study a some fundamental objects of all vector spaces: their bases. All vector isospaces, to have a vector space structure, must at least be associated with a basis. One might then ask about the concept of an isobasis; i.e., we should ask, if given a basis of a vector space, whether the isotopic lifting of the basis would be a basis of the corresponding vector isospace. If so, the isotopic lifting of the basis is called an *isobasis*. On the other hand, we can also ask ourselves if any basis of a vector isospace has an isobasis structure.

To answer both questions, let us assume that we are in the conditions of Proposition 4.1.2, while retaining all the notations used there. Suppose also that we have the sets $\beta = \{e_1, e_2, \ldots, e_n\}$ and $\widehat{\beta} = \{\widehat{e_1} = e_1 \Box \widehat{I}, \widehat{e_2} = e_2 \Box \widehat{I}, \ldots, \widehat{e_n} = e_n \Box \widehat{I}\}$. Then it interests us to study under what conditions we have that if β is a basis of U, then $\widehat{\beta}$ is a basis of $\widehat{U}$, and *vice versa*. We can do this as we have performed similar studies, i.e., imposing conditions for the isotopic elements involved in the lifting in question. Thus, what happens is that some possible cases do not appear. For example, if we had that β is a basis of U, $\widehat{\beta}$ is a basis of $\widehat{U}$, and $X = e_1 + e_2$, with $\widehat{X} = \widehat{e_3}$, in principle we could not establish any relationship between the isotopic elements involved to obtain other determined elements e_1, e_2, and $\widehat{e_3}$, then most likely we would restrict the possibility of being able to perform certain isotopic liftings of elements of U that otherwise could be done. If, however, we wanted to propose a method of lifting under which these two questions could be answered, the way of achieving it would not be unique, precisely because a generic lifting need not keep any relationship between the elements of the basis. In fact, this makes that, despite being able to find isotopic models that satisfy the conservation of bases, in general the answer to the two questions set forth before is negative.

In particular, one such model would be what has been given in Example 4.1.3, which is widely used in practice because of its property of

maintaining bases. To see it, we are going to start reducing the question of conservation of bases to generator systems and linearly independent systems. So we assume, first of all, that β is a generator of U and we want to see if $\widehat{\beta}$ is a generator system $\widehat{U}$, under the conditions of Example 4.1.3. For this purpose we take $\widehat{X} \in \widehat{U}$; then $X \in U$. However, as β is a generator system of U, we can write $X = (\lambda_1 \bullet e_1) \circ \ldots \circ (\lambda_n \bullet e_n)$, with $\lambda_1, \ldots, \lambda_n \in K$. Then $\widehat{X} = X \square \widehat{I} = ((\lambda_1 \bullet e_1) \circ \ldots \circ (\lambda_n \bullet e_n)) \square \widehat{I}$. Bearing in mind, then, that $\square \equiv \bullet$ and $\widehat{\circ} \equiv \circ, \widehat{I} \in K$ and the definition of the isoproduct $\widehat{\bullet}$, given in the construction of the isovector space, we will finally have that $\widehat{X} = ((\lambda_1 \bullet e_1) \circ \ldots \circ (\lambda_n \bullet e_n)) \bullet \widehat{I} = ((\lambda_1 \bullet e_1) \bullet \widehat{I}) \circ \ldots \circ ((\lambda_n \bullet e_n) \bullet \widehat{I}) = ((\lambda_1 \square e_1) \square \widehat{I}) \widehat{\circ} \ldots \widehat{\circ} ((\lambda_n \square e_n) \square \widehat{I}) = (\widehat{\lambda_1} \widehat{\bullet} \widehat{e_1}) \widehat{\circ} \ldots \widehat{\circ} (\widehat{\lambda_n} \widehat{\bullet} \widehat{e_n})$, with $\widehat{\lambda_1}, \ldots, \widehat{\lambda_n} \in \widehat{K}$. As a consequence, $\widehat{\beta}$ is a generator system of $\widehat{U}$.

We now suppose that β is a linearly independent system in U, and we are going to see if $\widehat{\beta}$ is one in $\widehat{U}$, always under the conditions of Example 4.1.3. We tak $\widehat{\lambda_1}, \ldots, \widehat{\lambda_n} \in \widehat{K}$, such that $(\widehat{\lambda_1} \widehat{\bullet} \widehat{e_1}) \widehat{\circ} \ldots \widehat{\circ} (\widehat{\lambda_n} \widehat{\bullet} \widehat{e_n}) = \overrightarrow{0} = \overrightarrow{0} \bullet \widehat{I}$ (which is the unit element of $(\widehat{U}, \widehat{\circ}) = (U, \circ)$). We would then have that: $(\widehat{\lambda_1} \widehat{\bullet} \widehat{e_1}) \widehat{\circ} \ldots \widehat{\circ} (\widehat{\lambda_n} \widehat{\bullet} \widehat{e_n}) = ((\lambda_1 \bullet e_1) \circ \ldots \circ (\lambda_n \bullet e_n)) \square \widehat{I} = \overrightarrow{0} = \overrightarrow{0} \square \widehat{I}$. However, with the conditions imposed in Example 4.1.3, the only element of U that can be lifted to $\overrightarrow{0}$ is $\overrightarrow{0}$, since if there exists another distinct element $X \in U$, such that $X \square \widehat{I} = X \bullet \widehat{I} = \overrightarrow{0}$, it should be $\widehat{I} = 0$, which is not possible, since in lifting the field K we took $* \equiv \times$, zero not being invertible (a necessary condition so that the isotopic element $T = \widehat{I}^{-I}$ exists and to construct the isofield). Therefore, we should maintain that $(\lambda_1 \bullet e_1) \circ \ldots \circ (\lambda_n \bullet e_n) = \overrightarrow{0}$. Then, applying that β is a linearly independent system, we would have that $\lambda_i = 0, \forall i = 1, \ldots, n$. So, $\widehat{\lambda_i} = \lambda_i * \widehat{I} = \lambda_i \times \widehat{I} = 0 \times \widehat{I} = 0$ (which is the unit element of $\widehat{K}(\widehat{a}, \widehat{+}) = K(a, +)$), $\forall i \in \{1, \ldots, n\}$, thus arriving at $\widehat{\beta}$ being a linearly independent system in $\widehat{U}$.

The consequence of all the above is as follows:

Proposition 4.1.5 *Let us consider the vector and isovector systems* $\beta = \{e_1, \ldots, e_n\}$ *and* $\widehat{\beta} = \{\widehat{e_1} = e_1 \square \widehat{I}, \ldots, \widehat{e_n} = e_n \square \widehat{I}\}$. *In the conditions of*

Proposition 4.1.2 and following the isotopic model utilized in Example 4.1.3, if β is a basis of U, then $\widehat{\beta}$ is an isobasis of $\widehat{U}$. □

For seeing if any basis of an isovector space can have an isobasis structure, we return to point out that this will depend on, at all times, the isotopic model that we are using. In particular, we will see again that the model of Example 4.1.3 gives an affirmative answer to this question, even though the same answer, in general, may have a negative response, depending on the lifting used.

Let us then consider the conditions of Proposition 4.1.2 and the isotopic model of Example 4.1.3. Let us suppose we have the sets β and $\widehat{\beta}$ indicated above. We are going to prove, in the first place, that if $\widehat{\beta}$ is a generator system of $\widehat{U}$, then β is a generator system of U. For this purpose, let us take $X \in U$; then $\widehat{X} = X \square \widehat{I} \in \widehat{U}$. As $\widehat{\beta}$ is a generator system of $\widehat{U}$, we can write $\widehat{X} = (\widehat{\lambda_1} \widehat{\bullet} \widehat{e_1}) \widehat{\circ} \ldots \widehat{\circ} (\widehat{\lambda_n} \widehat{\bullet} \widehat{e_n})$. Then, taking into account the conditions imposed in Example 4.1.3, the following would result: $\widehat{X} = ((\lambda_1 \bullet e_1) \circ \ldots \circ (\lambda_n \bullet e_n)) \square \widehat{I}$, from where we would see that $X = (\lambda_1 \bullet e_1) \circ \ldots \circ (\lambda_n \bullet e_n)$, for, supposing that an element $Y \in U$ exists, with $Y \neq X$, such that $\widehat{X} = \widehat{Y}$, we would have that $X \square \widehat{I} = Y \square \widehat{I} \Rightarrow X \bullet \widehat{I} = Y \bullet \widehat{I} \Rightarrow (X \bullet \widehat{I}) \circ (Y \bullet \widehat{I})^{-\vec{0}} = \vec{0} \Rightarrow (X \bullet \widehat{I}) \circ (Y^{-\vec{0}} \bullet \widehat{I}) = \vec{0} \Rightarrow (X \circ Y^{-\vec{0}}) \bullet \widehat{I} = \vec{0} = \vec{0} \bullet \widehat{I} \Rightarrow X \circ Y^{-\vec{0}} = \vec{0} \Rightarrow X = Y$, which is a contradiction to the fact that $X \neq Y$. Therefore, we would have then obtained X as a combination of the elements $e_1, \ldots, e_n$, which shows that β is a generator system of U.

Note that this feature of the isotopic model that we are considering, satisfying $X = Y$ if $\widehat{X} = \widehat{Y}$, is not a common property in isotopies. Moreover, this characteristic is also one of the main reasons why the answer to the question whether any basis of an isovector space has isobasis structure is negative. Because if we have, as in the example above, $\widehat{X} = ((\lambda_1 \bullet e_1) \circ \ldots \circ (\lambda_n \bullet e_n)) \square \widehat{I} \in \widehat{U}$, with $Y = (\lambda_1 \bullet e_1) \circ \ldots \circ (\lambda_n \bullet e_n) \in U$, it need not be that $X = Y$, since, in fact, we can find another, different combination of elements e_i which represents X or even that it need not have any. For this reason, it is convenient to give the following:

Definition 4.1.6 *In the conditions of Definition 3.1.3, an isotopic lifting of the structure E is called* injective *(or that* it corresponds to an injective isotopy*) if it satisfies $X = Y$ for all $X, Y \in G$ such that $\widehat{X} = \widehat{Y}$.*

Let us now suppose that $\widehat{\beta}$ is a linearly independent system in $\widehat{U}$, and we are going to prove that β is also one in U, always under the conditions of Example 4.1.3. We take for it $\lambda_1, \ldots, \lambda_n \in K$ such that $(\lambda_1 \bullet e_1) \circ \ldots \circ (\lambda_n \bullet e_n) = \overrightarrow{0}$. Then, $((\lambda_1 \bullet e_1) \circ \ldots \circ (\lambda_n \bullet e_n))\Box\widehat{I} = (\widehat{\lambda_1}\widehat{\bullet}\widehat{e_1})\widehat{\circ}\ldots\widehat{\circ}(\widehat{\lambda_n}\widehat{\bullet}\widehat{e_n}) = \widehat{\overrightarrow{0}} = \overrightarrow{0}$, and therefore we would have $\widehat{\lambda_i} = 0 = 0 \times \widehat{I} = 0 * \widehat{I} = 0$ (the unit element of $\widehat{K}(\widehat{a}, \widehat{+}) = K(a, +)$), $\forall i = 1, \ldots, n$, for $\widehat{\beta}$ being a basis of $\widehat{U}$. However, the foregoing implies that $\lambda_i = 0$, $\forall i = 1, \ldots, n$, so if another element $a \in K$ (distinct from 0) exists, such that $\widehat{a} = a * \widehat{I} = a \times \widehat{I} = 0$, then it would have to be $\widehat{I} = 0$, which is impossible by what was already seen previously, that there would be no isotopic element $T = \widehat{I}^{-I}$ necessary for the construction of the isofield $\widehat{K}$. Therefore, β is a linearly independent system.

All of the above proves the following:

Proposition 4.1.7 *Under the conditions of Proposition 4.1.2 and following the isotopic model used in Example 4.1.3, any basis of the isovector space $\widehat{U}$ is an isobasis.* □

We will finalize this section by studying isotopic liftings of substructures associated with vector spaces: vector subspaces. So, we will follow the usual procedure.

Definition 4.1.8 *Let $(U, \circ, \bullet)$ be a vector space over $K(a, +, \times)$, $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$ an isovector space associated with U over the field $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$, and $(W, \circ, \bullet)$ a vector subspace of U. $\widehat{W} \subseteq \widehat{U}$ is called an* isovector substance *of $\widehat{U}$ if, being an isotopy of W, $(\widehat{W}, \widehat{\circ}, \widehat{\bullet})$ is a vector subspace of $\widehat{U}$, i.e., if $(\widehat{W}, \widehat{\circ}, \widehat{\bullet})$ has isovector space structure over $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$ (given that we already have $\widehat{W} \subseteq \widehat{U}$).*

As it has been done so far in passing to the model of constructing an isotopy by means of an isounit and isoproduct, as we want that the associated operations for the future isovector subspace to be the same as

those associated with the starting isovector space, we will have to use the same isotopic elements as those used to construct $\widehat{U}$. Thus, we have in particular that, with a vector subspace W of the vector space $(U, \circ, \bullet)$ over $K(a, +, \times)$ fixed and the associated isovector space $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$ over $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$, $\widehat{W} \subseteq \widehat{U}$ will result, carrying out the corresponding isotopic lifting. Moreover, given that the other condition which $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$ must satisfy to be an isovector subspace is that it have isovector space structure, all that will be needed to make it one is to adjust the conditions of Proposition 4.1.2 for our set W, which in turn has vector space structure, being a vector subspace of U. We therefore have, similarly to the above proposition, the following:

Proposition 4.1.9 *Let $(U, \circ, \bullet)$ be a vector space defined over the field $K(a, +, \times)$. Let $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$ be the isovector space over the isofield $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$ associated with U, corresponding to the isotopy of elements $\widehat{I}, \widehat{S}, \widehat{S}', *, \star, \Box$, and $\diamond$ in the conditions of Proposition 4.1.2. Let $(W, \circ, \bullet)$ be a vector subspace of U. In these conditions, if $(W, \diamond, \Box)$ has vector subspace structure of $(U, \diamond, \Box)$ over the field $K(a, \star, *)$, $(W, \diamond)$ being a group with unit element $S' \in W$, then the isotopic lifting $(\widehat{W}, \widehat{\circ}, \widehat{\bullet})$ corresponding to the isotopy of elements indicated previously has isovector subspace structure $\widehat{U}$ over the field $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$.* $\Box$

Note that it is not necessary to assume here the rest of the assumptions required in Proposition 4.1.2, since they are all satisfied by the construction of $\widehat{U}$ (i.e., $\widehat{W}$ inherits them from $\widehat{U}$).

We end this section with an example of an isovector subspace.

Example 4.1.10 *We return to revisit the vector space $(M_{m\times n}(\mathbf{R}), +, \bullet)$ of real matrices of dimension $m \times n$, but now taken over the field of the rationals $(\mathbf{Q}, +, \times)$, with the usual sum and product. We can then carry out the isotopic lifting of this vector space corresponding to the isotopy of elements exactly the same as in Example 4.1.4, thus we would then obtain the isovector space $(M_{m\times n}(\mathbf{R}), +, \widehat{\bullet})$ over the field $(\mathbf{Q}, +, \widehat{\times})$, where the different isoproducts are defined similarly to the cited example.*

Now let the vector subspace $(M_{m\times n}(\mathbf{Q}), +, \bullet)$ of $M_{m\times n}(\mathbf{R})$ be of the rational matrices over the field $(\mathbf{Q}, +, \times)$. Since then (with the notation of

Example 4.1.4), $(M_{m\times n}(\mathbf{Q}), \diamond, \square) = (M_{m\times n}(\mathbf{Q}), +, \bullet)$ *is a vector subspace of* $(M_{m\times n}(\mathbf{R}), \diamond, \square) = (M_{m\times n}(\mathbf{R}), +, \bullet)$ *over* $(\mathbf{Q}, \star, *) = (\mathbf{Q}, +, \times)$, $(M_{m\times n}(\mathbf{Q}), \diamond) = (M_{m\times n}(\mathbf{Q}), +)$ *being a group with unit element* $S' = \mathbf{0} \in M_{m\times n}(\mathbf{Q})$, *we will have by Proposition 4.1.9 that the isotopic lifting* $(\widehat{M_{m\times n}(\mathbf{Q})_2}, \widehat{+}, \widehat{\bullet})$ *corresponding to the isotopy of the previously cited elements is an isovector subspace of* $(M_{m\times n}(\mathbf{R}), +, \widehat{\bullet})$ *over* $(\mathbf{Q}, +, \widehat{\times})$. *We verify then, in a manner analogous to the previously seen examples, that the vector subspace would be* $(\widehat{M_{m\times n}(\mathbf{Q})_2}, \widehat{+}, \widehat{\bullet}) = (M_{m\times n}(\mathbf{Q}), +, \widehat{\bullet})$. $\triangleleft$

We then move to the following subsection of this section, which will equip the isovector spaces with an *isometric*, similarly to how we endow conventional vector spaces with a metric.

4.1.2 Metric isovector spaces

We will continue our study with the isotopic lifting of metric vector spaces (see [110]), which implies, in turn, the isotopic lifting of conventional geometries, giving rise to the so-called *isogeometries* (see [127]).

To carry out this study we could follow the usual model so far, which consists in giving a general definition of the isotopic lifting in question and studying its possible construction. However, when we begin to study the structure of a metric vector space, there are a number of concepts which do not allow such a generalized study. This happens, for example, when studying the notion of scalar product or distance, the concept of well-orderedness in a field also appearing, which had not been necessary so far.

To solve these problems, we could impose conditions on the different structures with which we work, to obtain the desired results, in a similar manner to what has been done so far. However, we must not lose sight that the end goal of the study of metric vector spaces is isometric vector spaces. These, like the rest of existing geometries, require a practical study, to which over-generalizations do not contribute any important characteristic. In fact, in practice, the isogeometries that

have been studied so far consist mostly of a generalization of the conventional units of geometries which are isotopically lifted. It amounts to seeing how the fact of changing the conventional unit to another distinct one, albeit with identical topological properties, affects these geometries.

Therefore, to avoid too abstract generalizations, we restrict ourselves almost always in our study to the case of the metric isovector spaces and to the isogeometries coming from of the isotopies that follow the model given in Example 4.1.3. Under this model, the set of concepts related to the metric isovector spaces have an easy adaptation, as you will see throughout this section. Anyway, the definitions of the various concepts that will appear will be in the broadest sense possible.

We will start with the definitions of a number of basic concepts for the construction of a metric isovector space. This is followed along the lines of distinguishing isotopic notions from conventional ones, as it has been done so far:

Definition 4.1.11 *Let $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$ be an isovector space defined over an isofield $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$. We say that a function $f : \widehat{U} \times \widehat{U} \to \widehat{K}$ is an* isobilinear form *if for all $\widehat{a}, \widehat{b} \in \widehat{K}$ and for all $\widehat{X}, \widehat{Y}, \widehat{Z} \in \widehat{U}$ it satisfies the following conditions:*

1. $f((\widehat{a}\widehat{\bullet}\widehat{X})\widehat{\circ}(\widehat{b}\widehat{\bullet}\widehat{Y}), \widehat{Z}) = (\widehat{a}\widehat{\times}f(\widehat{X}, \widehat{Z}))\widehat{+}(\widehat{b}\widehat{\times}f(\widehat{Y}, \widehat{Z}))$.
2. $f(\widehat{X}, (\widehat{a}\widehat{\bullet}\widehat{Y})\widehat{\circ}(\widehat{b}\widehat{\bullet}\widehat{Z})) = (\widehat{a}\widehat{\times}f(\widehat{X}, \widehat{Y}))\widehat{+}(\widehat{b}\widehat{\times}f(\widehat{X}, \widehat{Z}))$.

Note that this concept given in the previous definition is the isotopic equivalent of the bilinear forms in vector spaces, fulfilling, in fact, the usual properties that these latter satisfy. So, for example, following the notation of the above definition, if $\beta = \{\widehat{e_1}, \widehat{e_2}, \ldots, \widehat{e_n}\}$ is a basis of the isovector space $\widehat{U}$, then the isobilinear form f would be determined by the n^2 isonumbers of the form $f_{i,j} = f(\widehat{e_i}, \widehat{e_j})$, with $i, j \in \{1, \ldots, n\}$. This is because, given the isovectors $\widehat{X} = \sum_{i=1}^{n} \widehat{x_i}\widehat{\bullet}\widehat{e_i}, \widehat{Y} = \sum_{j=1}^{n} \widehat{y_j}\widehat{\bullet}\widehat{e_j} \in \widehat{U}$, with $\widehat{x_i}, \widehat{y_j} \in \widehat{K}$, we have

$$f(\widehat{X},\widehat{Y}) = f(\sum_{i=1}^{n} \widehat{x_i}\widehat{\bullet}\widehat{e_i}, \sum_{j=1}^{n} \widehat{y_j}\widehat{\bullet}\widehat{e_j}) = \sum_{i=1}^{n} \widehat{x_i}\widehat{\bullet} f(\widehat{e_i}, \sum_{j=1}^{n} \widehat{y_j}\widehat{\bullet}\widehat{e_j}) =$$

$$\sum_{i=1}^{n} \widehat{x_i}\widehat{\bullet}(\sum_{j=1}^{n} \widehat{y_j}\widehat{\bullet} f(\widehat{e_i}, \widehat{e_j})) = \sum_{i,j=1}^{n} (\widehat{x_i}\widehat{\times}\widehat{y_j})\widehat{\bullet} f(\widehat{e_i}, \widehat{e_j}),$$

for all $i, j \in \{1, \ldots, n\}$, where the symbol $\sum$ denotes the usual summation, although with respect $\widehat{\circ}$, a notation that we will continue using later, with a similar meaning.

Let us see an example of a isobilinear form of great importance in the study of metric isovector spaces: the *isoscalar product*. It is the analogous concept for isovector spaces of the conventional scalar product.

Definition 4.1.12 *Let $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$ be an isofield associated with $K(a, +, \times)$, endowed with an order $\leq$, $0 \in \widehat{K}$ being the unit element of $\widehat{K}$ with respect to $\widehat{+}$, with the usual properties with respect to the order $\leq$. Let $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$ be an isovector space over $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$, associated with a Hilbert vector space $(U, \circ, \bullet)$, with scalar product $\langle ., . \rangle$, and with the element $\overrightarrow{0} \in \widehat{U}$ as the unit element with respect to $\widehat{\circ}$. We say that $\widehat{U}$ is a* Hilbert isovector space *if it is endowed with an* isoscalar product, $\widehat{\langle ., . \rangle} : \widehat{U} \times \widehat{U} \to \widehat{K}$, *satisfying for all $\widehat{a}, \widehat{b} \in \widehat{K}$ and for all $\widehat{X}, \widehat{Y}, \widehat{Z} \in \widehat{U}$ the following conditions:*

1. $0 \leq \widehat{\langle \widehat{X}, \widehat{X} \rangle}$; $\widehat{\langle \widehat{X}, \widehat{X} \rangle} = 0 \Leftrightarrow \widehat{X} = \overrightarrow{0}$.
2. $\widehat{\langle \widehat{X}, \widehat{Y} \rangle} = \overline{\widehat{\langle \widehat{Y}, \widehat{X} \rangle}}$, *where $\overline{\widehat{a}}$ represents the conjugate of $\widehat{a}$ in the isofield $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$, for all $\widehat{a} \in \widehat{K}$.*
3. $\widehat{\langle \widehat{X}, (\widehat{a}\widehat{\bullet}\widehat{Y})\widehat{\circ}(\widehat{b}\widehat{\bullet}\widehat{Z}) \rangle} = (\widehat{a}\widehat{\times}\widehat{\langle \widehat{X}, \widehat{Y} \rangle})\widehat{+}(\widehat{b}\widehat{\times}\widehat{\langle \widehat{X}, \widehat{Z} \rangle})$.

Note also that, as it is becoming usual in the isotopic level, we must distinguish between $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$ as a Hilbert isovector space over $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$ (whose definition we just saw) and $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$ as a Hilbert vector space over $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$, which, endowed with a scalar product $\widehat{\langle ., . \rangle}$, need not proceed from a Hilbert vector space $(U, \circ, \bullet)$ over $K(a, +, \times)$ (although from the vector space U over K).

One might then find an isovector space over which we can define an isoscalar product, but which however is not a Hilbert isovector space,

because it does not come from the isotopic lifting of a Hilbert vector space. Similarly, we can also give the case that the isotopic lifting of a Hilbert vector space is an isovector space, but not a Hilbert one, since we are unable to find an isoscalar product to give to said isovector space.

Let us then look at an example of a Hilbert isovector space:

Example 4.1.13 *Let $(U, \circ, \bullet)$ be a Hilbert space over $(\mathbf{R}, +, \times)$, endowed with the usual order $\leq$, with respect to the scalar product $\langle .,. \rangle$. Let us consider an isotopy of the vector space U that follows the model given in Example 4.1.3. We will then obtain an isovector space $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}) = (U, \circ, \widehat{\bullet})$ and an isofield $(\widehat{\mathbf{R}}, \widehat{+}, \widehat{\times}) = (\mathbf{R}, +, \widehat{\times})$.*

On this last, we can apply the usual order $\leq$ of real numbers, as what is required in Definition 4.1.12. We will impose, in addition, that the principal isounit used in such an isotopic lifting $\widehat{I} \in \mathbf{R}$ be such that $\widehat{I} > 0$.

Let us now consider the function $\widehat{\langle .,. \rangle} : \widehat{U} \times \widehat{U} \to \widehat{\mathbf{R}}$ such that $\widehat{\langle \widehat{X}, \widehat{Y} \rangle} = \langle X, Y \rangle * \widehat{I} = \langle X, Y \rangle \times \widehat{I}$ given $\widehat{X}, \widehat{Y} \in \widehat{U}$. Then, we will verify that $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}) = (U, \circ, \widehat{\bullet})$ is a Hilbert isovector space over $\widehat{\mathbf{R}} = \mathbf{R}$, the conditions of Definition 4.1.12 to be met for all $\widehat{a}, \widehat{b} \in \widehat{K}$ and for all $\widehat{X}, \widehat{Y}, \widehat{Z} \in \widehat{U}$. Indeed, we have

1. $\widehat{\langle \widehat{X}, \widehat{X} \rangle} = \langle X, X \rangle \times \widehat{I} \geq 0$, *with* $\widehat{I} > 0$ *and* $\langle X, X \rangle \geq 0$, $\langle .,. \rangle$ *being a scalar product of* $(U, \circ, \bullet)$ *over* $(\mathbf{R}, +, \times)$.

In addition, $\widehat{\langle \widehat{X}, \widehat{X} \rangle} = 0 \Leftrightarrow \langle X, X \rangle \times \widehat{I} = 0 \Leftrightarrow \langle X, X \rangle = 0$ *(since* $\widehat{I} \neq 0$ *and we have in the field* $(\mathbf{R}, +, \times)$, *endowed with the usual operations and order)* $\Leftrightarrow X = \overrightarrow{0}$ *(*$\langle .,. \rangle$ *being a scalar product)* $\Leftrightarrow \widehat{X} = \overrightarrow{0} * \widehat{I} = \overrightarrow{0} \times \widehat{I} = \overrightarrow{0}$.

2. $\widehat{\langle \widehat{X}, \widehat{Y} \rangle} = \langle X, Y \rangle \times \widehat{I} = \overline{\langle Y, X \rangle} \times \widehat{I} = (-\widehat{\langle Y, X \rangle}) \times \widehat{I} = -(\langle Y, X \rangle \times \widehat{I}) = -\widehat{\langle \widehat{Y}, \widehat{X} \rangle} = \overline{\widehat{\langle \widehat{Y}, \widehat{X} \rangle}}$.
3. $\langle \widehat{X}, (\widehat{a} \widehat{\bullet} \widehat{Y}) \widehat{\circ} (\widehat{b} \widehat{\bullet} \widehat{Z})) \rangle = \langle \widehat{X}, ((a \bullet Y) \bullet \widehat{I}) \circ (b \bullet Z) \bullet \widehat{I}) \rangle =$
$\langle \widehat{X}, (a \bullet Y) \widehat{\circ} (b \bullet Z) \rangle = \langle X, (a \bullet Y) \circ (b \bullet Z) \rangle \times \widehat{I} =$
$((a \times \langle X, Y \rangle) + (b \times \langle X, Z \rangle)) \times \widehat{I} = ((a \times \langle X, Y \rangle) \times \widehat{I}) +$
$((b \times \langle X, Z \rangle) \times \widehat{I}) = (\widehat{a} \widehat{\times} (\langle X, Y \rangle \times \widehat{I})) + (\widehat{b} \widehat{\times} (\langle X, Z \rangle \times \widehat{I})) =$

$$(\widehat{a}\widehat{\times}\widehat{\langle\widehat{X},\widehat{Y}\rangle})\widehat{+}(\widehat{b}\widehat{\times}\widehat{\langle\widehat{X},\widehat{Z}\rangle}). \qquad \triangleleft$$

The following concept to adapt to the isotopic level will be that of *metric distance*. We recall that any metric vector space is a vector space endowed with a metric, which in turn is associated with a metric distance. Therefore, to obtain the structure of a metric isovector space, we will also have to give the definition of *metric isodistance*.

To do so, as in a metric vector space, the metric distance can be given with respect to the elements of a basis of the space, if $\beta = \{e_1, e_2, \ldots, e_n\}$ is a basis of a metric vector space $(U, \circ, \bullet)$ over the field $K(a, +, \times)$, the possible associated distance with U would be given as a function d, represented by the n^2 numbers $d_{i,j} = d(e_i, e_j)$. In this way, if we have two elements $X = \sum_{i=1}^{n} x_i \bullet e_i, Y = \sum_{j=1}^{n} y_j \bullet e_j \in U$, with $x_i, y_j \in K$, we would have that $d(X, Y) = \sum_{i,j=1}^{n}(x_i \times y_j) \bullet d(e_i, e_j)$.

Now we could consider an $(n \times n)$-dimensional matrix of elements of the n^2 previous numbers $(d_{ij})_{i,j\in\{1,\ldots,n\}}$, which is what defines the metric associated with the distance d, which we symbolized by g. By convention it in fact shows $g \equiv (g_{ij})_{i,j\in\{1,\ldots,n\}} = (d_{i,j})_{i,j\in\{1,\ldots,n\}}$. In addition, taking into account the conditions that any metric distance must satisfy, we have that the matrix g represents a metric if and only if it is a regular, symmetric, and positive-definite matrix.

With these observations we are able to define the notion of *metric isovector space* and study how to achieve its construction.

Definition 4.1.14 *Let $U(X, g, K)$ be a metric vector space (with elements $X, Y, Z, \ldots$) over a field $K(a, +, \times)$, with a metric g associated with a metric distance d. We say that $\widehat{U}(\widehat{X}, \widehat{g}, \widehat{K})$ is a* metric isovector space *if, being an isotopy of U, it is an isovector space over the isofield $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$, endowed with an order $\leq$ and $0 \in \widehat{K}$ being its unit element with respect to $\widehat{+}$ (with the usual properties with respect to the order $\leq$), with elements $\widehat{X}, \widehat{Y}, \widehat{Z}, \ldots$ and endowed with a new* isometric *$\widehat{g}$, that will be an isotopy of the metric g satisfying the necessary properties to be a metric in $\widehat{U}$; i.e., that $\widehat{g}$ is associated with a metric isodistance $\widehat{d}$, which being an isotopy of the metric distance d, satisfies for all $\widehat{X}, \widehat{Y}, \widehat{Z} \in \widehat{U}$ the following conditions:*

1. $0 \leq \widehat{d}(\widehat{X}, \widehat{Y}) \quad ; \quad \widehat{d}(\widehat{X}, \widehat{Y}) = 0 \Leftrightarrow \widehat{X} = \widehat{Y}$.
2. $\widehat{d}(\widehat{X}, \widehat{Y}) = \widehat{d}(\widehat{Y}, \widehat{X})$.
3. *Triangle inequality:* $\widehat{d}(\widehat{X}, \widehat{Y}) \leq \widehat{d}(\widehat{X}, \widehat{Z}) \widehat{+} \widehat{d}(\widehat{Z}, \widehat{Y})$.

Definition 4.1.15 *If d, instead of being a metric distance, were a pseudometric distance (so $U(X, g, K)$ would be a pseudometric vector space), we say that $\widehat{U}(\widehat{X}, \widehat{g}, \widehat{K})$ is a* pseudometric isovector space *if it satisfies the three previous conditions except (1), satisfying instead the following:*

(1′) $0 \leq \widehat{d}(\widehat{X}, \widehat{Y}) \quad ; \quad \widehat{d}(\widehat{X}, \widehat{X}) = 0$, *for all* $\widehat{X}, \widehat{Y} \in \widehat{U}$.

In this case, we call $\widehat{g}$ the isopseudometric *and $\widehat{d}$ the* pseudometric isodistance.

Now let us look at an example of a metric isovector space.

Example 4.1.16 *Let $U(X, g, \mathbf{R})$ be a metric vector space associated with the n-dimensional vector space $(U, \circ, \bullet)$ over the field $(\mathbf{R}, +, \times)$, endowed with the usual order $\leq$, with a metric g. Let us consider an isotopy of the vector space U that follows the model given in Example 4.1.3, imposing in addition that $\widehat{I} \in \mathbf{R}$ be such that $\widehat{I} > 0$. We would then obtain the isovector space $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}) = (U, \circ, \widehat{\bullet})$ over the isofield $(\widehat{\mathbf{R}}, \widehat{+}, \widehat{\times}) = (\mathbf{R}, +, \widehat{\times})$. We can then equip the isofield $\widehat{\mathbf{R}} = \mathbf{R}$ with the same, usual, previous order $\leq$.*

Now we want to provide $\widehat{U}$ with a metric $\widehat{g}$, which is an isotopic lifting of the metric g. This possible metric $\widehat{g}$ is going to be associated with a matrix that in turn will come from a metric isodistance $\widehat{d}$, under conditions similar to the metric g and its metric distance d (following the general pattern seen before). Now, we saw in the section on vector isospaces that the model of Example 4.1.3 takes bases to bases. Therefore, as the starting isovector space U is assumed to be n-dimensional, the obtained isovector space $\widehat{U}$ will also be n-dimensional. Then, the matrix representing the isometric $\widehat{g}$ shall be $(n \times n)$, like the matrix representing the metric g. In addition, said matrix must have its elements in the isofield $\widehat{\mathbf{R}}$, which, following the notation of Example 4.1.3, are of the form $\widehat{a} = a * \widehat{I} = a \times \widehat{I}$, with $a \in \mathbf{R}$.

Then, if we want to obtain the isotopic lifting of the metric g following the model that is given by an isounit and isoproduct, a possibility would be

given by multiplying each element $(g_{i,j})_{i,j\in\{1,\ldots,n\}}$, *of the matrix that represents* g, *by the isounit* $\widehat{I}$ *that we had, with respect to the operation* $* \equiv \times$, *which was also already fixed in lifting the field* K. *We would thus define* $\widehat{g} \equiv (\widehat{g_{i,j}})_{i,j\in\{1,\ldots,n\}} = (g_{i,j} * \widehat{I})_{i,j\in\{1,\ldots,n\}} = (g_{i,j} \times \widehat{I})_{i,j\in\{1,\ldots,n\}}$.

To see then that $\widehat{g}$ *so defined is in fact a metric, we should check that the matrix that represents it is regular, symmetric, and positive-definite. However, the two last conditions are easily verified with the conditions imposed in Example 4.1.3, taking into account, in addition, that* $\widehat{I} > 0$, *since*

1. $\widehat{g_{i,j}} = g_{i,j} * \widehat{I} = g_{j,i} * \widehat{I} = \widehat{g_{j,i}}$, *for all* $i, j \in \{1, \ldots, n\}$.
2. $\widehat{g_{ii}} = g_{i,i} * \widehat{I} = g_{i,i} \times \widehat{I} > 0$ *for all* $i = 1, \ldots, n$, *with* $g_{i,i} > 0$ *for all* $i \in \{1, \ldots, n\}$ *(*g *being a metric and* $\widehat{I} > 0$*).*

We still need, then, to prove that the matrix $(\widehat{g_{i,j}})_{i,j\in\{1,\ldots,n\}}$ *is regular. For this, we observe that the way of obtaining the new matrix, multiplying each element of the old matrix by* $\widehat{I}$ *with respect to* $* \equiv \times$, *is equivalent to the usual product of said matrix by the matrix* $diag(\widehat{I}, \ldots, \widehat{I})$, *which we will denote by* $\widehat{H}$. *Since we also have not left the field* $\mathbf{R}$ *(with* $\widehat{\mathbf{R}} = \mathbf{R}$*), we see that the determinant of the matrix* $\widehat{H}$ *coincides with* $\widehat{I}$. *So* $\det((\widehat{g_{i,j}})_{i,j\in\{1,\ldots,n\}}) = \det((g_{i,j})_{i,j\in\{1,\ldots,n\}}) \times \det(diag(\widehat{I}, \ldots, \widehat{I})) = \det((g_{i,j})_{i,j\in\{1,\ldots,n\}}) \times \widehat{I} \neq 0$, *with* $\widehat{I} > 0$ *by supposition and the matrix that* g *represents being regular.*

We therefore arrive at that $\widehat{g}$ *is effectively a metric, being also the isodistance* $\widehat{d}$, *associated with the isometric* $\widehat{g}$, *in the following way:* $\widehat{d}(\widehat{X}, \widehat{Y}) = d(X, Y) \times \widehat{I}$, *for all* $\widehat{X}, \widehat{Y} \in \widehat{U}$. ◁

Note that in the preceding example, the matrix $\widehat{H}$ could be any matrix in $M_{n\times n}(\mathbf{R})$, provided that the resulting matrix for representing $\widehat{g}$ were regular, symmetric, and positive-definite. This is due to the fact that the isotopic model given in Example 4.1.3 makes the isofield $\widehat{K}$ coincide with the starting field K. In this way, regardless what the matrix $\widehat{H}$ be, the elements of the matrix $\widehat{g_{i,j}}$ will always be in $\widehat{K}$, which is the necessary condition that must be imposed.

We now return to the abstract level of the axioms in the subject that concerns us. Remember that at that level all vector spaces and isovector spaces are equivalent, and we can pass (as in the case of isofields)

from one vector space to another by means of a certain isotopy. We could ask then if this happens for metric isovector spaces. So that it be so, the only thing that would be needed would be to see that all metrics and isometrics are equivalent in the isotopic level, i.e., that we can pass from one to another by means of an isotopy. Now, as we have seen for isofields, the possible isotopies to use for this purpose need not follow the model of construction by an isounit and isoproduct, which is the model that we have seen so far.

However, the proposal that Santilli gave in 1983 to respond to this issue (see [110]) was to interpret any isometric $\widehat{g}$ as an isotopic lifting of the Euclidean metric $\delta \equiv \text{diag}(1, \ldots, 1)$, considering the matrix associated with $\widehat{g}$ as the isounit $\widehat{I}$, in such a way that we obtained as a result the lifting $\delta \to \delta * \widehat{I} = \widehat{I} \equiv \widehat{g}$. Thus, if we adapt this proposal to the model to which we are accustomed, we should impose that, with the usual notations, the operation $*$ have as a unit $I = \delta$. Let us see an example of the above.

Example 4.1.17 *Let us suppose that $U(X, \delta, \mathbf{R})$ is an n-dimensional metric vector space $(U, \circ, \bullet)$ over the field $(\mathbf{R}, +, \times)$, endowed with the usual order $\leq$ and the Euclidean metric $\delta \equiv diag(1, \ldots, 1)$.*

Let us also suppose that we have an isovector space $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$ associated with U, over an isofield $(\widehat{\mathbf{R}}, \widehat{+}, \widehat{\times})$, corresponding to an isotopy that follows the model of Example 4.1.3, i.e. $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}) = (U, \circ, \widehat{\bullet})$ and $(\widehat{\mathbf{R}}, \widehat{+}, \widehat{\times}) = (\mathbf{R}, +, \widehat{\times})$. Let us then consider that such an isovector space $\widehat{U}$ is endowed with a metric $\widehat{g} \equiv (\widehat{g_{i,j}})_{i,j \in \{1,\ldots,n\}}$ and we are going to prove that such a metric can be interpreted, in fact, as an isotopy of the Euclidean metric δ of the metric vector space U, following the model proposed by Santilli in 1983.

To do this, because the matrix associated with $\widehat{g}$ has elements in $\widehat{\mathbf{R}} = \mathbf{R}$, we will consider, as the isounit for such a lifting to such a matrix and as the necessary operation for the isotopy, the usual real $(n \times n)$-dimensional matrices that it has as unit element for the matrix $diag(1, \ldots, 1)$ (which is associated with the Euclidean metric δ). It makes sense also to talk about the matrix of $\widehat{g}$ as an isounit with respect to the indicated operation having an inverse matrix, by being regular. Such an inverse matrix would correspond

to the isotopic element of the isotopy in question of the metric δ. This way, we finally have the isotopic lifting sought:

$$\delta \to \delta \cdot (\widehat{g_{i,j}})_{i,j\in\{1,\ldots,n\}} = (\widehat{g_{i,j}})_{i,j\in\{1,\ldots,n\}} \equiv \widehat{g}$$

$\triangleleft$

With this example, together with 4.1.16, the fact that the isotopic lifting of the metric g of the metric vector space is not given uniquely is also patent. A possible model for such an isotopy is given in each of the two cited examples. In fact, if we did not have to look in Example 4.1.17 for an isotopic lifting that forcibly lifts g to $\widehat{g}$, we could provide the isospace $\widehat{U}$ with an isometric distinct from $\widehat{g}$. It would suffice to take the isotopic model seen in Example 4.1.16, multiply the matrix associated with g by the matrix $\mathrm{diag}(\widehat{I}, \ldots, \widehat{I})$, where $\widehat{I}$ is the principal isounit in the construction of the isovector space $\widehat{U}$. We have already even commented that we could multiply the matrix associated with g by any $\widehat{H} \in M_{n\times n}(\mathbf{R})$, provided that the resulting matrix be regular, symmetric, and positive-definite.

This observation tells us also that the isotopic model given in Example 4.1.3 is of vital importance in the construction of metric isovector spaces. In fact, we can get, by imposing new conditions, results of great interest such as the following:

Proposition 4.1.18 *Let $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$ be an isovector space defined over the isofield $(\widehat{\mathbf{R}}, \widehat{+}, \widehat{\times})$, associated with the n-dimensional vector space $(U, \circ, \bullet)$ over the field $(\mathbf{R}, +, \times)$, in the hypotheses of Proposition 4.1.2, following the isotopic model of Example 4.1.3 and also imposing that the principal isounit used, $\widehat{I} \in \mathbf{R}$, be such that $\widehat{I} > 0$, where we make use of the usual order $\leq$ in $\mathbf{R}$. Then we can endow $\widehat{U}$ with a metric if and only if $\widehat{U}$ has the structure of a metric isovector space, i.e., if and only if the departing vector space U is endowed with a metric g which can be lifted isotopically to an isometric $\widehat{g}$ in $\widehat{U}$.*

Proof

It will be done by double inclusion:

1. $\Leftarrow$

It is straightforward, for if $\widehat{U}$ has metric isospace structure, it is because it is equipped with an isometric $\widehat{g}$, which in turn satisfies the conditions required to be a metric.

2. $\Rightarrow$

Let us suppose that we can provide $\widehat{U}$ with a metric $\widehat{g} \equiv (\widehat{g_{i,j}})_{i,j\in\{1,\ldots,n\}}$. We then need to prove that a metric g in U exists which we can isotopically lift to the metric $\widehat{g}$, thus converting this into an isometric.

Therefore by the conditions imposed in Example 4.1.3, we know that $\widehat{U}$ is an n-dimensional isovector space, since this type of isotopy takes bases into bases, so the dimension of the vector space is preserved. We will have therefore that the matrix $\widehat{g} \equiv (\widehat{g_{i,j}})_{i,j\in\{1,\ldots,n\}}$ associated with $\widehat{g}$ will be $(n \times n)$-dimensional, with elements in $\widehat{\mathbf{R}} = \mathbf{R}$. We can then construct the matrix $(g_{i,j})_{i,j\in\{1,\ldots,n\}}$, $g_{i,j} \in \mathbf{R}$ being such that $g_{ij} * \widehat{I} = (g_{i,j}) \times \widehat{I} = \widehat{g_{i,j}}$ for all $i, j \in \{1, \ldots, n\}$, i.e., such that they are the real numbers that come from the n^2 isonumbers that constitute the matrix associated with $\widehat{g}$.

We then see that the new constructed matrix is regular, symmetric, and positive-definite; thus it will be a matrix associated with a metric g of the vector space U, that we could isotopically lift to $\widehat{g}$ without considering the model proposed in Example 4.1.16, of multiplying said matrix by $\mathrm{diag}(\widehat{I}, \ldots, \widehat{I})$. For it, as $\widehat{I} \in \mathbf{R}$ and $* \equiv \times$, we have that $T = \widehat{I}^{-I} = \widehat{I}^{-1} \in \mathbf{R}$, from where $g_{i,j} = \widehat{g_{i,j}} * T = \widehat{g_{i,j}} \times T$, for all $i, j \in \{1, \ldots, n\}$. Now, we prove that the matrix $(g_{i,j})_{i,j\in\{1,\ldots,n\}}$ is regular, symmetric, and positive-definite in a way totally analogous to what we did in Example 4.1.16 with respect to the isometric indicated there, given that $T > 0$ with $\widehat{I} > 0$ being considered the usual order in $\mathbf{R}$. By it, the matrix $(g_{i,j})_{i,j\in\{1,\ldots,n\}}$ is associated with a metric in U, which completes the proof. $\square$

An immediate consequence of this result is that, under the required hypotheses, an isovector space $\widehat{U}$ has a structure of a metric vector

space if and only if it has a metric isovector space structure, a result that we do not generally have, due to what was already mentioned in various occasions: the isotopic notions present differences with the conventional notions when we attempt to project the new theory onto the old one.

We end this section with a few brief notes on *isogeometry*. We have said that in the same way that conventional geometries appear after studying metric spaces, isogeometries should appear when studying the metric isospaces. It is said that the important aspect of isogeometries is that they allow us to study how the usual notions of conventional geometries vary when changing the usual units for other identical topological characteristics. An example of this is when we study the isotopic lifting of a Euclidean space $E(X, \delta, \mathbf{R})$ over the field $(\mathbf{R}, +, \times)$, equipped with the Euclidean metric $\delta \equiv \text{diag}(1, \ldots, 1)$. A decisive feature in such spaces is that if we concentrate on Euclidean axes, they all have as basic unit the same element $+1$. On the other hand, an important aspect of *Euclidean isospaces* (also called *isoeuclidean spaces* or *Euclid-Santilli spaces*, from which we obtain the *isoeuclidean geometry*) is that they permit us to alter the units of conventional space, including the axial units. Let us look at the following:

Example 4.1.19 *In the case of the 3-dimensional Euclidean space $E(X, \delta, \mathbf{R})$, the units for the three axes would be $I_k = +1$, for $k = 1, 2, 3$ ($\approx OX, OY$, and OZ axes).*

Now consider a Euclidean isospace corresponding to a 3-dimensional isotopy of class I, that is, an isotopic lifting where the isounit $\widehat{I}$ is a (3×3)-dimensional, Hermitian, and positive-definite matrix (see **[145]***). Due to this last characteristic, $\widehat{I}$ can be diagonalized into the form $\widehat{I} = diag(n_1^2, n_2^2, n_3^2)$, with $n_1, n_2, n_3 \in \mathbf{R}\backslash\{0\}$, thus obtaining as new isounits for the axes the elements $\widehat{I}_k = n_k^2$, for $k = 1, 2, 3$. This means that not only the original units are lifted to arbitrary invertible values, but the units of different axes generally have different values.* $\triangleleft$

These conditions allow for new applications. An example is the unification of all the possible ellipsoids

$$X^2 = \frac{X_1^2}{n_1^2} + \frac{X_2^2}{n_2^2} + \frac{X_3^2}{n_3^2} = r^2 \in \mathbf{R}$$

in the Euclidean space $E(X, \delta, \mathbf{R})$, in the so-called *isospheres*

$$\widehat{X}^{\widehat{2}} = \left(\frac{X_1^2}{n_1^2} + \frac{X_2^2}{n_2^2} + \frac{X_3^2}{n_3^2}\right) * \widehat{I} = r^2 * \widehat{I} \in \widehat{\mathbf{R}},$$

which are the spheres in the isospace $\widehat{E}(\widehat{X}, \widehat{\delta}, \widehat{\mathbf{R}})$ over the isofield $\widehat{\mathbf{R}}$ (see [145]).

In fact, the deformation of the axial units of the sphere, $I_k = +1$, into the new isounits $\widehat{I}_k = n_k^2$ retains the perfect sphericity. On the other hand, more advanced studies allow us to work with isotopies of class III so we may obtain the unification of all compact and non-compact conics. The use of class IV at the same time allows the inclusion of all conics (see [145]).

Finally, we note that these generalizations also have very important implications in physics. An example of this is that the changes in units, seen above, allow us to transform very large distances (lengths) into very small distances, and *vice versa*. Also, they have repercussions on the generalization of conventional mechanics (Newtonian, analytic, quantum, etc.) into new isotopic mechanics, just like in the classic problems of interior and exterior dynamics. The interested reader can see these themes in [145].

4.2 Isotransformations

Before proceeding with the study of new isostructures, we will dedicate this section to the different applications that we can establish between vector isospaces. In it, not only will we indicate the usual applications that exist when considering isovector spaces as vector spaces, as was already done for isogroups or isorings. We are going to go one step further, defining *isotransformations*, which will be those transformations that come from a type of isotopic lifting of a transformation

between vector spaces corresponding to the previous vector isospaces (see [121]).

On the other hand, the fact that Santilli isotopies are some very useful tools for the passage of a *linear and local* theory to another non-linear and non-local one, which is closely related to the previous one, will be apparent. We already said in Definition 3.1.2 that Santilli isotopies were those capable of lifting linear, local, and canonical structures, to non-linear, non-local and non-canonical forms, being able to reconstruct the linearity, locality, and canonicity of certain generalized spaces of departure. Well, we will see that in the event that we are working with vector spaces, the previous non-linearity and non-locality occurs only when the new theory is projected onto the original theory, since, in fact, the isotheory reconstructs the linearity and the locality in the constructed isovector space. With respect to the canonicity, we would have a similar result, which Santilli began to develop from the definition of an *isocalculus* (which is the isotopic lifting of conventional calculus) in 1996 (see [145]), but that due to its extension will be not treated in our study.

We therefore begin with the definition of functions between isovector spaces, considering them as vector spaces, and which together are called *transformations*:

Definition 4.2.1 *Let* $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$ *and* $(\widehat{U'}, \widehat{\triangle}, \widehat{\triangledown})$ *be two isovector spaces over an isofield* $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$. *A function* $f : \widehat{U} \to \widehat{U'}$ *is called a* homomorphism *of isovector spaces if for any* $\widehat{a} \in \widehat{K}$ *and for all* $\widehat{X}, \widehat{Y} \in \widehat{U}$,

1. $f(\widehat{X} \widehat{\circ} \widehat{Y}) = f(\widehat{X}) \widehat{\triangle} f(\widehat{Y})$
2. $f(\widehat{a} \widehat{\bullet} \widehat{X}) = \widehat{a} \widehat{\triangledown} f(\widehat{X})$

are satisfied.

If f *is bijective, it is called an* isomorphism, *and if* $\widehat{U} = \widehat{U'}$, *an* endomorphism. *In this latter case, if in addition* f *is bijective, it is called an* automorphism.

We also call any endomorphism a *linear operator*. This particular type of transformation has the property that when it occurs between con-

ventional vector spaces, it is given univocally by the associative product by an element $a \in K$, independent of local variables. So, if we have the linear operator $f : (U, \circ, \bullet) \to (U, \circ, \bullet)$, where U is an isovector space over a field $K(a, +, \times)$ and f is given by the element $a \in K$, then $f(X) = a \bullet X$ for all $X \in U$.

In the case that the element a has a dependence on local variables x, the transformation f will be called *nonlinear*. Then, $a = a(x)$, leaving f defined according to $f(X) = a(x) \bullet X$ for all $X \in U$. If this dependency is of integral type, it is then said that f is a *non-local* transformation, while we will call it *local* otherwise.

Then, in the same sense as we are considering the isovector spaces as vector spaces, given the linear operator $f : \widehat{U} \to \widehat{U}$, of Definition 4.2.1, we will have that f is given by an element $\widehat{a} \in \widehat{K}$, such that $f(\widehat{X}) = \widehat{a} \widehat{\bullet} \widehat{X}$ will be satisfied for all $\widehat{X} \in \widehat{U}$. We also speak of non-linear or non-local transformations in the vector isospace $\widehat{U}$ in a way analogous to what was seen previously for the vector space U.

We finally note that the form that the linear operators have permit us to establish a relationship between a linear operator on a vector space $(U, \circ, \bullet)$ over a field $K(a, +, \times)$ and a linear operator in an isovector space $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$ over $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$ associated with U. Let us then suppose, for example, that $f : U \to U$ is the first of these linear operators, given by the element $a \in K$, i.e., such that $f(X) = a \bullet X$ for all $X \in U$. We could then consider the lifting $\widehat{a} \in \widehat{K}$, corresponding to the element $a \in K$, and thus take the linear operator on the isovector space $\widehat{U}$, which is given by the element $\widehat{a}$. Calling such a linear operator $\widehat{f}$, it would result that $\widehat{f}(\widehat{X}) = \widehat{a} \widehat{\bullet} \widehat{X}$, for all $\widehat{X} \in \widehat{U}$. Then, $\widehat{f}$ is called an *isotransformation*, arising from the isotopic lifting of a transformation between vector spaces f and resulting in a transformation between the corresponding vector isospaces. We also have the following:

Definition 4.2.2 *An isotransformation $\widehat{f}$ given by an element $\widehat{a}$ from an injective isotopy is called* isolinear *or* isolocal *when the corresponding element a representing the transformation f is linear or local, respectively. It is*

called non-isolinear *or* non-isolocal *when said element a is non-linear or non-local, respectively.*

The fact of imposing in the above definition that the isotopy with which we are working is injective is that otherwise there could be a second element $b \in K$, with $b \neq a$, such that $\widehat{b} = \widehat{a}$, b being non-linear (non-local, respectively); thus, the isotransformation $\widehat{f}$ would be on the one hand isolinear (isolocal, respectively), while on the other hand it would be non-isolinear (non-isolocal, respectively), which would be a contradiction. However, we could expand the prior definition to isotopies that do not allow the existence of such a non-linear (non-local, respectively) element b, although they were not injective. However, to avoid problems we will impose hereinafter that the injectivity condition in the isotopies be used. Thus, this definition allows us to distinguish the concepts of *isolinearity* and *isolocality* from the conventional concepts of linearity and locality. We can in fact have that a certain isotransformation $\widehat{f}$ be non-linear and non-local, the element $\widehat{a}$ being such, and instead be isolinear and isolocal, the corresponding element a being such, just like any other possible combination of these concepts. Nevertheless, the following result is verified:

Proposition 4.2.3 *Let $(U, \circ, \bullet)$ be a vector space over $K(a, +, \times)$, $f : U \to U$ being a transformation of linear operator type, given by the element $a \in K$. Let $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$ be an isospace associated with U, over the field $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$, corresponding to an isotopy that is injective. Then f is a linear (local, respectively) transformation if and only if the corresponding isotransformation $\widehat{f}$ in $\widehat{U}$, given by the element $\widehat{a} \in \widehat{K}$, is isolinear (isolocal, respectively).*

Proof

It suffices to take into account Definition 4.2.2, since then $\widehat{f}$ will be isolinear (isolocal, respectively) if and only if the element $a \in K$ is linear (local, respectively), which is equivalent, in turn, to saying that f is a linear (local, respectively) transformation. □

Therefore the above results show us that we need to lift isotopically the linear and local functions to maintain linearity and locality in the

corresponding isospaces, and that the non-linearity and non-locality of the isotheory only occurs when it is projected into the starting theory. In our case we have such a projection when we seek to obtain the linearity and locality of the transformation $\widehat{f}$ in the element $\widehat{a} \in \widehat{K}$ which is given univocally (as is done in the conventional theory), rather than in the element $a \in K$, as it has been done in Definition 4.2.2.

So, going back to the abstract level of the axioms (which we already discussed in the section on isofields), it results that the vector spaces and isovector spaces can be considered equivalent at that level; the linear transformations between the first and the isolinear isotransformations between the second can also be considered equivalent, as well as the local transformations and the isolocal isotransformations.

We will then study an example of the isotransformations, which will consist in studying those isotransformations resulting from the Santilli isotopy, a consequence of the construction given by an isounit and isoproduct:

Example 4.2.4 *Let us suppose a vector space $(U, \circ, \bullet)$ defined over a field $K(a, +, \times)$. Let $f : U \to U$ be the linear operator existing in U, given by a fixed element $a \in K$. Now let $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$ be the isovector space over the isofield $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$, corresponding to the isotopy of elements $\widehat{I}, \widehat{S}, \widehat{S}', *, \star, \Box$, and $\diamond$, in the conditions of Proposition 4.1.2. We will then have that the lifting of the element $a \in K$ with respect to the previous isotopy is $\widehat{a} = a * \widehat{I} \in \widehat{K}$ and that the linear operator existing in $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$, which is given by the element $\widehat{a}$, is $\widehat{f} : \widehat{U} \to \widehat{U}$, such that $\widehat{f}(\widehat{X}) = \widehat{a}\widehat{\bullet}\widehat{X} = (a \Box X) \Box \widehat{I}$, for all $\widehat{X} \in \widehat{U}$. We also already know, by the previous section, that the isoproduct $\widehat{\bullet}$ is associative, verifying the condition (2.a) of Definition 4.1.1, by construction. We should therefore have that $\widehat{f}$ is effectively a linear operator, which comes from the isotopic lifting of the linear operator f, and therefore $\widehat{f}$ is an isotransformation.* $\triangleleft$

Let us finally, to finish this section, see a particular result in this isotopic model which corroborates something already cited in the general case (see [175]):

Proposition 4.2.5 *Let $(U, \circ, \bullet)$ be a vector space defined over the field $K(a, +, \times)$ and $f : U \to U$ a non-linear (non-local, respectively) transformation given by the element $a = a(x)$, where $a(x) \in K$ for any local variable x. Then, in these conditions, and under the adequate topological conditions, a linear (local, respectively) transformation $\widehat{f}$ exists in an isospace $\widehat{U}$ associated with U.*

Proof

The topological conditions to find will be those that permit finding a decomposition $a = b * T$, with $b \in K$ linear (local, respectively) and where $*$ is the principal isotopic element from the isotopy that lifts to the field K and has T as isotopic element. T will therefore be invertible, $\widehat{I} = T^{-I}$ (if I is the unit element with respect to $*$) being the principal isounit that we use in our isotopic lifting.

Under an isotopy compatible with the previous elements, then $\widehat{a} = \widehat{b * T} = (b * T) * \widehat{I} = b * T * \widehat{I} = b \in \widehat{K}$ will be the isotopically lifted element. So, the isotransformation $\widehat{f}$, which would be defined in the corresponding isovector space $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$, would be given by the previous element $b \in \widehat{K}$, being defined according to $f(\widehat{X}) = b \widehat{\bullet} \widehat{X}$, for all $\widehat{X} \in \widehat{U}$. So, $\widehat{f}$ would thus be linear (local, respectively) for b; thus, we have the desired result. □

We finally note that the decomposition $a = b * T$, realized in the previous demonstration, what is actually done is to go beyond the non-linearity (non-locality, respectively) of the element $a \in K$ to the isotopic element T. The problem that arises would be to find in each specific case the most favorable decomposition, imposing sufficient conditions for this. For example, imposing it be invertible with respect to $*$, then we would take $T = b^{-I} * a$, needing also to impose that a should be invertible with respect to $*$, so that T would also be.

The following isostructure to study will be the isotopic lifting of modules. To do this we will continue the normal development, beginning with the definition of *isomodules*.

4.3 Isomodules

Following a development in this section similar to that above, we will study isomodules, isosubmodules, and functions between these isostructures (see [103]).

Definition 4.3.1 *Let $(A, \circ, \bullet)$ be a ring and $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$ and associated isoring. Let $(M, +)$ be an A-module with external product $\times$. We call an* iso-$\widehat{A}$-module $\widehat{M}$ *any isotopy of M equipped with a new internal operation $\widehat{+}$ and a new external operation $\widehat{\times}$, satisfying the axioms of an $\widehat{A}$-module, i.e., such that*

1. *$(\widehat{M}, \widehat{+})$ is an isogroup, with $\widehat{a}\widehat{\times}\widehat{m} \in \widehat{M}, \forall \widehat{a} \in \widehat{A}, \forall \widehat{m} \in \widehat{M}$.*
2. *Axioms of the external operation:*

   a. $\widehat{a}\widehat{\times}(\widehat{b}\widehat{\times}\widehat{m}) = (\widehat{a}\widehat{\bullet}\widehat{b})\widehat{\times}\widehat{m}, \forall \widehat{a}, \widehat{b} \in \widehat{A}, \forall \widehat{m} \in \widehat{M}$,
   b. $\widehat{a}\widehat{\times}(\widehat{m}\widehat{+}\widehat{n}) = (\widehat{a}\widehat{\times}\widehat{m})\widehat{+}(\widehat{a}\widehat{\times}\widehat{n}), \forall \widehat{a} \in \widehat{A}, \forall \widehat{m}, \widehat{n} \in \widehat{M}$,
   c. $(\widehat{a}\widehat{\circ}\widehat{b})\widehat{\times}\widehat{m} = (\widehat{a}\widehat{\times}\widehat{m})\widehat{+}(\widehat{b}\widehat{\times}\widehat{m}), \forall \widehat{a}, \widehat{b} \in \widehat{A}, \forall \widehat{m} \in \widehat{M}$,
   d. $\widehat{I}\widehat{\times}\widehat{m} = \widehat{m}, \forall \widehat{m} \in \widehat{M}$,

   $\widehat{I}$ being the isounit associated with $\widehat{A}$ with respect to the operation $\widehat{\bullet}$.

We observe first that, in the terms of the above definition, if A were a field and $\widehat{A}$ an isofield associated with A, we would have that M would have a vector space structure over A and $\widehat{M}$ would have an isovector space structure associated with M, over $\widehat{A}$. Therefore, here, as in the case of conventional structures, an isomodule is no more than a generalization of an isovector space.

Another example which is deducted from the conventional case is as follows:

Example 4.3.2 *Let $\Im$ be an ideal of the ring A, $\widehat{\Im}$ being an isoideal of $\widehat{A}$ associated with $\Im$. Given that $\Im$ has an A-module structure for the internal multiplication of the ring A, $\bullet$, it is obvious that $\widehat{\Im}$ has an iso-$\widehat{A}$-module structure for the internal multiplication $\widehat{\bullet}$ of the isoring $\widehat{A}$, since it is verified that:*

1. $(\widehat{\Im}, \widehat{\circ})$ *is an isogroup,* $\widehat{\Im}$ *being an isoideal of* $\widehat{A}$.
2. $\widehat{a}\widehat{\bullet}(\widehat{b}\widehat{\bullet}\widehat{m}) = (\widehat{a}\widehat{\bullet}\widehat{b})\widehat{\bullet}\widehat{m}, \forall \widehat{a}, \widehat{b} \in \widehat{A}, \forall \widehat{m} \in \widehat{\Im}$.
3. $\widehat{a}\widehat{\bullet}(\widehat{m}\widehat{\circ}\widehat{n}) = (\widehat{a}\widehat{\bullet}\widehat{m})\widehat{\circ}(\widehat{a}\widehat{\bullet}\widehat{n}), \forall \widehat{a} \in \widehat{A}, \forall \widehat{m}, \widehat{n} \in \widehat{M}$.
4. $(\widehat{a}\widehat{\circ}\widehat{b})\widehat{\bullet}\widehat{m} = (\widehat{a}\widehat{\bullet}\widehat{m})\widehat{\circ}(\widehat{b}\widehat{\bullet}\widehat{m}), \forall \widehat{a}, \widehat{b} \in \widehat{A}, \forall \widehat{m} \in \widehat{M}$.
5. $\widehat{I}\widehat{\bullet}\widehat{m} = \widehat{m}, \forall \widehat{m} \in \widehat{M}$; $\widehat{I}$ *being the isounit associated with* $\widehat{A}$ *with respect to multiplication* $\widehat{\bullet}$. $\triangleleft$

We observe, however, that, despite $A/\Im$ having an A-module structure for the usual multiplication by a scalar ($A \times A/\Im \to A/\Im$, such that $(a, b + \Im) \to (a \bullet b) + \Im$), we cannot go from this conventional example to the case of isomodules, since in general $\widehat{A}/\widehat{\Im}$ need not coincide with the isotopic lifting of any quotient ring, as we saw in the section on isorings. However, $\widehat{A}/\widehat{\Im}$ does have an $\widehat{A}$-module structure by means of the usual product by an isoscalar, thus the usual differences between concepts and conventional and isotopic properties appear again.

We now turn to consider the procedure for constructing isotopies from an isounit and the isoproduct. Just as in the case of the isovector spaces, such a construction must have some differences in the case of isomodules with respect to the previous isostructures. In fact, since the only difference between an isoring and an isofield is that we can talk about an isoinverse with respect to the second operation, which does not occur in the isorings, and given that this property does not intervene directly in the construction of the isotopic lifting of a vector space or module, the model of constructing both must be entirely analogous. We would therefore arrive, similarly to Proposition 4.1.2 of isovector spaces, at the following:

Proposition 4.3.3 *Let* $(A, \circ, \bullet)$ *be a ring and* $(M, +)$ *an A-module with external product* $\times$. *Let* $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$ *be the isoring with respect to multiplication associated with A, corresponding to the isotopy of principal elements* $\widehat{I}$ *and* $*$ *(of unit element I) and secondary elements* $\widehat{S}$ *and* $\star$ *(of unit element S, with* $\widehat{S}^{-S} = \widehat{R} = R * \widehat{I}$*), in the conditions of Proposition 3.4.2. Let* $\square$ *(of unit element I) be* $\widehat{S'}$ *and* $\diamond$ *(of unit element S', with* $\widehat{S'}^{-S'} = \widehat{R'} = R' \square \widehat{I}$*) be isotopic elements that, together with* $\widehat{I}$*, are in the conditions of Definition 3.1.3,*

the general associated set V' being such that $A \cup U \subseteq V'$. In the conditions, if it is satisfied that

1. *$(M, \diamond)$ has a module structure with respect to the ring $(A, \star, *)$, with external product $\square$,*
2. *$(M, \diamond)$ has a group structure with unit element $S' \in M$,*
3. *$\widehat{a} \widehat{\times} \widehat{R'} = \widehat{R'}, \forall \widehat{a} \in \widehat{K}$ and*
4. *$\widehat{R} \widehat{\times} \widehat{m} = \widehat{R'}$, for all $\widehat{m} \in \widehat{M}$,*

then the isotopic lifting $(\widehat{M}, \widehat{+})$ corresponding to the isotopy of principal elements $\widehat{I}$ and $\square$ and secondary elements $\widehat{S}$ and $\diamond$, by the isoproduct procedure, is an iso-$\widehat{A}$-module for the external product $\widehat{\times}$ (also constructed by the previous isotopy). $\square$

Noting that this proposition has its equivalent in the case that $\widehat{A}$ be an isoring with respect to the sum, we obtain a similar result.

Now let us look at some examples of isomodules. We have already mentioned at the beginning of this section that both isovector spaces and isoideals can be given an isomodule structure. In fact, all of the examples we have seen of isovector spaces are valid examples of isomodules. We will then see a particular case of an isoideal arising from an isomodule, and we will generalize the isotopic model of Example 4.1.3 to the case of isomodules, giving a a concrete example of it:

Example 4.3.4 *Let us consider the ring $(\mathbf{Z}, +, \times)$ and the isoring $(\mathbf{P}, +, \widehat{\times})$ seen in Example 3.4.5 and the ideal $(\mathbf{P}, +, \times)$ and the isoideal $(\mathbf{Z_4}, +, \widehat{\times})$ of Example 3.4.13.*

We know that in the conventional structure, the ideal $(\mathbf{P}, +, \times)$ can be equipped with $\mathbf{Z}$-module structure, taking as an external product the second operation $\times$. We see that, as we have said before, we can do the same with the isoideal $(\mathbf{Z_4}, +, \widehat{\times})$, giving it an iso-$\mathbf{P}$-module structure, with respect to the external product $\widehat{\times}$. Therefore, we will check that the conditions of Proposition 4.3.3 are satisfied.

We will begin by adapting the notation used in Example 3.4.13 to that used for the construction of an isomodule. We should thus have that the isotopic elements used for the construction of the iso-$\mathbf{P}$-module would be the principal

elements $\widehat{I} = 2$ *and* $\square \equiv * \equiv \times$ *and the secondary elements* $\widehat{S'} = 0 = \widehat{S}$ *and* $\diamond \equiv \star \equiv +$ *(of unit element* $S' = 0 = S$*, the unit element with respect to* $\star$*, which is the law used in the construction of the isoring* $(\mathbf{Z}, +, \widehat{\times})$*). In this way, the general set that we will use in the isotopy will be* $V' = \mathbf{Z} = \mathbf{Z} \cup \mathbf{P}$*, which is under the conditions imposed by Proposition 4.3.3.*

Trivially, both the isotopic set and the operations associated with the iso-$\mathbf{P}$*-module must be equal to the corresponding ones associated with the isoideal* $(\mathbf{Z_4}, +, \widehat{\times})$*. We note that the operation* $\widehat{\times}$ *defined in Example 3.4.5 poses no problems with the construction of the isoproduct given for an isomodule (which is equivalent to the one given, in turn, to an isovector space), since it would be* $\widehat{a}\widehat{\times}\widehat{b} = \widehat{a} * \frac{1}{2} * \widehat{b} = \widehat{a} \times \frac{1}{2} \times \widehat{b} = (a \times b) \times 2 = (a\square b)\square 2$*, for all* $\widehat{a} \in \mathbf{P} = \widehat{\mathbf{Z}}_2$ *and for all* $\widehat{b} \in \mathbf{Z_4} = \widehat{\mathbf{P}}_2$*.*

Finally, we see that the following are satisfied:

1. $(\mathbf{P}, \diamond) = (\mathbf{P}, +)$ *has a module structure with respect to the ring* $(\mathbf{Z}, \star, *) = (\mathbf{Z}, +, \times)$*, with external product* $\square \equiv * \equiv \times$ *(since it corresponds precisely to the conventional way of equipping a determined ideal with a module structure).*
2. $(\mathbf{P}, \diamond) = (\mathbf{P}, +)$ *has a group structure with respect to the unit element* $S' = 0 \in \mathbf{P}$*.*
3. *As* $\widehat{R'} = \widehat{S'}^{-S'} = 0^{-0} = 0 = 0 \times 2 = 0\square\widehat{I}$*, we would have that:* $\widehat{a}\widehat{\times}\widehat{R'} = (a \times 2)\widehat{\times}(0 \times 2) = (a \times 0) \times 2 = 0 \times 2 = \widehat{R'}$*, for all* $\widehat{a} \in \mathbf{P} = \widehat{\mathbf{Z}}_2$*.*
4. *As* $\widehat{R} = \widehat{S'}^{-S'} = 0^{-0} = 0 = 0 \times 2 = \widehat{R'}$*, we would have that* $\widehat{R}\widehat{\times}\widehat{m} = (0 \times 2)\widehat{\times}(m \times 2) = (0 \times m) \times 2 = 0 \times 2 = \widehat{R'}$*, for all* $\widehat{m} \in \mathbf{Z_4} = \widehat{\mathbf{P}}_2$*.*

Then applying Proposition 4.3.3, we have that $(\widehat{\mathbf{P}}_2, \widehat{+}) = (\mathbf{Z_4}, +)$ *has an iso-*$\mathbf{P}$*-module structure with respect to the external product* $\widehat{\times}$*. In this way we equip* $(\mathbf{Z_4}, +, \widehat{\times})$ *with a new structure because we remember that we already endowed it with an isoideal structure with respect to the isotopy of the same elements.* $\triangleleft$

We will now generalize to the case of isomodules given in Example 4.1.3. We note that Example 4.3.4 is in fact a concrete case of such a generalization, since the isoring $(\mathbf{P}, +, \widehat{\times})$ associated with $(\mathbf{Z}, +, \times)$ corresponds to the isotopy of principal elements $\widehat{I} = 2 \in \mathbf{Z}$ and $* \equiv \times$

(of unit element 1, the same as that of $(\mathbf{Z}, \times)$) and secondary elements $\widehat{S} = 0$ and $\star \equiv +$ (of unit element $S = 0$, the same as that of $(\mathbf{Z}, +)$); thus, we would be facing an isotopy of the same conditions as the one used for the lifting of the field K of Example 4.1.3. Regarding the isotopic elements used for obtaining the iso-**P**-module $(\mathbf{Z_4}, +, \widehat{\times})$, they are also in the same conditions as those used for obtaining the isovector space $\widehat{U}$ of Example 4.1.3, for we would have as principal elements to $\widehat{I}$ and $\Box \equiv \times$, which is the external product of the starting **Z**-module, $(\mathbf{P}, +, \times)$, which in turn is equivalent to the second operation $\bullet$ of the vector space U of Example 4.1.3, and as secondary elements to $\widehat{S'} = \widehat{S}$ and $\diamond \equiv +$ (which is equivalent to the first operation $\circ$ of the vector space U).

Regarding the conditions (equivalent to those of Proposition 4.1.2) of Proposition 4.3.3, we have already seen in Example 4.3.4 that they occur, so finally the isotopic model of Example 4.1.3 comes to have its equivalent for isomodules.

However, a difference, compared to the model of Example 4.1.3 (which also occurs in Example 4.3.4), is that in general the isotopic set corresponding to the lifting of an A-module M need not correspond with it itself, unlike in the case in the above example, where $\widehat{U} = U$ always. This is due to the fact that in a ring we cannot speak of an inverse element with respect to the second operation, thus it would not be a demonstration similar to the one given in Example 4.1.3, of which $X \in U$, then $X = X \bullet T \bullet \widehat{I} \in \widehat{U}$, if $T = \widehat{I}^{-I} \in A$, as in our case it can be that $T \notin A$.

We finally note that this specific case of Example 4.3.4 can be generalized, not only to the rest of the ideals of a given ring, but to cases in which the module of departure in question is under similar conditions. Let us see an example of the latter:

Example 4.3.5 *Let us consider the ring* $(\mathbf{Z}, +, \times)$ *and the isoring* $(\mathbf{P}, +, \widehat{\times})$ *seen in Example 3.4.5. Let us take, moreover, the* **Z***-module* $(\mathbf{Q}, +)$ *of the rational numbers with the usual sum, with external product* $\bullet$ *(the usual product of the rational numbers). Then, realizing the isotopy of principal el-*

ements $\widehat{I} = 2$ *and* $\square \equiv \bullet$ *and secondary elements* $\widehat{S} = 0$ *and* $\diamond \equiv +$ *(sum of rationals), we would arrive, in a manner analogous to Example 4.3.4, and using Proposition 4.3.3, at the iso-*$\mathbf{P}$*-module* $(\mathbf{Q}, +, \widehat{\bullet})$*, where the isoproduct* $\widehat{\bullet}$ *would be defined according to* $\widehat{m}\widehat{\bullet}\widehat{n} = (m \bullet n) \bullet 2$ *for all* $\widehat{m}, \widehat{n} \in \widehat{\mathbf{Q}}_2 = \mathbf{Q}$*.*
$\triangleleft$

We will now continue our study with the lifting of associated substructures to modules: the submodules. We will give first the definition of such liftings, which will be called *isosubmodules*:

Definition 4.3.6 *Let* $(A, \circ, \bullet)$ *be a ring and* $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$ *an associated isoring. Let* $(M, +)$ *be an* A*-module with external product* $\times$ *and* $(\widehat{M}, \widehat{+})$ *an iso-*$\widehat{A}$*-module with external product* $\widehat{\times}$ *associated with* M*. Let* N *be a submodule of* M*. We say that* $\widehat{N} \subseteq \widehat{M}$ *is an* isosub-$\widehat{A}$-module *of* $\widehat{M}$ *if, being an isotopy of* N*, it has a sub-*$\widehat{A}$*-module structure of* $\widehat{M}$*, i.e., if* $(\widehat{N}, \widehat{+})$ *has an iso-*$\widehat{A}$*-module structure, with external product* $\widehat{\times}$ *(given that we already have that* $\widehat{N} \subseteq \widehat{M}$*).*

We now turn to the model of constructing isotopies by means of an isounit and the isoproduct, making sure the laws associated with the future isosubmodule are the same as those of the starting isomodule. We therefore need that isotopic elements be exactly the same as those used to construct said isomodule. With this we will also obtain that the resulting isotopic set of the lifting of the starting submodule is contained in the one resulting from the lifting of the corresponding module. We still need, however, to give such a set an isomodule structure. For this purpose it would suffice to adapt the conditions of Proposition 4.3.3 to the starting submodule, which in turn has a module structure. We have thus, analogously to said proposition, the following:

Proposition 4.3.7 *Let* $(A, \circ, \bullet)$ *be a ring and* $(M, +)$ *an* A*-module with external product* $\times$*. Let* $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$ *be an isoring associated with* A *and* $(\widehat{M}, \widehat{+})$ *the iso-*$\widehat{A}$*-module with external product* $\widehat{\times}$*, corresponding to the isotopy of elements* $\widehat{I}, \widehat{S}, \widehat{S}', *, \star, \square$*, and* $\diamond$*, in the conditions of Proposition 4.3.3. Let* N *be a submodule of* M*. In these conditions, if* $(N, \diamond)$ *has a submodule structure of* $(M, \diamond)$*, both with external product* $\square$ *with respect to the ring* $(A, \star, *)$*,*

$(N, \diamond)$ being a group with unit element $S' \in N$, then the isotopic lifting $(\widehat{N}, \widehat{+})$ coupled with the product $\widehat{\times}$ corresponding to the isotopy of the previously indicated elements has an isosub-$\widehat{A}$-module structure. □

Let us note that we do not have to assume here the rest of the hypotheses of Proposition 4.3.3, given that they hold by the construction itself of $\widehat{M}$, i.e., that $\widehat{N}$ inherits them from $\widehat{M}$.

Let us see an example of an isosubmodule:

Example 4.3.8 *Let us consider the ring $(\mathbf{Z}, +, \times)$, the isoring $(\mathbf{P}, +, \cdot)$, the $\mathbf{Z}$-module $(\mathbf{Q}, +)$ with external product $\bullet$, and the iso-$\mathbf{P}$-module $(\mathbf{Q}, \widehat{+})$ with external product $\widehat{\bullet}$, all given in Example 4.3.5. Let us also take the $\mathbf{Z}$-module $(\mathbf{P}, +)$ with external product $\times$ (which is also a submodule of $\mathbf{Q}$) and the iso-$\mathbf{P}$-module $(\mathbf{P}, +)$ with external product $\widehat{\times}$ of Example 4.3.4. We will then have that, with the notation of these examples, $(\mathbf{P}, \diamond) = (\mathbf{P}, +)$ has a submodule structure of $(\mathbf{Q}, \diamond) = (\mathbf{Q}, +)$, both with external product $\square \equiv \times$, with respect to the ring $(\mathbf{Z}, \star, *) = (\mathbf{Z}, +, \times)$, $(\mathbf{P}, \diamond) = (\mathbf{P}, +)$ being a group with unit element $0 \in \mathbf{P}$.*

Then, by Proposition 4.3.7, the isotopic lifting $(\mathbf{P}, +)$ with external product $\widehat{\times}$, corresponding to the isotopy of elements used for the rest of the designated isostructures, has an isosub-$\mathbf{P}$-module structure. ◁

We will finish this section by giving the definition of the various existing applications among isomodules:

Definition 4.3.9 *Let $(\widehat{A}, \widehat{\circ}, \widehat{\bullet})$ be an isoring and let $(\widehat{M}, \widehat{+})$ and $(\widehat{M'}, \widehat{\triangle})$ be two iso-$\widehat{A}$-modules with respective external products $\widehat{\times}$ and $\widehat{\triangledown}$. A function $f : \widehat{M} \to \widehat{M'}$ is called a* iso-$\widehat{A}$ -module homomorphism *if, $\forall \widehat{a} \in \widehat{A}$ and $\forall \widehat{m}, \widehat{n} \in \widehat{M}$, it is satisfied that:*

1. $f(\widehat{m} \widehat{+} \widehat{n}) = f(\widehat{m}) \widehat{\triangle} f(\widehat{n})$.
2. $f(\widehat{a} \widehat{\times} \widehat{m}) = \widehat{a} \widehat{\triangledown} f(\widehat{m})$.

In addition, as in the other applications already seen above, if f is bijective, it is called an isomorphism*; if $\widehat{M} = \widehat{M'}$, an* endomorphism*. In the latter case, if in addition f is bijective, it is called an* automorphism*.*

We finally note that, given the similarity already mentioned between vector spaces and modules, we can generalize to the case of isomodules all the theory studied in the isotransformations section. It would then be completely analogous to the previous theory, so everything said about isotransformations between isovector spaces can also be applied to the case of isotransformations of isomodules.

Chapter 5
LIE-SANTILLI ISOTHEORY: ISOTOPIC STRUCTURES (III)

We are going to study in this chapter the isotopic lifting of a new structure: algebras. In the first section we will treat *isoalgebras* in general, according to the procedure already used in previous chapters (see [98]). The lifting of associated substructures (the *isosubalgebras*) will also be studied. In the second section, we will study how to lift isotopically a particular type of algebra, Lie algebras, giving rise to the *Lie-Santilli algebras,* being then able to see the improvements Santilli isotheory introduced in Lie theory. In a third section we will explore some types of isotopic Lie isoalgebras, including *isosimples, isosemisimples, isoresolvables, isonilpotents,* and *isofiliforms.*

5.1 Isoalgebras

Definition 5.1.1 *Let $(U, \circ, \bullet, \cdot)$ be an algebra with internal laws $\circ$ and $\cdot$ and with external product $\bullet$ over a field $K(a, +, \times)$. We call an* isoalgebra *any isotopy $\widehat{U}$ of U equipped with two new internal laws $\widehat{\circ}$ and $\widehat{\cdot}$ and with an external isoproduct $\widehat{\bullet}$ over the isofield $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$ satisfying the axioms of the algebra, i.e., such that $\forall \widehat{a}, \widehat{b} \in \widehat{K}$ and $\forall \widehat{X}, \widehat{Y}, \widehat{Z} \in \widehat{U}$, they may satisfy:*

1. *$(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$ has an isovector space structure defined over the isofield $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$.*

2. $(\widehat{a}\widehat{\bullet}\widehat{X})\widehat{\cdot}\widehat{Y} = \widehat{X}\widehat{\cdot}(\widehat{a}\widehat{\bullet}\widehat{Y}) = \widehat{a}\widehat{\bullet}(\widehat{X}\widehat{\cdot}\widehat{Y})$.
3. $\widehat{X}\widehat{\cdot}(\widehat{Y}\widehat{\circ}\widehat{Z}) = (\widehat{X}\widehat{\cdot}\widehat{Y})\widehat{\circ}(\widehat{X}\widehat{\cdot}\widehat{Z})$, *and* $(\widehat{X}\widehat{\circ}\widehat{Y})\widehat{\cdot}\widehat{Z} = (\widehat{X}\widehat{\cdot}\widehat{Z})\widehat{\circ}(\widehat{Y}\widehat{\cdot}\widehat{Z})$.

In the case that the law $\widehat{\cdot}$ *be commutative, i.e., if* $\widehat{X}\widehat{\cdot}\widehat{Y} = \widehat{Y}\widehat{\cdot}\widehat{X}, \forall \widehat{X}, \widehat{Y} \in \widehat{U}$, *we will say that* $\widehat{U}$ *is an* isocommutative isoalgebra.

If $\widehat{\cdot}$ *is associative, i.e., if* $\widehat{X}\widehat{\cdot}(\widehat{Y}\widehat{\cdot}\widehat{Z}) = (\widehat{X}\widehat{\cdot}\widehat{Y})\widehat{\cdot}\widehat{Z}$ *for all* $\widehat{X}, \widehat{Y}, \widehat{Z} \in \widehat{U}$, *we will say that* $\widehat{U}$ *is an* isoassociative isoalgebra.

If $\widehat{X}\widehat{\cdot}(\widehat{Y}\widehat{\cdot}\widehat{Y}) = (\widehat{X}\widehat{\cdot}\widehat{Y})\widehat{\cdot}\widehat{Y}$ *and* $(\widehat{X}\widehat{\cdot}\widehat{X})\widehat{\cdot}\widehat{Y} = \widehat{X}\widehat{\cdot}(\widehat{X}\widehat{\cdot}\widehat{Y})$ *for all* $\widehat{X}, \widehat{Y} \in \widehat{U}$, $\widehat{U}$ *will be called an* isoalternate.

Finally, if $\widehat{S} \in \widehat{U}$ *is the unit element of* $\widehat{U}$ *with respect to the law* $\widehat{\circ}$, *we will say that* $\widehat{U}$ *is an* isodivision isoalgebra *if for all* $\widehat{A}, \widehat{B} \in \widehat{U}$, *with* $\widehat{A} \neq \widehat{S}$, *the equation* $\widehat{A}\widehat{\cdot}\widehat{X} = \widehat{B}$ *always has a solution.*

Let us look at some observations regarding the previous definition. To begin with, we note that, in a way analogous to the conventional case, the isoalgebras are a special case of isovector spaces, for which we define a second internal operation with the designated properties. This second internal operation does not have a unit element, so in the case of using the isotopic lifting with respect to an isounit and the isoproduct, the isounit which is used need not become an isounit of $\widehat{\cdot}$.

On the other hand, we also note that, again in a way analogous to the conventional case, any isoassociative isoalgebra is an isoalternate isoalgebra, but not *vice versa* in general, as the property of isoassociativity implies isoalternancy immediately, but not so in a reciprocal way.

We shall now proceed to the construction of isoalgebras by means of the model taken from an isounit and the isoproduct. Due to the particular nature of the isovector space which the isoalgebras have, it results that the construction of these, following the aforementioned model, is entirely analogous to the construction of isovector spaces, with the only exception that, in addition, we must carry out the isotopic lifting of the second internal operation. Let us take into account for this purpose what was already mentioned, that this operation need not have a unit element nor even its corresponding isotopic lifting; consequently, we cannot speak at all of the inverse element with respect to these operations. For these reasons, when in an isoalgebra we speak of the

main isounit used in its construction, this will refer to the isounit used in the lifting of the isovector space $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$ (following the notations of Definition 5.1.1). In this way, it is clear that when we seek the isotopic lifting of an algebra $(U, \circ, \bullet, \cdot)$ over a field $K(a, +, \times)$, the first thing that must be done is to raise isotopically the vector space $(U, \circ, \bullet)$ to an isovector space $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$ over an isofield $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$, thus resulting in condition (1) of the definition. With this, assuming that they have been used as principal isotopic elements for $\widehat{I}$ and $\square$, the isotopic set $\widehat{U} = \widehat{U}_{\widehat{I}}$ is also already fixed.

For all the above, even if we could make use of a new isounit and a new law (provided they were compatible with the isotopic set $\widehat{U}$ already obtained), the most convenient way to perform the lifting of the external product $\cdot$ will be to define it directly with the isotopic elements for what we already have. It would then suffice to define the product $\widehat{\cdot}$ according to: $\widehat{X}\widehat{\cdot}\widehat{Y} = (X\square\widehat{I})\widehat{\cdot}(Y\square\widehat{I}) = (X \cdot Y)\square\widehat{I} = \widehat{X \cdot Y}$ for all $\widehat{X}, \widehat{Y} \in \widehat{U}$. Note that $\widehat{X}\widehat{\cdot}\widehat{Y} \in \widehat{U}$, since for $\cdot$ being an internal operation, $X \cdot Y \in U$.

Now we should verify that $\widehat{U}$ with this new operation $\widehat{\cdot}$ satisfies the rest of the required conditions in the definition for having an isoalgebra structure. However, as usual, we will have to impose some property for the used elements to achieve these conditions. We are going to prove that together with the conditions of Proposition 4.1.2, it suffices to impose that $(U, \diamond, \square, \cdot)$ also has an algebra structure over the field $K(a, \star, *)$ and that $\widehat{X}\widehat{\cdot}\widehat{R'} = \widehat{R'}\widehat{\cdot}\widehat{X} = \widehat{R'}$, for all $\widehat{X} \in \widehat{U}$.

Indeed, for all $\widehat{a} \in \widehat{K}$ and for all $\widehat{X}, \widehat{Y}, \widehat{Z} \in \widehat{U}$, we have:

1. $(\widehat{a}\widehat{\bullet}\widehat{X})\widehat{\cdot}\widehat{Y} = ((a\square X)\square\widehat{I})\widehat{\cdot}\widehat{Y} = ((a\square X) \cdot Y)\square\widehat{I} = (X \cdot (a\square Y))\square\widehat{I} = \widehat{X}\widehat{\cdot}(\widehat{a}\widehat{\bullet}\widehat{Y}) = (a\square(X \cdot Y))\square\widehat{I} = \widehat{a}\widehat{\bullet}((X \cdot Y)\square\widehat{I}) = \widehat{a}\widehat{\bullet}(\widehat{X}\widehat{\cdot}\widehat{Y})$.
2. $\widehat{X}\widehat{\cdot}(\widehat{Y}\widehat{\circ}\widehat{Z}) = \widehat{X}\widehat{\cdot}((Y \diamond R' \diamond Z)\square\widehat{I}) = (X \cdot (Y \diamond R' \diamond Z))\square\widehat{I} = ((X \cdot Y) \diamond (X \cdot R') \diamond (X \cdot Z))\square\widehat{I} = ((X \cdot Y)\square\widehat{I}) \diamond ((X \cdot R')\square\widehat{I}) \diamond ((X \cdot Z)\square\widehat{I}) = (\widehat{X}\widehat{\cdot}\widehat{Y}) \diamond (\widehat{X}\widehat{\cdot}\widehat{R'}) \diamond (\widehat{X}\widehat{\cdot}\widehat{Z}) = (\widehat{X}\widehat{\cdot}\widehat{Y}) \diamond \widehat{R'} \diamond (\widehat{X}\widehat{\cdot}\widehat{Z}) = (\widehat{X}\widehat{\cdot}\widehat{Y})\widehat{\circ}(\widehat{X}\widehat{\cdot}\widehat{Z})$.
   $(\cdot)(\widehat{X}\widehat{\circ}\widehat{Y})\widehat{\cdot}\widehat{Z} = ((X \diamond R' \diamond Y) \cdot Z)\square\widehat{I} = ((X \cdot Z) \diamond (R' \cdot Z) \diamond (Y \cdot Z))\square\widehat{I} = (\widehat{X}\widehat{\cdot}\widehat{Z}) \diamond (\widehat{R'}\widehat{\cdot}\widehat{Z}) \diamond (\widehat{Y}\widehat{\cdot}\widehat{Z}) = (\widehat{X}\widehat{\cdot}\widehat{Z}) \diamond \widehat{R'} \diamond (\widehat{Y}\widehat{\cdot}\widehat{Z}) = (\widehat{X}\widehat{\cdot}\widehat{Z})\widehat{\circ}(\widehat{Y}\widehat{\cdot}\widehat{Z})$.

We will also have that if U is a commutative algebra, $\widehat{U}$ will be an isocommutative isoalgebra, since $\widehat{X}\widehat{\cdot}\widehat{Y} = (X \cdot Y)\square\widehat{I} = (Y \cdot X)\square\widehat{I} = \widehat{Y}\widehat{\cdot}\widehat{X}$, for all $\widehat{X}, \widehat{Y} \in \widehat{U}$.

If U is associative, $\widehat{U}$ will be isoassociative, since $\widehat{X}\widehat{\cdot}(\widehat{Y}\widehat{\cdot}\widehat{Z}) = \widehat{X}\widehat{\cdot}((Y \cdot Z)\square\widehat{I}) = (X \cdot (Y \cdot Z))\square\widehat{I} = ((X \cdot Y) \cdot Z)\square\widehat{I} = ((X \cdot Y)\square\widehat{I})\widehat{\cdot}\widehat{Z} = (\widehat{X}\widehat{\cdot}\widehat{Y})\widehat{\cdot}\widehat{Z}$, for all $\widehat{X}, \widehat{Y}, \widehat{Z} \in \widehat{U}$.

If U is alternate, $\widehat{U}$ will be isoalternate, since $\widehat{X}\widehat{\cdot}(\widehat{Y}\widehat{\cdot}\widehat{Y}) = \widehat{X}\widehat{\cdot}((Y \cdot Y)\square\widehat{I}) = (X \cdot (Y \cdot Y))\square\widehat{I} = ((X \cdot Y) \cdot Y)\square\widehat{I} = (\widehat{X}\widehat{\cdot}\widehat{Y})\widehat{\cdot}\widehat{Y}$ and $(\widehat{X}\widehat{\cdot}\widehat{X})\widehat{\cdot}\widehat{Y} = ((X \cdot X)\square\widehat{I})\widehat{\cdot}\widehat{Y} = ((X \cdot X) \cdot Y)\widehat{I} = (X \cdot (X \cdot Y))\square\widehat{I} = \widehat{X}\widehat{\cdot}(\widehat{X}\widehat{\cdot}\widehat{Y})$ are satisfied for all $\widehat{X}, \widehat{Y} \in \widehat{U}$.

Finally, so that $\widehat{U}$ would be an isodivision isoalgebra, it will not be enough just that U be a division algebra; rather, it will also be necessary to impose that $\widehat{S} = \widehat{\overrightarrow{0}}$, $\overrightarrow{0}$ being the unit element of U with respect to the operation $\circ$. In this way, given $\widehat{A}, \widehat{B} \in \widehat{U}$, with $\widehat{A} \neq \widehat{\overrightarrow{0}}$, we have that the equation $\widehat{A}\widehat{\cdot}\widehat{X} = \widehat{B}$ always has a solution, since it would be equivalent to $(A \cdot X)\square\widehat{I} = B\square\widehat{I}$, which would have a possible solution for an element $X \in U$ that satisfies $A \cdot X = B$, which we know exists, U being a division algebra.

With all of the above we have shown the following results:

Proposition 5.1.2 *Let $(U, \circ, \bullet, \cdot)$ be an algebra defined over $K(a, +, \times)$. Let $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$ be an isovector space over the isofield $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$, constructed in the conditions of Proposition 4.1.2. If, in addition, it satisfies that $(U, \diamond, \square, \cdot)$ has an algebra structure over the field $K(a, \star, *)$ and for all $\widehat{X} \in \widehat{U}$ we have that $\widehat{X}\widehat{\cdot}\widehat{R'} = \widehat{R'}\widehat{\cdot}\widehat{X} = \widehat{R'}$, then the isotopic lifting $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$, corresponding to the isotopy of principal elements $\widehat{I}$ and $\square$ and secondary elements $\widehat{S}$ and $\diamond$, by means of the isoproduct procedure, has an isoalgebra structure over $\widehat{K}$. Also, said lifting preserves the type of the initial algebra, i.e., if U is a commutative, associative, or alternate algebra, then $\widehat{U}$ we be an isocommutative, isoassociative, or isoalternate isoalgebra, respectively. If, in addition, $\widehat{S} = \widehat{\overrightarrow{0}}$, $\overrightarrow{0}$ being the unit element of U with respect to $\circ$, we have that if U is a division algebra, then $\widehat{U}$ is an isodivision isoalgebra.* $\square$

Another important aspect for the isoalgebras is that of their bases. Because of the particular nature of isoalgebras and isovector spaces, all the theory regarding bases and isobases studied for the latter serves for the new isostructure. In particular, we would thus have, since the isotopic model given in Example 4.1.3 is fully compatible for the case of isoalgebras, that, in case of using such a model, the isotopy obtained also possesses the property of preserving bases, in the sense of carrying bases into bases, completely analogously to the case of isovector spaces.

Making use of the bases of an isoalgebra, the isotopic concept of norm (*isonorm*) and *normed algebra* also appears (see **[128]**):

Definition 5.1.3 *Let* $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ *be an isoalgebra over the isofield* $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$ *in the conditions of Proposition 5.1.2. Let* $\beta = \{\widehat{e_1}, \ldots, \widehat{e_n}\}$ *be a basis of* $\widehat{U}$ *and let* $\widehat{X} \in \widehat{U}$. *If* $X = \sum_{i=1}^{n} \widehat{x_i} \widehat{\times} \widehat{e_i}$ *with regard to* β, $\widehat{x_i} \in \widehat{K}$ *for all* $i \in \{1, \ldots, n\}$, *we define the* isonorm *of* $\widehat{X}$ *in* $\widehat{U}$, *according to:* $\widehat{|\widehat{X}|} = (\sum_{i=1}^{n} x_i \times x_i)^{1/2} * \widehat{I} = (\sum_{i=1}^{n} x_i^2) * \widehat{I} = |X| * \widehat{I} = |\widehat{X}| \in \widehat{U}$ *Also, the isoalgebra* $\widehat{U}$ *will be called* isonormed *if for all* $\widehat{X}, \widehat{Y} \in \widehat{U}$ *and for all* $\widehat{a} \in \widehat{K}$, *the isonorm satisfies the two following conditions:*

1. $\widehat{|\widehat{X} \cdot \widehat{Y}|} = \widehat{|\widehat{X}|} \widehat{\times} \widehat{|\widehat{Y}|} \in \widehat{K}$.
2. $\widehat{|\widehat{a} \bullet \widehat{X}|} = \widehat{|\widehat{a}|} \widehat{\times} \widehat{|\widehat{X}|}$, *where* $\widehat{|\widehat{a}|}$ *represents the isonorm of the element* $\widehat{a}$ *corresponding to the isofield* $\widehat{K}$.

If we are not working with an isotopy under the conditions of Proposition 3.5.2, following the model of Example 4.1.3, we immediately see that if U is a normed algebra, then $\widehat{U}$ is an isonormed isoalgebra, since $\forall \widehat{a} \in \widehat{K}$; and, $\forall \widehat{X}, \widehat{Y} \in \widehat{U}$, we have

1. $\widehat{|\widehat{X} \cdot \widehat{Y}|} = \widehat{|(X \cdot Y) \square \widehat{I}|} = \widehat{|X \cdot Y|} = \widehat{|X| \times |Y|} = \widehat{|X|} \widehat{\times} \widehat{|Y|}$.
2. $\widehat{|\widehat{a} \bullet \widehat{X}|} = \widehat{|(a \square X) \square \widehat{I}|} = \widehat{|\widehat{a \square X}|} = \widehat{|\widehat{a \bullet X}|} = \widehat{|a \bullet X|} = \widehat{|a| \times |X|} = \widehat{|a|} \widehat{\times} \widehat{|X|} = \widehat{|\widehat{a}|} \widehat{\times} \widehat{|\widehat{X}|}$.

Let us then look at some examples of all the above. In the first of these, we are going to prove that we can provide the isoreal numbers with a 1-dimensional isocommutative, isoassociative (and therefore isoalternate), isodivision, and isonormed isoalgebra structure.

Example 5.1.4 *Let $(\mathbf{R}, +, \times)$ be the field of real numbers with the usual sum and product. Considering the product $\times$ as an external product over $\mathbf{R}$ itself, we can give $(\mathbf{R}, +, \times)$ a vector space structure over $\mathbf{R}$ itself. In turn, considering the product $\times$ as a second internal operation, we can give $(\mathbf{R}, +, \times, \times)$ an algebra structure over $\mathbf{R}$ itself, being, in addition, commutative, associative (and therefore alternate), of division, and normed, using the conventional operations.*

Realizing now the isotopy of the field $(\mathbf{R}, +, \times)$ of principal elements $\widehat{I} \in \mathbf{R}$ and $* \equiv \times$ and secondary elements $\widehat{S} = 0$ and $\star \equiv +$, we would obtain $(\widehat{\mathbf{R}}_{\widehat{I}}, +, \widehat{\times})$, where $\widehat{\mathbf{R}}_{\widehat{I}} = \{a \times \widehat{I} \mid a \in \mathbf{R}\} = \mathbf{R}$, $\widehat{\times}$ being defined according to: $\widehat{a}\widehat{\times}\widehat{b} = (a \times b) \times \widehat{I}$, for all $a, b \in \mathbf{R}$.

On the other hand, taking the elements $\square \equiv \times$, $\widehat{S'} = 0$, and $\diamond \equiv +$, it would result that $(\mathbf{R}, \diamond, \square, \times) = (\mathbf{R}, +, \times, \times)$ has an algebra structure over $(\mathbf{R}, \star, *) = (\mathbf{R}, +, \times)$, $(\mathbf{R}, \diamond, \square)$ then having a vector space structure over the same field, and $(\mathbf{R}, \diamond) = (\mathbf{R}, +)$ a group structure with unit element $S' = 0$, which coincides with the unit element of $\mathbf{R}$ with respect to $\star$. In this way $\widehat{R} = \widehat{S}^{-0} = 0 = 0 \times \widehat{I} = \widehat{0} = \widehat{S'}^{-0} = \widehat{R'}$. Also, for all $\widehat{a}, \widehat{X} \in \widehat{\mathbf{R}} = \mathbf{R}$ we have:

1. $\widehat{a}\widehat{\times}\widehat{R'} = \widehat{a}\widehat{\times}\widehat{0} = (a \times 0) \times \widehat{I} = 0 \times \widehat{I} = \widehat{R'}$.
2. $\widehat{R}\widehat{\times}\widehat{a} = \widehat{0}\widehat{\times}\widehat{a} = \widehat{0} = \widehat{R}$.
3. $\widehat{X}\widehat{\times}\widehat{R'} = \widehat{X}\widehat{\times}\widehat{0} = \widehat{0} = \widehat{R'} = \widehat{0}\widehat{\times}\widehat{X} = \widehat{R'}\widehat{\times}\widehat{X}$.

Therefore, as we have the necessary hypotheses to be able to apply Proposition 5.1.2, we arrive at that the isotopic lifting $(\widehat{\mathbf{R}}_{\widehat{I}}, \widehat{+}, \widehat{\times}, \widehat{\times}) = (\mathbf{R}, +, \widehat{\times}, \widehat{\times})$, corresponding to the isotopy with the aforementioned elements, has an isoalgebra structure over the isofield $(\mathbf{R}, +, \widehat{\times})$. Also, given the characteristics of the starting algebra $(\mathbf{R}, +, \times, \times)$, the obtained isoalgebra is isocommutative and isoassociative (and therefore isoalternate).

On the other hand, given that $\widehat{S} = 0 = \widehat{0}$ (0 being the unit element of $\mathbf{R}$ with respect to $+$) and that the algebra was a division one, the resulting isoalgebra is also of isodivision.

Also, given that to obtain the vector isospace $(\mathbf{R}, +, \widehat{\times})$ what has been done is to follow the isotopic model of Example 4.1.3, we get that by the indicated

observation after Definition 5.1.3, and by the starting algebra being a normed algebra, the isoalgebra $(\mathbf{R}, +, \widehat{\times}, \widehat{\times})$ *is isonormed.*

Finally, as the isotopy utilized follows the model of Example 4.1.3, we also have that our isotopic lifting retains bases and therefore dimensions, thus the starting algebra $\mathbf{R}$ *being* 1*-dimensional, the isoalgebra obtained will also be of this dimension. Moreover, as a basis for the initial algebra would be* $\beta = \{1\}$ *and taking into account that* $\widehat{1} = 1 * \widehat{I} = 1 \times \widehat{I} = \widehat{I}$*, we would reach that an isobasis associated with* β *would be* $\widehat{\beta} = \{\widehat{I}\}$. $\triangleleft$

Example 5.1.5 *Let us consider the algebra* $(M_{n\times n}(\mathbf{R}), +, \bullet, \cdot)$ *defined over the field* $(\mathbf{R}, +, \times)$ *of the real* $(n \times n)$*-dimensional matrices (with the usual matrix sum and product,* $+$ *and* $\cdot$ *and the usual product of a matrix with a scalar* $\bullet$*). Let us now consider the isovector space* $(M_{n\times n}(\mathbf{R}), +, \widehat{\bullet})$ *over the isofield* $(\mathbf{R}, +, \widehat{\times})$ *given in Example 4.1.4, adapted to matrices of dimension* $(n\times n)$*, in place of* $(m\times n)$*. We will then have that, with the notation of said example,* $(M_{n\times n}(\mathbf{R}), \diamond, \Box, \cdot) = (M_{n\times n}(\mathbf{R}), +, \bullet, \cdot)$ *is an algebra over the field* $(\mathbf{R}, \star, *) = (\mathbf{R}, +, \times)$.

We now define the operation $\widehat{\cdot}$ *according to:* $\widehat{A}\widehat{\cdot}\widehat{B} = (A \cdot B)\Box 2 = (A \cdot B) \bullet 2 \in M_{n\times n}(\mathbf{R})$*, for all* $A, B \in M_{n\times n}(\mathbf{R})$.

Then, as $\widehat{R'} = \widehat{S'}^{-S'} = \mathbf{0}^{-\mathbf{0}} = \mathbf{0} = \mathbf{0} \bullet 2 = \widehat{\mathbf{0}}$ *(the null* $(n \times n)$*-dimensional matrix), we would have that* $\widehat{A}\widehat{\cdot}\widehat{R'} = \widehat{A}\widehat{\cdot}\widehat{\mathbf{0}} = (A \cdot \mathbf{0}) \bullet 2 = \mathbf{0} \bullet 2 = \mathbf{0} = \widehat{R'} = \widehat{R'}\widehat{\cdot}\widehat{A}$*, for all* $\widehat{A} \in \widehat{M_{n\times n}(\mathbf{R})_2} = M_{n\times n}(\mathbf{R})$.

Therefore, since we have the necessary conditions to be able to apply Proposition 5.1.2, it is evident that the isotopic lifting given by $(\widehat{M_{n\times n}(\mathbf{R})_2}, \widehat{+}, \widehat{\bullet}, \widehat{\cdot}) = (M_{n\times n}(\mathbf{R}), +, \widehat{\bullet}, \widehat{\cdot})$*, corresponding to the isotopy of elements previously indicated, is an isoalgebra defined over the isofield* $(\widehat{\mathbf{R}}_2, \widehat{+}, \widehat{\times}) = (\mathbf{R}, +, \widehat{\times})$*, which will also be isoassociative by the initial algebra being associative.* $\triangleleft$

We will then study the isotopic lifting of substructures associated with the algebras: the subalgebras. We will start by giving the definition of isosubalgebra:

Definition 5.1.6 *Let* $(U, \circ, \bullet, \cdot)$ *be an algebra over* $K(a, +, \times)$ *and let* $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ *be an isoalgebra associated with* U*, over the isofield* $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$.

Let $(W, \circ, \bullet, \cdot)$ be a subalgebra of U. We say that $\widehat{W} \subseteq \widehat{U}$ is an isosubalgebra *of $\widehat{U}$ if, being an isotopy of W, $(\widehat{W}, \hat{\circ}, \hat{\bullet}, \hat{\cdot})$ has a subalgebra structure of $\widehat{U}$, i.e., if $(\widehat{W}, \hat{\circ}, \hat{\bullet}, \hat{\cdot})$ has an isoalgebra structure over $\widehat{K}(\hat{a}, \hat{+}, \hat{\times})$ (given that we already have that $\widehat{W} \subseteq \widehat{U}$).*

Let us now pass to the model constructing isotopies by means of an isounit and the isoproduct, making sure that the laws associated with the future isosubalgebra are the same as those associated with the starting isoalgebra. We must therefore use the same isotopic elements as those used to construct $\widehat{U}$, thus arriving in particular at the isotopic set associated with the subalgebra W being contained in the one corresponding to U. In turn, to get that $(\widehat{W}, \hat{\circ}, \hat{\bullet}, \hat{\cdot})$ has an isoalgebra structure, it simply suffices to adapt the conditions of Proposition 5.1.2 to the set W, which in turn has an algebra structure, being a subalgebra of U. We will thus have, analogously to the above proposition, the following:

Proposition 5.1.7 *Let $(U, \circ, \bullet, \cdot)$ be an algebra defined over the field $K(a, +, \times)$. Let $(\widehat{U}, \hat{\circ}, \hat{\bullet}, \hat{\cdot})$ be an isoalgebra associated with U, over the isofield $\widehat{K}(\hat{a}, \hat{+}, \hat{\times})$, associated with U, corresponding to the isotopy of elements $\widehat{I}, \widehat{S}, \widehat{S'}, *, \star, \square$, and $\diamond$, in the conditions of Proposition 5.1.2. In these conditions, if $(W, \diamond, \square, \cdot)$ has a subalgebra structure of $(U, \diamond, \square, \cdot)$ over $K(a, \star, *)$, $(W, \diamond)$ being a group with unit element $S' \in W$, then the isotopic lifting $(\widehat{W}, \hat{\circ}, \hat{\bullet}, \hat{\cdot})$ corresponding to the isotopy of the aforementioned elements,has isosubalgebra structure of $\widehat{U}$ over the isofield $\widehat{K}(\hat{a}, \hat{+}, \hat{\times})$.* $\square$

Note that it is not necessary to require in the previous proposition the rest of the hypotheses that are called for in Proposition 5.1.2 because $\widehat{W}$ would inherit them from $\widehat{U}$.

Here below is an example of an isosubalgebra:

Example 5.1.8 *Let us again consider the algebra $(M_{n\times n}(\mathbf{R}), +, \bullet, \cdot)$, now over the field $(\mathbf{Q}, +, \times)$, and perform the isotopic lifting of elements exactly the same as those of Example 5.1.5. We will then obtain, in this way, the isoalgebra $(M_{n\times n}(\mathbf{R}), +, \hat{\bullet}, \hat{\cdot})$ over the isofield $(\mathbf{Q}, +, \hat{\times})$, where the different isoproducts are defined similarly to those of said example.*

Let us now consider the subalgebra $(M_{n\times n}(\mathbf{Q}),+,\bullet,\cdot)$ *of* $M_{n\times n}(\mathbf{R})$. *With the notation of Example 5.1.5,* $(M_{n\times n}(\mathbf{Q}),\diamond,\Box,\cdot) = (M_{n\times n}(\mathbf{Q}),+,\bullet,\cdot)$ *is a subalgebra of* $(M_{n\times n}(\mathbf{R}),\diamond,\Box,\cdot) = (M_{n\times n}(\mathbf{R}),+,\bullet,\cdot)$ *over* $(\mathbf{Q},\star,*) = (\mathbf{Q},+,\times)$, $(M_{n\times n}(\mathbf{Q}),\diamond) = (M_{n\times n}(\mathbf{Q}),+)$ *then being a group with unit element* $\mathbf{0} \in M_{n\times n}(\mathbf{Q})$ *(which is the unit element of* $(M_{n\times n}(\mathbf{R}),\diamond) = (M_{n\times n}(\mathbf{R}),+)$, *which will be had by Proposition 5.1.7, that the isotopic lifting* $(\widehat{M_{n\times n}(\mathbf{Q})_2},\widehat{+},\widehat{\bullet},\widehat{\cdot}) = (M_{n\times n}(\mathbf{Q}),+,\widehat{\bullet},\widehat{\cdot})$, *corresponding to the isotopy of the aforementioned elements, is an isosubalgebra of* $(M_{n\times n}(\mathbf{R}),+,\widehat{\bullet},\widehat{\cdot})$ *over* $(\mathbf{Q},+,\widehat{\times})$, *with* $\widehat{M_{n\times n}(\mathbf{Q})_2} = M_{n\times n}(\mathbf{Q})$. $\triangleleft$

Analogously to the isostructures already considered above, we will now continue this section with the definition of the various functions existing between isoalgebras:

Definition 5.1.9 *Let* $(\widehat{U},\widehat{\circ},\widehat{\bullet},\widehat{\cdot})$ *and* $(\widehat{U'},\widehat{\triangle},\widehat{\nabla},\widehat{\triangleright})$ *be two isoalgebras over the same isofield* $\widehat{K}(\widehat{a},\widehat{+},\widehat{\times})$. *A function* $f:\widehat{U}\rightarrow\widehat{U'}$ *is called an isoalgebra* homomorphism *if,* $\forall\widehat{X},\widehat{Y}\in\widehat{U}$, *it is verified that:*

1. *f is a homomorphism of isovector spaces restricted to the operations* $\widehat{\circ}$ *and* $\widehat{\bullet}$.
2. $f(\widehat{X}\widehat{\cdot}\widehat{Y}) = f(\widehat{X})\widehat{\triangleright}f(\widehat{Y})$.

The concepts of isomorphism, endomorphism, *and* automorphism *are defined completely analogously as in the case of the isostructures already considered above.*

We will finish this section by mentioning an interesting result on the existence of possible isonormed isoalgebras with the isomultiplicative isounit in the isoreals (see [128]).

Historically, the four types of numbers that constitute associative algebras of respective dimensions 1, 2, 4, and 8 (i.e., the real, complex, quaternion, and octonion numbers, respectively) were obtained as the only solutions of the following equation: $(a_1^2+a_2^2+...+a_n^2)\times(b_1^2+b_2^2+...+b_n^2) = A_1^2+A_2^2+...+A_n^2$, for a certain fixed $n\in\mathbf{N}$, with $A_k = \sum_{r,s=1}^{n} c_{krs}\times a_r\times b_s$, $\forall k\in\{1,\ldots,n\}$, the elements a_r, b_s and

c_{krs} belonging to a field $K(a, +, \times)$, with the usual operations $+$ and $\times$. On the other hand, all possible normed algebras with multiplicative unit in the real numbers are given by algebras of dimension 1 (real), 2 (complex), 4 (quaternions), and 8 (octonions).

Then, if we reformulate our problem under the usual isotopy given by an isounit and the isoproduct, following the model of Example 4.1.3, it would result that:

$$(\widehat{a_1}^{\widehat{2}}\widehat{+}\widehat{a_2}^{\widehat{2}}\widehat{+}\ldots\widehat{+}\widehat{a_n}^{\widehat{2}})\widehat{\times}(\widehat{b_1}^{\widehat{2}}\widehat{+}\widehat{b_2}^{\widehat{2}}\widehat{+}\ldots\widehat{+}\widehat{b_n}^{\widehat{2}}) = \widehat{A_1}^{\widehat{2}}\widehat{+}\widehat{A_2}^{\widehat{2}}\widehat{+}\ldots\widehat{+}\widehat{A_n}^{\widehat{2}},$$

with $\widehat{A_k} = \sum_{r,s=1}^{n} \widehat{c_{krs}}\widehat{\times}\widehat{a_r}\widehat{\times}\widehat{b_s}$, $\forall k \in \{1, \ldots, n\}$ and $a_r, b_s, c_{krs} \in \widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$.

Utilizing the results that are deduced from Example 4.1.3, it results that, if we use the principal isounit for $\widehat{I} \in K$, the previous problem is equivalent to:

$$((a_1^2 + a_2^2 + \ldots + a_n^2) \times (b_1^2 + b_2^2 + \ldots + b_n^2))\widehat{I} = (A_1^2 + A_2^2 + \ldots + A_n^2)\widehat{I},$$

obtaining then as possible solutions of it those coming from the initial problem, isotopically lifted. From this, the following is deduced (see [128]):

Proposition 5.1.10 *All the possible isonormed isoalgebras with isomultiplicative isounit over the isoreals are isoalgebras of dimension* 1 *(isoreals),* 2 *(isocomplex),* 4 *(isoquaternions), and* 8 *(isooctonions).* □

5.2 Lie Isotopic Isoalgebras

In this section we are going to study a particular type of isoalgebra that will already allow us to introduce ourselves directly to the generalization of Lie theory carried out by Santilli. We treat the *Lie isotopic isoalgebras* (see [98]), the isotopic generalization of the so-called Lie algebras, the basic elements of the conventional theory of Lie algebras.

We will start with some prior definitions, pointing out that, an isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ given, we will denote by $-\widehat{X} \in \widehat{U}$ the inverse element $\widehat{X}$ in $\widehat{U}$ with respect to $\widehat{\circ}$, i.e., with respect to the element $-\widehat{X} = \widehat{X}^{-\widehat{S}}$, if $\widehat{S}$ is the isounit of $\widehat{U}$ with respect to $\widehat{\circ}$. Similarly, $-X \in U$ will denote the inverse element of X in an algebra $(U, \circ, \bullet, \cdot)$ with respect to the operation $\circ$.

Definition 5.2.1 *Let $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \cdot)$ be an isoalgebra defined over an isofield $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$. $\widehat{U}$ is called* Lie isotopic *if it satisfies the Lie axioms; i.e., if for all $\widehat{a}, \widehat{b} \in \widehat{K}$ and for all $\widehat{X}, \widehat{Y}, \widehat{Z} \in \widehat{U}$, the following are satisfied:*

1. *$\widehat{\cdot}$ is a bilinear operation, i.e.:*
   a. $((\widehat{a}\widehat{\bullet}\widehat{X})\widehat{\circ}(\widehat{b}\widehat{\bullet}\widehat{Y}))\widehat{\cdot}\widehat{Z} = (\widehat{a}\widehat{\bullet}(\widehat{X}\widehat{\cdot}\widehat{Z}))\widehat{\circ}(\widehat{b}\widehat{\bullet}(\widehat{Y}\widehat{\cdot}\widehat{Z}))$.
   b. $\widehat{X}\widehat{\cdot}((\widehat{a}\widehat{\bullet}\widehat{Y})\widehat{\circ}(\widehat{b}\widehat{\bullet}\widehat{Z})) = (\widehat{a}\widehat{\bullet}(\widehat{X}\widehat{\cdot}\widehat{Y}))\widehat{\circ}(\widehat{b}\widehat{\bullet}(\widehat{X}\widehat{\cdot}\widehat{Z}))$.
2. *$\widehat{\cdot}$ is anticommutative: $\widehat{X}\widehat{\cdot}\widehat{Y} = -(\widehat{Y}\widehat{\cdot}\widehat{X})$.*
3. *Isotopic Jacobi identity (or Jacobi isoidentity): $((\widehat{X}\widehat{\cdot}\widehat{Y})\widehat{\cdot}\widehat{Z})\widehat{\circ}((\widehat{Y}\widehat{\cdot}\widehat{Z})\widehat{\cdot}\widehat{X})\widehat{\circ}((\widehat{Z}\widehat{\cdot}\widehat{X})\widehat{\cdot}\widehat{Y}) = \widehat{S}$ where $\widehat{S}$ is the isounit of $\widehat{U}$ with respect to $\widehat{\circ}$.*

Definition 5.2.2 *An isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ over an isofield $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$ is called* isoreal *(*isocomplex*, respectively) depending on the isofield associated with it. Likewise, we call the* dimension *of a Lie isotopic isoalgebra $\widehat{U}$ the dimension that $\widehat{U}$ has as an isovector space.*

Definition 5.2.3 *If $\{\widehat{e_1}, \ldots, \widehat{e_n}\}$ is an isobasis of $\widehat{U}$, with $\widehat{e_i}\widehat{\cdot}\widehat{e_j} = \sum \widehat{c_{ij}^h}\widehat{\bullet}\widehat{e_h}$, $\forall\, 1 \leq i, j \leq n$, we call the coefficients $\widehat{c_{ij}^h} \in \widehat{K}$* structure isoconstants *or* Maurer-Cartan isoconstants *of the isoalgebra.*

Definition 5.2.4 *An isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ over an isofield $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$ is called* Lie isoadmissible *if, with the commutator product $[.,.]_{\widehat{U}}$ associated with $\widehat{\cdot}$ (defined according to: $[\widehat{X}, \widehat{Y}]_{\widehat{U}} = (\widehat{X}\widehat{\cdot}\widehat{Y}) - (\widehat{Y}\widehat{\cdot}\widehat{X})$, for all $\widehat{X}, \widehat{Y} \in \widehat{U}$), $\widehat{U}$ is a Lie isotopic isoalgebra.*

We will continue our study comparing concepts of Lie isotopic isoalgebras and Lie isoadmissible isoalgebras, defined above, with similar

conventional concepts. If we study the construction of isotopies by means of an isounit and the isoproduct, we observe that in general there does not have to be an isotopic lifting of a Lie or a Lie admissible algebra that is respectively a Lie isotopic algebra or a Lie isoadmissible isoalgebra.

To see this, let us start by fixing an algebra $(U, \circ, \bullet, \cdot)$ defined over a field $K(a, +, \times)$ and an isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ associated with U over the isofield $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$, under an isotopy of principal isounit $\widehat{I}$. In general, we will then have $(-X)\widehat{I} \neq -\widehat{X}$ with $X \in U$, since the unit elements and operations under which we take the corresponding inverse element are distinct. In this way, although U is Lie, we have in general that $\widehat{X}\widehat{\cdot}\widehat{Y} = (X \cdot Y)\widehat{I} = (-(Y \cdot X))\widehat{I} \neq -(\widehat{Y}\widehat{\cdot}\widehat{X})$, thus it does not verify the necessary anticommutativity so that $\widehat{U}$ is a Lie isotopic isoalgebra. The same drawback would present itself in the event that U were Lie admissible, so when lifting to $\widehat{U}$, it was a Lie isoadmissible isoalgebra. We should therefore impose another condition in the isotopic lifting of the algebra U, so we could keep the types of algebras above.

One possibility would consist in imposing that in the isotopic lifting the operation $\widehat{\circ}$ is defined according to: $\widehat{X}\widehat{\circ}\widehat{Y} = (X \circ Y)\widehat{I} = \widehat{X \circ Y}$, for all $X, Y \in U$. In this way, if $\overrightarrow{0} \in U$ is the unit element of U with respect to $\circ$, $\widehat{S} = \widehat{\overrightarrow{0}} \in \widehat{U}$ would be the unit element of $\widehat{U}$ with respect to $\widehat{\circ}$, since given $X \in U$, we would have $\widehat{X}\widehat{\circ}\widehat{\overrightarrow{0}} = (X \circ \overrightarrow{0})\widehat{I} = X\widehat{I} = \widehat{X} = \widehat{\overrightarrow{0}}\widehat{\circ}\widehat{X}$. So, we would therefore reach the sought result that given $X \in U$, then $(-X)\widehat{I} = -\widehat{X}$, since $\widehat{X}\widehat{\circ}((-X)\widehat{I}) = (X - X)\widehat{I} = \overrightarrow{0}\widehat{I} = \widehat{\overrightarrow{0}}$.

Let us then prove that, in fact, the previous condition is sufficient so that if U is a Lie algebra, then $\widehat{U}$ is a Lie isotopic isoalgebra. We will check for this, that the conditions of Definition 5.2.1 are satisfied. To do so, given $\widehat{a}, \widehat{b} \in \widehat{K}$ and $\widehat{X}, \widehat{Y}, \widehat{Z} \in \widehat{U}$, we have

1. $((\widehat{a}\widehat{\bullet}\widehat{X})\widehat{\circ}(\widehat{b}\widehat{\bullet}\widehat{Y}))\widehat{\cdot}\widehat{Z} = ((\widehat{a}\widehat{\bullet}\widehat{X})\widehat{\cdot}\widehat{Z})\widehat{\circ}((\widehat{b}\widehat{\bullet}\widehat{Y})\widehat{\cdot}\widehat{Z}) = (\widehat{a}\widehat{\bullet}(\widehat{X}\widehat{\cdot}\widehat{Z}))\widehat{\circ}(\widehat{b}\widehat{\bullet}(\widehat{Y}\widehat{\cdot}\widehat{Z}))$.
2. $\widehat{X}\widehat{\cdot}((\widehat{a}\widehat{\bullet}\widehat{Y})\widehat{\circ}(\widehat{b}\widehat{\bullet}\widehat{Z})) = (\widehat{X}\widehat{\cdot}(\widehat{a}\widehat{\bullet}\widehat{Y}))\widehat{\circ}(\widehat{X}\widehat{\cdot}(\widehat{b}\widehat{\bullet}\widehat{Z})) = (\widehat{a}\widehat{\bullet}(\widehat{X}\widehat{\cdot}\widehat{Y}))\widehat{\circ}(\widehat{b}\widehat{\bullet}(\widehat{X}\widehat{\cdot}\widehat{Z}))$.
3. $\widehat{X}\widehat{\cdot}\widehat{Y} = (X \cdot Y)\widehat{I} = (-(Y \cdot X))\widehat{I} = -(\widehat{Y}\widehat{\cdot}\widehat{X})$.

4. $((\widehat{X}\widehat{\cdot}\widehat{Y})\widehat{\cdot}\widehat{Z})\widehat{\circ}((\widehat{Y}\widehat{\cdot}\widehat{Z})\widehat{\cdot}\widehat{X})\widehat{\circ}((\widehat{Z}\widehat{\cdot}\widehat{X})\widehat{\cdot}\widehat{Y}) = ((X \cdot Y) \cdot Z) \circ ((Y \cdot Z) \cdot X) \circ ((Z \cdot X) \cdot Y)))\widehat{I} = \widehat{\vec{0}}$

Similarly, maintaining the condition imposed before, if $(U, \circ, \bullet, \cdot)$ is a Lie admissible algebra, $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ will be a Lie isoadmissible algebra, for the commutator will be given by: $[\widehat{X}, \widehat{Y}]_{\widehat{U}} = (\widehat{X}\widehat{\cdot}\widehat{Y}) - (\widehat{Y}\widehat{\cdot}\widehat{X}) = (X \cdot Y - Y \cdot X)\widehat{I} = [X, Y]_U \widehat{I}$, for all $\widehat{X}, \widehat{Y} \in \widehat{U}$, $[.,.]_U$ being the commutator product U associated with $\cdot$. In this way, as $(U, \circ, \bullet, [.,.]_U)$ would be a Lie algebra ($(U, \circ, \bullet, \cdot)$ being a Lie admissible algebra), we would have that $(\widehat{U}, \widehat{\circ}, \widehat{\cdot}, [.,.]_{\widehat{U}})$ would be a Lie isotopic isoalgebra, as we would be in the same conditions of the situation that we have just seen. Finally, we would have by definition that $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ would be a Lie isoadmissible isoalgebra.

With all of the above we have shown the following:

Proposition 5.2.5 *Under the conditions of Proposition 5.1.2 and the supposed operation $\widehat{\circ}$ of the isoalgebra $\widehat{U}$ defined as $\widehat{X}\widehat{\circ}\widehat{Y} = (X \circ Y)\Box\widehat{I}$, for all $X, Y \in U$, we have that if U is a Lie algebra (an Lie admissible algebra, respectively), then $\widehat{U}$ is a Lie isotopic isoalgebra (a Lie isoadmissible isoalgebra, respectively).* □

Therefore, from now on we will assume at all times that the construction of Lie isotopic isoalgebras shall be done according to the model of the previous proposition, because under this model we get the usual properties of any Lie algebra, adapted to a Lie isotopic isoalgebra. Thus, we have:

Proposition 5.2.6 *Let $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ be a Lie isotopic isoalgebra over an isofield $\widehat{K}$, associated with a Lie algebra $(U, \circ, \bullet, \cdot)$. If $\widehat{K}$ is of null isocharacteristic, the following results are satisfied:*

1. *$\widehat{X}\widehat{\cdot}\widehat{X} = \widehat{S}$, for all $\widehat{X} \in \widehat{U}$, where $\widehat{S} = \widehat{\vec{0}}$ is the unit element of $\widehat{U}$ with respect to $\widehat{\circ}$.*
2. *$\widehat{X}\widehat{\cdot}\widehat{S} = \widehat{S}\widehat{\cdot}\widehat{X} = \widehat{S}$, for all $\widehat{X} \in \widehat{U}$.*
3. *If the three isovectors that form a Jacobi isoidentity are equal or proportional, each addend of the isoidentity is null.*

4. *The structure isoconstants $\widehat{U}$ define the isoalgebra and satisfy:*

   a. $\widehat{c_{ij}^h} = -\widehat{c_{ij}^h}$.

   b. $\sum(\widehat{c_{ij}^r}\widehat{c_{rh}^s}\hat{+}\widehat{c_{jh}^r}\widehat{c_{ri}^s}\hat{+}\widehat{c_{hi}^r}\widehat{c_{rj}^s}) = \widehat{0}$, *where $\widehat{0}$ is the unit element of $\widehat{K}$ with respect to $\hat{+}$.*

Proof

Given the anticommutativity of $\hat{\cdot}$, we have that $\widehat{X}\hat{\cdot}\widehat{X} = -\widehat{X}\hat{\cdot}\widehat{X}$ for all $\widehat{X} \in \widehat{U}$, which implies condition (1), for $\widehat{K}$ being of null isocharacteristic.

Now let $\widehat{X} \in \widehat{U}$. Then, for U being a Lie algebra, $\widehat{X}\hat{\cdot}\widehat{S} = (X \cdot S)\widehat{I} = S\widehat{I} = \widehat{S}$, which is condition (2).

To prove (3) let $\widehat{X}, \widehat{Y}, \widehat{Z} \in \widehat{U}$ be such that $\widehat{Y} = \widehat{\lambda}\widehat{X}$, $\widehat{Z} = \widehat{\mu}\widehat{X}$, with $\widehat{\lambda}, \widehat{\mu} \in \widehat{K}$. Let us consider the first addend of the Jacobi isoidentity. Using the bilinearity of $\hat{\cdot}$ and the previous results, we have that $((\widehat{X}\hat{\cdot}\widehat{Y})\hat{\cdot}\widehat{Z}) = ((\widehat{X}\hat{\cdot}(\widehat{\lambda}\widehat{X}))\hat{\cdot}(\widehat{\mu}\hat{\cdot}\widehat{X})) = (\widehat{\lambda}\hat{\times}\widehat{\mu})\hat{\bullet}((\widehat{X}\hat{\cdot}\widehat{X})\hat{\cdot}\widehat{X}) = (\widehat{\lambda}\hat{\times}\widehat{\mu})\hat{\bullet}(\widehat{S}\hat{\cdot}\widehat{X}) = (\widehat{\lambda}\hat{\times}\widehat{\mu})\hat{\bullet}\widehat{S} = \widehat{S}$. For the rest of the addends of the Jacobi isoidentity, the procedure is analogous.

To prove (4) we see in the first place that the structure isoconstants define the isoalgebra. For this, given $\widehat{X} = \sum\widehat{\lambda_i}\widehat{e_i}$, $\widehat{Y} = \sum\widehat{\mu_j}\widehat{e_j}$, two isovectors of $\widehat{U}$, we have $\widehat{X}\hat{\cdot}\widehat{Y} = \sum(\widehat{\lambda_i}\hat{\times}\widehat{\mu_j})\hat{\bullet}(\widehat{e_i}\hat{\cdot}\widehat{e_j}) = \sum((\widehat{\lambda_i}\hat{\times}\widehat{\mu_j})\hat{\times}\widehat{c_{ij}^h}\hat{\bullet}\widehat{e_h}$.

In addition, these isoconstants satisfy:

1. $\widehat{e_i}\hat{\cdot}\widehat{e_j} = \sum\widehat{c_{ij}^h}\widehat{e_h} = -\widehat{e_j}\hat{\cdot}\widehat{e_i} = -\sum\widehat{c_{ji}^h}\widehat{e_h} \Rightarrow \widehat{c_{ij}^h} = -\widehat{c_{ji}^h}$, taking into account the uniqueness of writing in an isobasis of an isovector space.
2. According to the Jacobi isoidentity, we have that

$$\widehat{S} = ((\widehat{e_i}\hat{\cdot}\widehat{e_j})\hat{\cdot}\widehat{e_h})\hat{\circ}((\widehat{e_j}\hat{\cdot}\widehat{e_h})\hat{\cdot}\widehat{e_i})\hat{\circ}((\widehat{e_h}\hat{\cdot}\widehat{e_i})\hat{\cdot}\widehat{e_j}) =$$

$$((\sum\widehat{c_{ij}^r}\widehat{e_r})\hat{\cdot}\widehat{e_h})\hat{\circ}((\sum\widehat{c_{jh}^r}\widehat{e_r})\hat{\cdot}\widehat{e_i})\hat{\circ}((\sum\widehat{c_{hi}^r}\widehat{e_r})\hat{\cdot}\widehat{e_j}) =$$

$$(\sum(\widehat{c_{ij}^r}\hat{\times}\widehat{c_{rh}^p})\hat{\bullet}\widehat{e_p})\hat{\circ}(\sum(\widehat{c_{jh}^r}\hat{\times}\widehat{c_{ri}^p})\hat{\bullet}\widehat{e_p})\hat{\circ}(\sum(\widehat{c_{hi}^r}\hat{\times}\widehat{c_{rj}^p})\hat{\bullet}\widehat{e_p}) =$$

$$\sum((\widehat{c_{ij}^r}\hat{\times}\widehat{c_{rh}^p})\hat{+}(\widehat{c_{jh}^r}\hat{\times}\widehat{c_{ri}^p})\hat{+}(\widehat{c_{hi}^r}\hat{\times}\widehat{c_{rj}^p}))\hat{\bullet}\widehat{e_p},$$

which implies that $(\widehat{c_{ij}^r}\hat{\times}\widehat{c_{rh}^p})\hat{+}(\widehat{c_{jh}^r}\hat{\times}\widehat{c_{ri}^p})\hat{+}(\widehat{c_{hi}^r}\hat{\times}\widehat{c_{rj}^p}) = \hat{0}$, $\forall p \in \{1,\ldots,n\}$, from which the result is deduced. □

Note that as an immediate consequence of this proposition, the operation $\hat{\cdot}$ is isodistributive and not isoassociative.

With respect to the issue of functions, the morphisms between Lie isotopic isoalgebras and isosubalgebras of these as particular cases of definitions 5.1.9 and 5.1.6, respectively, also appear. We will pause some more at the concept of an isoideal of a Lie isotopic isoalgebra, studying the properties that it must satisfy as an isotopic lifting of an ideal of a Lie algebra and seeing some examples:

Definition 5.2.7 *Let $(\hat{U},\hat{\circ},\hat{\bullet},\hat{\cdot})$ be a Lie isotopic isoalgebra over the isofield $\hat{K}(\hat{a},\hat{+},\hat{\times})$, associated with an algebra $(U,\circ,\bullet,\cdot)$ over the field $K(a,+,\times)$. We say that $\hat{\Im}$ is an* isoideal *of $\hat{U}$ if, being the isotopic lifting of an ideal $\Im$ of U, it is an isosubalgebra of $\hat{U}$ such that for all $\hat{X} \in \hat{\Im}$, $\hat{X}\hat{\cdot}\hat{Y} \in \hat{\Im}$, for all $\hat{Y} \in \hat{U}$, i.e., if $\hat{\Im}\hat{\cdot}\hat{U} \subset \hat{\Im}$.*

Here below are some examples of isoideals:

Example 5.2.8 *Any Lie isotopic isoalgebra $\hat{u}$ associated with a Lie algebra U is an isoideal of itself, as $\hat{U}\hat{\cdot}\hat{U} \subset \hat{U}$, $\hat{U}$ being an isosubalgebra of $\hat{U}$ and U an ideal of U, trivially.* ◁

Example 5.2.9 *Under the conditions of Proposition 5.2.6, the set $\{\hat{S}\}$, where $\hat{S}$ is the unit of a Lie isotopic isoalgebra $(\hat{U},\hat{\circ},\hat{\bullet},\hat{\cdot})$ with respect to $\hat{\circ}$, is an isoideal of $\hat{U}$.*

Indeed, in the first place, $\{\hat{S}\}$ is an isosubalgebra of $\hat{U}$, since as has been seen, $\hat{S} = \widehat{\overrightarrow{0}}$, so we have that $(\{\hat{S}\},\hat{\circ},\hat{\bullet},\hat{\cdot})$ is the isotopic lifting of the subalgebra (and ideal) $(\{\overrightarrow{0}\},\circ,\bullet,\cdot)$ of the Lie algebra $(U,\circ,\bullet,\cdot)$ to which $\hat{U}$ is associated. As, in addition, $(\{\hat{S}\},\hat{\circ},\hat{\bullet},\hat{\cdot})$ is clearly a subalgebra of $\hat{U}$, we arrive at it effectively being an isosubalgebra.

It remains now to see that $\{\hat{S}\}\hat{\cdot}\hat{U} \subset \{\hat{S}\}$, but it is evident by result (2) of Proposition 5.2.6. With this, $\{\hat{S}\}$ being an isoideal of $\hat{U}$ is demonstrated. ◁

We call the two previous ideals, $\{\hat{S}\}$ and $\hat{U}$, *principal* isoideals of the isoalgebra $\hat{U}$.

Example 5.2.10 *A third example of isoideals comes from the morphisms between Lie isotopic isoalgebras. In the conventional theory of Lie algebras, we know that if $\Phi : U \to U'$ is a morphism of Lie algebras, then the kernel of the morphism is an ideal of U.*

However, in general, if $\varphi : \widehat{U} \to \widehat{U'}$ is a morphism of Lie isotopic isoalgebras, ker φ *does not have be an isoideal of $\widehat{U}$ (even if it will be an ideal of $\widehat{U}$, considering the algebraic structure of the latter). This is because the condition that is usual in these cases may fail, i.e., although* ker φ *is an ideal, it is not necessarily the ideal of a determined Lie algebra.*

Nevertheless, let us prove that, with the appropriate restrictions, we can accomplish that ker φ *be an isoideal. So, it suffices to restrict ourselves to the conditions of Proposition 5.2.5. So, if we have two Lie isotopic isoalgebras, $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ and $(\widehat{U'}, \widehat{\triangle}, \widehat{\triangledown}, \widehat{\triangleright})$, associated with the Lie algebras $(\widehat{U}, \circ, \bullet, \cdot)$ and $(U', \triangle, \triangledown, \triangleright)$, respectively, and a Lie algebra morphism $\Phi : U \to U'$, we will then be able to define $\widehat{\Phi} : \widehat{U} \to \widehat{U'}$ according to: $\widehat{\Phi}(\widehat{X}) = \widehat{\Phi(X)}$, for all $\widehat{X} \in \widehat{U}$. Then, if $\widehat{S'} = \widehat{\overrightarrow{0}'}$ is the unit element of $\widehat{U'}$ with respect to $\widehat{\triangle}$, we will have that $\widehat{\Phi}(\widehat{X}) = \widehat{S'} \Leftrightarrow \widehat{\Phi(X)} = \widehat{S'} = \widehat{\overrightarrow{0}'} \Leftrightarrow X \in \ker \Phi \Leftrightarrow \widehat{X} \in \widehat{\ker \Phi}$. Therefore, under the conditions of Proposition 5.2.5, $\ker \widehat{\Phi} = \widehat{\ker \Phi}$ is satisfied, and thus $\ker \widehat{\Phi}$ is an isotopy of an ideal of the Lie algebra U. As it also has an ideal structure of $\widehat{U}$, considering $\widehat{\Phi}$ as a morphism between Lie algebras, we arrive at $\ker \widehat{\Phi}$ being an isoideal of $\widehat{U}$, as we wanted to prove.* $\triangleleft$

In order to see a fourth example of an isoideal of a Lie isotopic isoalgebra, we first give the following definition:

Definition 5.2.11 *We call the* center *of a Lie isotopic isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ the set of isovectors $\widehat{X} \in \widehat{U}$ such that $\widehat{X} \widehat{\cdot} \widehat{Y} = \widehat{S}$ for all $\widehat{Y} \in \widehat{U}$, where $\widehat{S}$ is the unit element of $\widehat{U}$ with respect to $\widehat{\circ}$. In what follows, the center will be denoted by* cen $\widehat{U}$.

Example 5.2.12 *We have that* cen $\widehat{U}$ *is an ideal of $\widehat{U}$, considering this latter with algebra structure. However, to make it an isoideal, it is necessary that it be the isotopic lifting of an ideal of a determined Lie algebra.*

However, the model we have been using so far, based on the hypotheses of Proposition 5.2.5, will solve this problem again. Indeed, if $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ is an

isoalgebra associated with the Lie algebra $(U, \circ, \bullet, \cdot)$ *under said hypotheses,* cen U *being the center of the Lie algebra* U*, then* $\widehat{X} \in \text{cen}\,\widehat{U} \Leftrightarrow \widehat{X}\widehat{\cdot}\widehat{Y} = \widehat{X \cdot Y} = \widehat{S} = \widehat{\vec{0}}$*, for all* $\widehat{Y} \in \widehat{U} \Leftrightarrow X \cdot Y = \vec{0}$*, for all* $Y \in U \Leftrightarrow X \in \text{cen}\,U$*. Therefore,* $\text{cen}\,\widehat{U} = \widehat{\text{cen}\,U}$*, from where it is deduced that* cen $\widehat{U}$ *is the isotopic lifting of an ideal of a Lie algebra, which, in turn, implies that* cen $\widehat{U}$ *is an isoideal of* $\widehat{U}$*.* ◁

We can collect all the previous examples into the following:

Proposition 5.2.13 *Under the hypotheses of Proposition 5.2.5, if we have an isoalgebra* $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ *associated with the Lie algebra* $(U, \circ, \bullet, \cdot)$*, then the following sets:* $\widehat{U}$*,* $\{\widehat{S}\}$*, and* cen $\widehat{U}$ *are isoideals of* $\widehat{U}$*. Also, if* $\widehat{\Phi} : \widehat{U} \to \widehat{U'}$ *is a morphism of isoalgebras associated with a morphism* $\Phi : U \to U'$*, of Lie algebras, then* $\ker \widehat{\Phi}$ *is an isoideal of* $\widehat{U}$*.* □

However, a more general result shall be given in the following:

Proposition 5.2.14 *In the hypotheses of Proposition 5.1.7, given a Lie isotopic isoalgebra* $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ *associated with a Lie algebra* $(U, \circ, \bullet, \cdot)$ *and given* $\Im$ *an ideal of* U*, then the corresponding isotopic lifting* $\widehat{\Im}$ *is an isoideal of* $\widehat{U}$*.*

Proof

For $\Im$ being an ideal of U, it will be in particular a subalgebra of U and therefore, to verify the hypotheses of Proposition 5.1.7, we have that $\widehat{\Im}$ is an isosubalgebra of $\widehat{U}$. Also, by construction, given $\widehat{X} \in \widehat{\Im}$, we will have that $\widehat{X}\widehat{\cdot}\widehat{Y} = \widehat{X \cdot Y}$, for all $\widehat{Y} \in \widehat{U}$. Then $X \in \Im$ and $Y \in U$, and for $\Im$ being an ideal of U, we will have that $X \cdot Y \in \Im$, from where $\widehat{X}\widehat{\cdot}\widehat{Y} = \widehat{X \cdot Y} \in \widehat{\Im}$, which is the condition that was missing in order that $\widehat{\Im}$ was an isoideal of $\widehat{U}$, since $\widehat{X}$ was an arbitrary isovector of $\widehat{\Im}$. □

Next, we will see how to get new isoideals from some givens. For this purpose, we first give the following:

Definition 5.2.15 *Let* $(\widehat{U_1}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ *and* $(\widehat{U_2}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ *be two Lie isotopic isoalgebras. The* sum of both *is the set* $\{\widehat{X} = \widehat{X_1}\widehat{\circ}\widehat{X_2} \mid \widehat{X_1} \in \widehat{U_1}$ *and* $\widehat{X_2} \in \widehat{U_2}\}$*.*

We say that the sum is direct *if* $\widehat{U_1} \cap \widehat{U_2} = \{\widehat{S}\}$ *and* $\widehat{U_1}\widehat{\cdot}\widehat{U_2} = \widehat{S}$*. The set of isovectors of the direct sum will be denoted by* $\widehat{U_1} \oplus \widehat{U_2}$*.*

Note that the notation of a direct sum of Lie isotopic isoalgebras is unique, for if $\widehat{X} \in \widehat{U_1} \oplus \widehat{U_2}$ is such that $\widehat{X} = \widehat{X_1} \widehat{\circ} \widehat{X_2} = \widehat{Y_1} \widehat{\circ} \widehat{Y_2}$, with $\widehat{X_1}, \widehat{Y_1} \in \widehat{U_1}$ and $\widehat{X_2}, \widehat{Y_2} \in \widehat{U_2}$, then $\widehat{X_1} - \widehat{Y_1} = \widehat{Y_2} - \widehat{X_2}$, with $\widehat{X_1} - \widehat{Y_1} \in \widehat{U_1}$ and $\widehat{Y_2} - \widehat{X_2} \in \widehat{U_2}$. As $\widehat{U_1} \cap \widehat{U_2} = \{\widehat{S}\}$, they will be $\widehat{X_1} = \widehat{Y_1}$ and $\widehat{Y_2} = \widehat{X_2}$, which shows that the notation is unique.

The following is also verified:

Proposition 5.2.16 *Let $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ be a Lie isotopic isoalgebra. Given two isoideals $\widehat{\Im_1}$ and $\widehat{\Im_2}$ of $\widehat{U}$, it is verified that:*

1. *$\widehat{\Im_1} \cap \widehat{\Im_2}$ is an isoideal of $\widehat{U}$.*
2. *Also, under the hypotheses of Proposition 5.2.5, it is verified that*
   a. *$\widehat{\Im_1} \widehat{\circ} \widehat{\Im_2}$ is an isoideal of $\widehat{U}$.*
   b. *$\widehat{\Im_1} \widehat{\cdot} \widehat{\Im_2}$ is an isoideal of $\widehat{U}$.*

Proof

From conventional Lie theory, we know that giving $\widehat{U}$ a Lie algebra structure and giving $\widehat{\Im_1}, \widehat{\Im_2}$ ideal structures of $\widehat{U}$, then $\widehat{\Im_1} \cap \widehat{\Im_2}$, $\widehat{\Im_1} \widehat{\circ} \widehat{\Im_2}$, $\widehat{\Im_1} \widehat{\cdot} \widehat{\Im_2}$ are ideals of $\widehat{U}$.

It then remains to be seen that they are isotopic liftings of ideals of a determined Lie algebra. Let us suppose, for this purpose, that $\widehat{\Im_j}$ is the isotopic lifting of $\Im_j$, an ideal of a Lie algebra U, which will be associated with $\widehat{U}$, for $j = 1, 2$. Let $\widehat{X} \in \widehat{\Im_1} \cap \widehat{\Im_2}$. Then, $\widehat{X} \in \widehat{\Im_j}$ $(j = 1, 2) \Leftrightarrow X \in \Im_j (j = 1, 2) \Leftrightarrow X \in \Im_1 \cap \Im_2$. Therefore, $\widehat{\Im_1} \cap \widehat{\Im_2} = \widehat{\Im_1 \cap \Im_2}$. As we are also under the hypotheses of Proposition 5.2.5, we will have that, on the one hand, $\widehat{\Im_1} \widehat{\circ} \widehat{\Im_2} = \widehat{\Im_1 \circ \Im_2}$ and on the other, $\widehat{\Im_1} \widehat{\cdot} \widehat{\Im_2} = \widehat{\Im_1 \cdot \Im_2}$, from where it is proved that in all three cases we have isotopic liftings, and therefore new isoideals are obtained. □

Note that to prove the last condition, only the hypotheses of Proposition 5.1.2 were necessary, since indeed Proposition 5.2.5 improves the hypotheses regarding the operation $\widehat{\circ}$, while the first proposition improves only $\widehat{\cdot}$.

We will end this section with the isotopic lifting of the algebras derived from a Lie algebra $(U, \circ, \bullet, \cdot)$, that is, with the isotopic generalization of the set $U \cdot U$. As it is becoming customary, given a Lie isotopic

isoalgebra, we can consider its Lie algebra structure, working with the algebra $\widehat{U}\hat{\cdot}\widehat{U}$, derived from $\widehat{U}$. Now, restricting ourselves to the model of construction of Proposition 5.1.2, considering that $\widehat{U}$ is the isotopic lifting of the Lie algebra $(U, \circ, \bullet, \cdot)$, we will have that $\widehat{U}\hat{\cdot}\widehat{U} = \widehat{U \cdot U}$, and therefore the algebra derived from $\widehat{U}$ is, in turn, an isoalgebra, by being the isotopic lifting of an algebra (which is the algebra derived from U). We can then speak of the *derived isoalgebra* of $\widehat{U}$. We will have, in addition, that the isoalgebra derived from $\widehat{U}$ is an isoideal of $\widehat{U}$, for it is the isotopic lifting of the algebra derived from U, which, in turn, is an ideal of U.

Moreover, it will also interest us to study, analogously to the conventional case, when it is verified that $\widehat{U}\hat{\cdot}\widehat{U} = \widehat{S}$. Thus, we give the following in advance:

Definition 5.2.17 *An isoideal $\widehat{\Im}$ of a Lie isotopic isoalgebra $(\widehat{U}, \hat{\circ}, \hat{\bullet}, \hat{\cdot})$ is* isocommutative *if $\widehat{X}\hat{\cdot}\widehat{Y} = \widehat{S}$, for all $\widehat{X} \in \widehat{\Im}$ and for all $\widehat{Y} \in \widehat{U}$. In turn, a Lie isotopic isoalgebra is called* isocommutative *if, considered as an isoideal, it is isocommutative.*

The following results are verified:

Proposition 5.2.18 *A Lie isotopic isoalgebra is isocommutative if and only if its derived isoalgebra is null.*

Proof

Let $(\widehat{U}, \hat{\circ}, \hat{\bullet}, \hat{\cdot})$ be a Lie isotopic isoalgebra. Then $\widehat{U}$ is isocommutative $\Leftrightarrow \widehat{X}\hat{\cdot}\widehat{Y} = \widehat{S}, \forall \widehat{X}, \widehat{Y} \in \widehat{U} \Leftrightarrow \widehat{U}\hat{\cdot}\widehat{U} = \widehat{S}$. □

Proposition 5.2.19 *Under the hypotheses of Proposition 5.2.5, an isoalgebra $(\widehat{U}, \hat{\circ}, \hat{\bullet}, \hat{\cdot})$ associated with a Lie algebra $(U, \circ, \bullet, \cdot)$ is isocommutative if and only if U is commutative.*

Proof

Under the conditions of Proposition 5.2.5, we know that given $\widehat{X}, \widehat{Y} \in \widehat{U}$, then $\widehat{X}\hat{\cdot}\widehat{Y} = \widehat{X \cdot Y} = \widehat{S} = \widehat{\vec{0}} \Leftrightarrow X \cdot Y = \vec{0}$. Therefore, as $\widehat{X}, \widehat{Y}$ are arbitrary isovectors in U, we have that $\widehat{U}$ is isocommutative $\Leftrightarrow U$ is commutative. □

5.3 Types of Lie isotopic isoalgebras

In this section we will give some examples of Lie isotopic isoalgebras. We will begin studying the *Lie-Santilli algebras* (see [98] and [174]) and continue with the study of certain Lie isotopic isoalgebras, including the *isosimple, isosemisimple, isoresolvable, isonilpotent,* and *isofiliform.*

5.3.1 Lie-Santilli Algebras

Definition 5.3.1 *Let* $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ *be an isoassociative isoalgebra over an isofield* $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$. *The* Lie-Santilli product bracket *with respect to* $\widehat{\cdot}$, $[.,.]_S$, *is the product* $\widehat{U}$ *associated with* $\widehat{\cdot}$, *i.e., the operation defined according to:* $[\widehat{X}, \widehat{Y}]_S = (\widehat{X} \widehat{\cdot} \widehat{Y}) - (\widehat{Y} \widehat{\cdot} \widehat{X})$, *for all* $\widehat{X}, \widehat{Y} \in \widehat{U}$. *The isoalgebra* $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, [.,.]_S)$ *is called the* Lie-Santilli algebra.

We are next going to check that the name given in this definition for the isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, [.,.]_S)$ is not arbitrary. Therefore, we begin observing that $(\widehat{U}, \widehat{\circ}, \widehat{\bullet})$ is a vector space over $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$, for $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ being an isoalgebra over said field. Also, for all $\widehat{a} \in \widehat{K}$ and $\widehat{X}, \widehat{Y}, \widehat{Z} \in \widehat{U}$, we have

1. $[\widehat{a} \widehat{\bullet} \widehat{X}, \widehat{Y}]_S = ((\widehat{a} \widehat{\bullet} \widehat{X}) \widehat{\cdot} \widehat{Y}) - (\widehat{Y} \widehat{\cdot} (\widehat{a} \widehat{\bullet} \widehat{X})) = (\widehat{X} \widehat{\cdot} (\widehat{a} \widehat{\bullet} \widehat{Y})) - ((\widehat{a} \widehat{\bullet} \widehat{Y}) \widehat{\cdot} \widehat{X}) = [\widehat{X}, \widehat{a} \widehat{\bullet} \widehat{Y}]_S = (\widehat{a} \widehat{\bullet} (\widehat{X} \widehat{\cdot} \widehat{Y})) - (\widehat{a} \widehat{\bullet} (\widehat{Y} \widehat{\cdot} \widehat{X})) = \widehat{a} \widehat{\bullet} ((\widehat{X} \widehat{\cdot} \widehat{Y}) - (\widehat{Y} \widehat{\cdot} \widehat{X})) = \widehat{a} \widehat{\bullet} [\widehat{X}, \widehat{Y}]_S$,
2. $[\widehat{X}, \widehat{Y} \widehat{\circ} \widehat{Z}]_S = (\widehat{X} \widehat{\cdot} (\widehat{Y} \widehat{\circ} \widehat{Z})) - ((\widehat{Y} \widehat{\circ} \widehat{Z}) \widehat{\cdot} \widehat{X}) = ((\widehat{X} \widehat{\cdot} \widehat{Y}) \widehat{\circ} (\widehat{X} \widehat{\cdot} \widehat{Z})) - ((\widehat{Y} \widehat{\cdot} \widehat{X}) \widehat{\circ} (\widehat{Z} \widehat{\cdot} \widehat{X})) = (\widehat{X} \widehat{\cdot} \widehat{Y}) \widehat{\circ} (\widehat{X} \widehat{\cdot} \widehat{Z}) - (\widehat{Y} \widehat{\cdot} \widehat{X}) - (\widehat{Z} \widehat{\cdot} \widehat{X}) = ((\widehat{X} \widehat{\cdot} \widehat{Y}) - (\widehat{Y} \widehat{\cdot} \widehat{X})) \widehat{\circ} ((\widehat{X} \widehat{\cdot} \widehat{Z}) - (\widehat{Z} \widehat{\cdot} \widehat{X})) = [\widehat{X}, \widehat{Y}]_S \widehat{\circ} [\widehat{X}, \widehat{Z}]_S$,
3. $[\widehat{X} \widehat{\circ} \widehat{Y}, \widehat{Z}]_S = ((\widehat{X} \widehat{\circ} \widehat{Y}) \widehat{\cdot} \widehat{Z}) - (\widehat{Z} \widehat{\cdot} (\widehat{X} \widehat{\circ} \widehat{Y})) = ((\widehat{X} \widehat{\cdot} \widehat{Z}) \widehat{\circ} (\widehat{Y} \widehat{\cdot} \widehat{Z})) - ((\widehat{Z} \widehat{\cdot} \widehat{X}) \widehat{\circ} (\widehat{Z} \widehat{\cdot} \widehat{Y})) = (\widehat{X} \widehat{\cdot} \widehat{Z}) \widehat{\circ} (\widehat{Y} \widehat{\cdot} \widehat{Z}) - (\widehat{Z} \widehat{\cdot} \widehat{X}) - (\widehat{Z} \widehat{\cdot} \widehat{Y}) = ((\widehat{X} \widehat{\cdot} \widehat{Z}) - (\widehat{Z} \widehat{\cdot} \widehat{X})) \widehat{\circ} ((\widehat{Y} \widehat{\cdot} \widehat{Z}) - (\widehat{Z} \widehat{\cdot} \widehat{Y})) = [\widehat{X}, \widehat{Z}]_S \widehat{\circ} [\widehat{Y}, \widehat{Z}]_S$,

and therefore $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, [.,.]_S)$ is an algebra over $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$. Also, it is a Lie algebra, since for $\widehat{a}, \widehat{b} \in \widehat{K}$ and $\widehat{X}, \widehat{Y}, \widehat{Z} \in \widehat{U}$, the following are satisfied:

4. $[(\widehat{a}\widehat{\bullet}\widehat{X})\widehat{\circ}(\widehat{b}\widehat{\bullet}\widehat{Y}),\widehat{Z}]_S = [\widehat{a}\widehat{\bullet}\widehat{X},\widehat{Z}]_S\widehat{\circ}[\widehat{b}\widehat{\bullet}\widehat{Y},\widehat{Z}]_S = (\widehat{a}\widehat{\bullet}[\widehat{X},\widehat{Z}]_S)\widehat{\circ}(\widehat{b}\widehat{\bullet}[\widehat{Y},\widehat{Z}]_S)$.
5. $[\widehat{X},(\widehat{a}\widehat{\bullet}\widehat{Y})\widehat{\circ}(\widehat{b}\widehat{\bullet}\widehat{Z})]_S = [\widehat{X},\widehat{a}\widehat{\bullet}\widehat{Y}]_S\widehat{\circ}[\widehat{X},\widehat{b}\widehat{\bullet}\widehat{Z}]_S = (\widehat{a}\widehat{\bullet}[\widehat{X},\widehat{Y}]_S)\widehat{\circ}(\widehat{b}\widehat{\bullet}[\widehat{X},\widehat{Z}]_S)$.
6. $[\widehat{X},\widehat{Y}]_S = (\widehat{X}\widehat{\cdot}\widehat{Y}) - (\widehat{Y}\widehat{\cdot}\widehat{X}) = -((\widehat{Y}\widehat{\cdot}\widehat{X}) - (\widehat{X}\widehat{\cdot}\widehat{Y})) = -[\widehat{Y},\widehat{X}]_S$.
7. Jacobi isoidentity:

$$[[\widehat{X},\widehat{Y}]_S,\widehat{Z}]_S\widehat{\circ}[[\widehat{Y},\widehat{Z}]_S,\widehat{X}]_S\widehat{\circ}[[\widehat{Z},\widehat{X}]_S,\widehat{Y}]_S = [(\widehat{X}\widehat{\cdot}\widehat{Y}) - (\widehat{Y}\widehat{\cdot}\widehat{X}),\widehat{Z}]_S\widehat{\circ}$$
$$[(\widehat{Y}\widehat{\cdot}\widehat{Z}) - (\widehat{Z}\widehat{\cdot}\widehat{Y}),\widehat{X}]_S\widehat{\circ}[(\widehat{Z}\widehat{\cdot}\widehat{X}) - (\widehat{X}\widehat{\cdot}\widehat{Z}),\widehat{Y}]_S = (((\widehat{X}\widehat{\cdot}\widehat{Y})-$$
$$(\widehat{Y}\widehat{\cdot}\widehat{X})\widehat{\cdot}\widehat{Z}) - (\widehat{Z}\widehat{\cdot}((\widehat{X}\widehat{\cdot}\widehat{Y}) - (\widehat{Y}\widehat{\cdot}\widehat{X})))\widehat{\circ}(((\widehat{Y}\widehat{\cdot}\widehat{Z}) - (\widehat{Z}\widehat{\cdot}\widehat{Y}))\widehat{\cdot}\widehat{X})-$$
$$(\widehat{X}\widehat{\cdot}((\widehat{Y}\widehat{\cdot}\widehat{Z})-(\widehat{Z}\widehat{\cdot}\widehat{Y})))\widehat{\circ}(((\widehat{Z}\widehat{\cdot}\widehat{X})-(\widehat{X}\widehat{\cdot}\widehat{Z}))\widehat{\cdot}\widehat{Y})-(\widehat{Y}\widehat{\cdot}((\widehat{Z}\widehat{\cdot}\widehat{X})-(\widehat{X}\widehat{\cdot}\widehat{Z}))) =$$
$$(\widehat{X}\widehat{\cdot}\widehat{Y}\widehat{\cdot}\widehat{Z}) - (\widehat{Y}\widehat{\cdot}\widehat{X}\widehat{\cdot}\widehat{Z}) - (\widehat{Z}\widehat{\cdot}\widehat{X}\widehat{\cdot}\widehat{Y})\widehat{\circ}(\widehat{Z}\widehat{\cdot}\widehat{Y}\widehat{\cdot}\widehat{X})\widehat{\circ}(\widehat{Y}\widehat{\cdot}\widehat{Z}\widehat{\cdot}\widehat{X}) - (\widehat{Z}\widehat{\cdot}\widehat{Y}\widehat{\cdot}\widehat{X})-$$
$$(\widehat{X}\widehat{\cdot}\widehat{Y}\widehat{\cdot}\widehat{Z})\widehat{\circ}(\widehat{X}\widehat{\cdot}\widehat{Z}\widehat{\cdot}\widehat{Y})\widehat{\circ}(\widehat{Z}\widehat{\cdot}\widehat{X}\widehat{\cdot}\widehat{Y}) - (\widehat{X}\widehat{\cdot}\widehat{Z}\widehat{\cdot}\widehat{Y}) - (\widehat{Y}\widehat{\cdot}\widehat{Z}\widehat{\cdot}\widehat{X})\widehat{\circ}(\widehat{Y}\widehat{\cdot}\widehat{X}\widehat{\cdot}\widehat{Z}) = \widehat{S},$$

$\widehat{S}$ being the unit element of $\widehat{U}$ with respect to $\widehat{\circ}$).

However, what cannot in principle be ensured is that $(\widehat{U},\widehat{\circ},\widehat{\bullet},[.,.]_S)$ is Lie isotopic, because we in fact do not know even if it is an isoalgebra, since for it $[.,.]_S$ should be the isotopic lifting from the second internal operation of an algebra U. However, this can be solved by restricting ourselves to the conditions of Proposition 5.2.5 which in particular eliminated the problems relating to the inverse elements with respect to the first operations of an algebra and an associated isoalgebra. Under such conditions, if $[.,.]$ represents the Lie product bracket for $\cdot$ in the algebra U (defined according to: $[X,Y] = (X\cdot Y) - (Y\cdot X)$, for all $X,Y \in U$ and therefore coinciding with the commutator product associated with $\cdot$), we have that the Lie-Santilli product bracket is defined according to: $[\widehat{X},\widehat{Y}]_S = (\widehat{X}\widehat{\cdot}\widehat{Y}) - (\widehat{Y}\widehat{\cdot}\widehat{X}) = ((X\cdot Y)-(Y\cdot X))\Box\widehat{I} = [X,Y]\Box\widehat{I} = \widehat{[X,Y]}$, for all $\widehat{X},\widehat{Y} \in \widehat{U}$.

Therefore, we would have that the Lie-Santilli product bracket is nothing more than the isotopic lifting by the model of the isoproduct of the Lie product bracket. If it is imposed in addition that $(U,\circ,\bullet,\cdot)$ be an associative algebra, we have, in a way analogous to the isotopic case

which we have just seen, that $(U, \circ, \bullet, [.,.])$ is a Lie algebra. (Conventionally, U equipped with this new internal operation is represented by U^- and it is called the *commutator algebra* of the associative algebra U.) On the other hand, where U is a Lie admissible algebra, we also would arrive, by the definition of Lie admissibility, at $(U, \circ, \bullet, [.,.])$ being a Lie algebra.

Therefore, in either of the previous two cases, we would finally arrive at $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, [.,.]_S)$ being an isoalgebra on the isofield $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$, corresponding to the isotopic lifting of $(U, \circ, \bullet, [.,.])$ over the field $K(a, +, \times)$, under an isotopy in the conditions of Proposition 5.2.5. As the necessary conditions to be a Lie isotopic isoalgebra (verifying the axioms of Definition 5.2.1 for being a Lie algebra) are satisfied, the following result is finally proved:

Proposition 5.3.2 *Let $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ be an isoassociative isoalgebra over an isofield $\widehat{K}(\widehat{a}, \widehat{+}, \widehat{\times})$, associated with an algebra $(U, \circ, \bullet, \cdot)$ defined over the field $K(a, +, \times)$, under the conditions of Proposition 5.2.5. Then the Lie-Santilli algebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, [.,.]_S)$, corresponding to the previous isoalgebra, is a Lie isotopic isoalgebra if the algebra U is either associative or Lie admissible. In this case, the Lie-Santilli algebra will be an isotopic lifting associated with the Lie algebra $(U, \circ, \bullet, [.,.])$.* □

Note also that under the conditions of Proposition 5.2.5, if U is a commutative algebra and $\overrightarrow{0}$ is the unit element of U with respect to $\circ$, then the Lie-Santilli product bracket is constant and equal to $\widehat{\overrightarrow{0}}$, since, for all $X, Y \in U$, we have that $[\widehat{X}, \widehat{Y}]_S = ((X \cdot Y) - (Y \cdot X)) \Box \widehat{I} = ((X \cdot Y) - (X \cdot Y)) \Box \widehat{I} = \overrightarrow{0} \Box \widehat{I} = \widehat{\overrightarrow{0}}$. It is then also corroborated that $\widehat{U}$ would be an isocommutative isoalgebra (as Proposition 5.1.2 indicates, whose conditions we are under); then, for all $X, Y \in U$, we would have that $[\widehat{X}, \widehat{Y}]_S = \widehat{\overrightarrow{0}} = ((Y \cdot X) - (Y \cdot X)) \Box \widehat{I} = ((Y \cdot X) - (X \cdot Y)) \Box \widehat{I} = [\widehat{Y}, \widehat{X}]_S$.

On the other hand, if under the conditions of Proposition 5.3.2 we follow the isotopic model given in Example 4.1.3, it should be noted that the isotopic lifting $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, [.,.]_S)$ retains the structure constants of the algebra $(U, \circ, \bullet, [.,.])$ (see **[174]**) in the following sense: as this

isotopic model preserves the systems of generators, then it indeed preserves the bases, if $\{X_k\}_{k=1,\ldots,n}$ is a generator system of U; then $\{\widehat{X_k} = X_k \Box \widehat{I}\}_{k=1,\ldots,n}$ will be a generator system of $\widehat{U}$. Then, these two generators fixed, $X_i, X_j \in U$ such that $[X_i, X_j] = (c^1_{i,j} \bullet X_1) \circ \ldots \circ (c^n_{i,j} \bullet X_n)$ with $c^k_{i,j} \in K$ for all $k \in \{1, \ldots, n\}$, we have that $[\widehat{X_i}, \widehat{X_j}]_S = [X_i, X_j] \Box \widehat{I} = ((c^1_{ij} \bullet X_1) \circ \ldots \circ (c^n_{ij} \bullet X_n)) \Box \widehat{I} = (\widehat{c^1_{ij}} \widehat{\bullet} \widehat{X_1}) \widehat{\circ} \ldots \widehat{\circ} (\widehat{c^n_{ij}} \widehat{\bullet} \widehat{X_n}) = \widehat{(c^1_{ij} \bullet X_1) \circ \ldots \circ (c^n_{ij} \bullet X_n)}$. Therefore, if $\{c^k_{i,j}\}_{k=1,\ldots,n}$ are the structure constants associated with the generators X_i and X_j of U, then the structure isoconstants associated with the generators $\widehat{X_i}$ and $\widehat{X_j}$ of $\widehat{U}$ are $\{\widehat{c^k_{i,j}} = c^k_{i,j} * \widehat{I}\}_{k=1,\ldots,n}$.

Let us next look at an example of a Lie-Santilli algebra:

Example 5.3.3 *Let us consider the algebra $(M_{n\times n}(\mathbf{R}), +, \bullet, \cdot)$ defined over the field $(\mathbf{R}, +, \times)$ and the isoalgebra $(M_{n\times n}(\mathbf{R}), +, \widehat{\bullet}, \widehat{\cdot})$ defined over the isofield $(\mathbf{R}, +, \times)$ given in Example 5.1.5. As the isotopy followed is the one given in this example, which satisfies the hypotheses of Proposition 5.1.2, the lifting $\widehat{+} \equiv +$ also being defined according to: $\widehat{A} \widehat{+} \widehat{B} = (A+B) \bullet 2$, for all $A, B \in M_{n\times n}(\mathbf{R})$, we find ourselves in the conditions of Proposition 5.2.5.*

As, in addition, the algebra $(M_{n\times n}(\mathbf{R}), +, \bullet, \cdot)$ is associative, we have by Proposition 5.1.2 that the isoalgebra $(M_{n\times n}(\mathbf{R}), +, \widehat{\bullet}, \widehat{\cdot})$ is isoassociative, and therefore $(M_{n\times n}(\mathbf{R}), +, \widehat{\bullet}, [.,.]_S)$ is a Lie-Santilli algebra, the Lie-Santilli product bracket being defined according to: $[\widehat{A}, \widehat{B}]_S = (\widehat{A} \widehat{\cdot} \widehat{B}) - (\widehat{B} \widehat{\cdot} \widehat{A}) = ((A \cdot B) - (B \cdot A)) \bullet 2$, for all $A, B \in M_{n\times n}(\mathbf{R})$.

Also, by $(M_{n\times n}(\mathbf{R}), +, \bullet, \cdot)$ being associative, Proposition 5.3.2 ensures that $(M_{n\times n}(\mathbf{R}), +, \widehat{\bullet}, [.,.]_S)$ is a Lie isotopic isoalgebra over $(\mathbf{R}, +, \widehat{\times})$ (associated with the commutator algebra $(M_{n\times n}(\mathbf{R}), +, \bullet, [.,.])$, where $[.,.]$ denotes the Lie product bracket with respect to the operation $\cdot$). $\triangleleft$

5.3.2 Isosimple and Isosemisimple Lie Isotopic Isoalgebras

We will next study the isotopic lifting of simple and semisimple Lie algebras. Let us begin with some preliminary definitions:

Definition 5.3.4 *A Lie isotopic isoalgebra $\widehat{U}$ is called* isosimple *if an isotopy of a Lie algebra is simple, it also satisfies that it is not isocommutative and that the only isoideals that it contains are the trivial ones. Similarly, $\widehat{U}$ is called* isosemisimple *if an isotopy of a Lie algebra being semisimple, it does not contain non-trivial isocommutative isoideals.*

From the definition above and given that any semisimple Lie algebra is a simple Lie algebra, we will have that any isosemisimple Lie isotopic isoalgebra is an isosimple Lie isotopic isoalgebra. In addition, we have that if $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ is an isosimple Lie isotopic isoalgebra, then $\widehat{U}\widehat{\cdot}\widehat{U} = \widehat{U}$; then we have already seen that the derived isoalgebra is an isoideal of $\widehat{U}$, which for it not being commutative, cannot be $\widehat{U}\widehat{\cdot}\widehat{U} = \{\widehat{S}\}$ and therefore, necessarily, $\widehat{U}\widehat{\cdot}\widehat{U} = \widehat{U}$, this being the other isoideal existing in $\widehat{U}$.

The next step is to study when we can be assured that a Lie isotopic isoalgebra of Lie is isosimple or isosemisimple, depending on the algebra from which has been lifted. A first attempt would be to see what happens in the conditions of Proposition 5.2.5.

Let us then suppose an isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ associated with a Lie algebra $(U, \circ, \bullet, \cdot)$. If we assume that U is a simple Lie algebra, then U is not commutative and the unique ideals that it contains are the trivial ones. Now Proposition 5.2.19 says then that $\widehat{U}$ is not isocommutative. Therefore, so that $\widehat{U}$ is isosimple, it suffices to see that the only isoideals that it contains are the trivial ones. Now, if $\widehat{\Im}$ is an isoideal of $\widehat{U}$, it should be associated with, by definition, an ideal $\Im$ or U, which should be either $\{\overrightarrow{0}\}$ or U itself, U being simple and these being the trivial ideals.

Then, if $\Im = \{\overrightarrow{0}\}$, we have that $\widehat{\Im} = \widehat{\{\overrightarrow{0}\}} = \{\widehat{S}\}$; and if $\Im = U$, then $\widehat{\Im} = \widehat{U}$, so the only isoideals of $\widehat{U}$ are also the trivial ones; thus

we finally arrive at $\widehat{U}$ being an isosimple Lie isotopic isoalgebra. In addition, as the latter reasoning can be done similarly for isosemisimple Lie isotopic isoalgebras, we deduce the following:

Proposition 5.3.5 *In the conditions of Proposition 5.2.5, if $(U, \circ, \bullet, \cdot)$ is a simple (semisimple, respectively) Lie algebra, then the isotopic lifting $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ is an isosimple (isosemisimple, respectively) Lie isotopic isoalgebra.* □

Finally, we are going to prove that, equivalently to what happens in conventional Lie theory, the following are satisfied:

Proposition 5.3.6

1. *Under the conditions of Proposition 5.1.2, any isosemisimple Lie isotopic isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ satisfies $\widehat{U} \widehat{\cdot} \widehat{U} = \widehat{U}$.*
2. *Under the conditions of Proposition 5.2.5, any isosemisimple Lie isotopic isoalgebra is a direct sum of isosimple Lie isotopic isoalgebras.*

Proof

To prove (1), let us consider $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ an isosemisimple Lie isotopic isoalgebra, which is then associated with a semisimple Lie algebra $(U, \circ, \bullet, \cdot)$. According to conventional Lie theory, we know that any semisimple Lie algebra satisfies that its derived algebra coincides with itself. Then, in our case, $U \cdot U = U$. But then, given that we are in the conditions of Proposition 5.1.2, we will have that $\widehat{U} \widehat{\cdot} \widehat{U} = \widehat{U \cdot U} = \widehat{U}$, which demonstrates the result.

To prove (2), let $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ now be an isosemisimple Lie isotopic isoalgebra, which will then be associated with a semisimple Lie algebra $(U, \circ, \bullet, \cdot)$. Also by conventional Lie theory, we know that any semisimple Lie algebra is a direct sum of simple Lie algebras. Let us suppose in the first place that said semisimple algebra is a direct sum of two simple Lie algebras. Then, $U = U_1 \oplus U_2$ is satisfied, with $(U_1, \circ, \bullet, \cdot)$, $(U_2, \circ, \bullet, \cdot)$ being two simple Lie algebras. We then see that $\widehat{U} = \widehat{U_1} \oplus \widehat{U_2}$, which would prove (2). To do this, let us take $\widehat{X} \in \widehat{U}$. Then $X \in U$ and we will be able to write $X = X_1 \circ X_2$, with $X_1 \in U_1$

and $X_2 \in U_2$. As we are still in the conditions of Proposition 5.2.5, we would have that $\widehat{X} = \widehat{X_1 \circ X_2} = \widehat{X_1} \hat{\circ} \widehat{X_2}$, where $\widehat{X_1} \in \widehat{U_1}$ and $\widehat{X_2} \in \widehat{U_2}$. As $\widehat{X}$ is an arbitrary isovector of $\widehat{U}$ and $\widehat{U_1} \cap \widehat{U_2} = \widehat{U_1 \cap U_2} = \widehat{\overrightarrow{0}} = \widehat{S}$ and $\widehat{U_1} \hat{\cdot} \widehat{U_2} = \widehat{U_1 \cdot U_2} = \widehat{\overrightarrow{0}} = \widehat{S}$, we will finally have that $\widehat{U} = \widehat{U_1} \oplus \widehat{U_2}$, which is what we sought, since $\widehat{U_1}$ and $\widehat{U_2}$ are isosimple Lie isotopic isoalgebras by Proposition 5.3.5. In the event that the semisimple algebra is a direct sum of more than two simple Lie algebras, the reasoning would be analogous. □

5.3.3 Irresolvable Lie isotopic isoalgebras

We will now see the isotopic lifting of the resolvable Lie algebras.

Definition 5.3.7 *A Lie isotopic isoalgebra* $(\widehat{U}, \hat{\circ}, \hat{\bullet}, \hat{\cdot})$ *is called* isoresolvable *if, an isotopy of a Lie algebra being resolvable into a sequence*

$$\widehat{U}_1 = \widehat{U}, \quad \widehat{U}_2 = \widehat{U} \hat{\cdot} \widehat{U}, \quad \widehat{U}_3 = \widehat{U}_2 \hat{\cdot} \widehat{U}_2, \ldots, \widehat{U}_i = \widehat{U}_{i-1} \hat{\cdot} \widehat{U}_{i-1}, \ldots$$

(called an isoresolvability succession*), a number n exists such that $\widehat{U}_n = \{\widehat{S}\}$. The least of these numbers is called the* isoresolvability index *of the isoalgebra.*

This same definition is also valid in the case of the isoideals of the isoalgebra.

The following is satisfied:

Proposition 5.3.8 *Under the conditions of Proposition 5.2.5, the isotopic lifting* $(\widehat{U}, \hat{\circ}, \hat{\bullet}, \hat{\cdot})$ *of a resolvable Lie algebra* $(U, \circ, \bullet, \cdot)$ *is an isoresolvable Lie isotopic isoalgebra.*

Proof

It is evident, since by U being resolvable, there exists an $n \in \mathbf{N}$ such that $U_n = \overrightarrow{0}$. But then, by construction, we have that $\widehat{U}_n = \widehat{U_n} = \widehat{\overrightarrow{0}} =$

$\widehat{S}$, which implies that $\widehat{U}$ is irresolvable, in particular as a Lie isotopic isoalgebra which is in the conditions of Proposition 5.2.5. □

Isocommutative Lie isotopic isoalgebras constitute a simple example of an isoresolvable isoalgebra, as they satisfy by definition that $\widehat{U}\widehat{\cdot}\widehat{U} = \widehat{U}_2 = \{\widehat{S}\}$. With it we also have that any non-null isocommutative isoalgebra has an index of isoresolvability 2, the trivial isoalgebra $\{\widehat{S}\}$ being 1.

We will now prove that, as it happens in the conventional theory of Lie, the following results are satisfied:

Proposition 5.3.9 *Let $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ be a Lie isotopic isoalgebra associated with a Lie algebra $(U, \circ, \bullet, \cdot)$. Under the hypotheses of Proposition 5.1.2, the following are satisfied:*

1. *$\widehat{U}_i$ is an isoideal of $\widehat{U}$ and of $\widehat{U}_{i-1}$, for all $i \in \mathbf{N}$.*
2. *If $\widehat{U}$ is isoresolvable, U being resolvable, then any isosubalgebra of $\widehat{U}$ is isoresolvable.*
3. *The intersection and the product of two isoresolvable isoideals of $\widehat{U}$ are isoresolvable isoideals. In addition, under the assumptions of Proposition 5.2.5, the sum of isoresolvable isoideals will also be.*

Proof

To prove (1), following the model of construction of Proposition 5.1.2, we have that $\widehat{U}_i = \widehat{U_i}$ for all $i \in \mathbf{N}$. Now, according to conventional Lie theory, U_i is an ideal of U and of U_{i-1}. Therefore, using Proposition 5.2.14, we arrive at $\widehat{U_i}$ being an isoideal of $\widehat{U}$ and $\widehat{U}_{i-1}$.

To prove (2), let $\widehat{W}$ be an isosubalgebra of $\widehat{U}$ (that will then be associated with a subalgebra W of U). Conventional Lie theory ensures that W is resolvable. Therefore, the reasoning used in the demonstration of Proposition 5.3.8 leads us also to $\widehat{W}$ being isoresolvable, as we wanted to prove. Note that this reasoning can be used even if we are not in the hypotheses of Proposition 5.2.5, since what matters in our particular case is the model of constructing the isoproduct $\widehat{\cdot}$.

To prove (3), Proposition 5.2.16 ensures that in all three proposed cases they become new isoideals. It is sufficient therefore to use again the

reasoning used in the demonstration of Proposition 5.3.8 to deduce that the three new isoideals are also isoresolvable. □

Using this last result (3) we arrive at the sum of all the isoresolvable isoideals of $\widehat{U}$ being another isoresolvable isoideal. This new isoideal is called the *isoradical* of $\widehat{U}$, thus distinguishing it from the radical of $\widehat{U}$ which would be the sum of all the resolvable ideals of $\widehat{U}$. It is denoted by isorad $\widehat{U}$, so as not to confuse it with rad $\widehat{U}$, and it will always contain $\{\widehat{S}\}$, being a trivial isoresolvable isoideal of any isoalgebra. Observe also that, given that any isoresolvable isoideal of $\widehat{U}$ is a resolvable ideal of $\widehat{U}$, we have that isorad $\widehat{U} \subset$ rad $\widehat{U}$. Hence, if $\widehat{U}$ is isoresolvable, then $\widehat{U} =$ isorad $\widehat{U} =$ rad $\widehat{U}$, for in particular $\widehat{U}$ would be resolvable.

We will also need the following:

Proposition 5.3.10 *Under the hypotheses of Proposition 5.2.6, if $\widehat{U}$ is an isosemisimple Lie isotopic isoalgebra, then* isorad $\widehat{U} = \{\widehat{S}\}$.

Proof

In effect, by definition $\widehat{U}$ will be the isotopic lifting of a semisimple Lie algebra U. Then, according to conventional Lie theory, we will have that rad $U = \{\overrightarrow{0}\}$. But as $\{\widehat{S}\} \subset$ isorad $\widehat{U} \subset \widehat{\text{rad}\, U}$, we finally arrive at isorad $\widehat{U} = \{\widehat{S}\}$. □

5.3.4 Isofiliform and isonilpotent Lie isotopic isoalgebras

We will finish this subsection, and with it this introduction to the Lie-Santilli isotheory, by performing the isotopic lifting of nilpotent Lie algebras and of a particular case, the filiform Lie algebras. We start with the definition of *isonilpotent isoalgebras*.

Definition 5.3.11 *A Lie isotopic isoalgebra $(\widehat{U}, \widehat{\circ}, \widehat{\bullet}, \widehat{\cdot})$ is called* isonilpotent *if, being an isotopy of a nilpotent Lie algebra, in the sequence*

$$\widehat{U}^1 = \widehat{U}, \quad \widehat{U}^2 = \widehat{U}\hat{\cdot}\widehat{U}, \quad \widehat{U}^3 = \widehat{U}^2\hat{\cdot}\widehat{U}, \ldots, \widehat{U}^i = \widehat{U}^{i-1}\hat{\cdot}\widehat{U}, \ldots$$

(which is called the isonilpotency sequence*), a natural number n exists such that $\widehat{U}^n = \{\widehat{S}\}$. The least of these natural numbers is called the* isonilpotency index *of the isoalgebra.*

This same definition is also valid in the case of the isoideals of the isoalgebra.

From this definition it follows immediately that any isonilpotent Lie isotopic isoalgebra is isoresolvable, so any nilpotent Lie algebra is resolvable and, in addition, $\widehat{U}_i \subset \widehat{U}^i$ for all $i \in \mathbf{N}$. It must also be evident that any non-null isocommutative Lie isotopic isoalgebra is isonilpotent, of isopotency index 2, 1 being the isopotency index of the isoalgebra $\{\widehat{S}\}$.

The following is also verified:

Proposition 5.3.12 *Under the hypotheses of Proposition 5.2.5, the isotopic lifting $(\widehat{U}, \hat{\circ}, \hat{\bullet}, \hat{\cdot})$ of a nilpotent Lie algebra $(U, \circ, \bullet, \cdot)$ is an isonilpotent Lie isotopic isoalgebra.*

Proof

The same Proposition 5.2.5 already guarantees that $\widehat{U}$ is a Lie isotopic isoalgebra. Then, as for U being nilpotent there exists an $n \in \mathbf{N}$ such that $U^n = \{\vec{0}\}$ using the model of construction utilized in Proposition 5.2.5 that is the usual one of the isoproduct, we will have that $\widehat{U}^n = \widehat{U^n} = \{\widehat{\vec{0}}\} = \{\widehat{S}\}$; thus, we finally arrive at $\widehat{U}$ being isonilpotent. □

On the other hand, adapting the results of conventional Lie theory to this new situation, we will prove the following:

Proposition 5.3.13 *Let $(\widehat{U}, \hat{\circ}, \hat{\bullet}, \hat{\cdot})$ be a Lie isotopic isoalgebra associated with a Lie algebra $(U, \circ, \bullet, \cdot)$. It is then satisfied that:*

1. *Under the hypotheses of Proposition 5.2.5, the sum of two isonilpotent isoideals of $\widehat{U}$ is another isonilpotent isoideal.*
2. *If, in addition, $\widehat{U}$ is isonilpotent, U being nilpotent, then*

a. Under the hypotheses of Proposition 5.1.7, any isosubalgebra of $\widehat{U}$ is isonilpotent.

b. Under the hypotheses of Proposition 5.2.5, if $\widehat{U}$ is non-null isonilpotent, its center is non-null.

Proof

To prove (1), we already have by Proposition 5.2.16 that the sum is a new isoideal. The same reasoning of the demonstration of Proposition 5.3.12 also helps to ensure that this new isoideal is also isonilpotent.

To prove (2.a), let $\widehat{W}$ be an isosubalgebra of $\widehat{U}$ (which will be associated with a subalgebra W of U). As for conventional Lie theory, W is a nilpotent subalgebra of U and as the construction used is that of the model of the isoproduct $\hat{\cdot}$, we arrive, as seen in the demonstration of Proposition 5.3.12, at the sought result.

To prove (2.b), let us suppose $\widehat{U} \neq \{\widehat{S}\}$. Then it must be that $U \neq \{\overrightarrow{0}\}$. One the one hand, conventional Lie theory ensures that $\text{cen}\ U \neq \{\overrightarrow{0}\}$. On the other hand, we have that $\text{cen}\ \widehat{U} = \widehat{\text{cen}\ U}$ and $\{\widehat{\overrightarrow{0}}\} = \{\widehat{S}\}$ by construction, and finally we are working on the basis of an isounit that is invertible with respect to the operation $*$ (which are the isotopic elements with which we have realized the isotopic lifting in question). From all this we deduce that $\text{cen}\ \widehat{U} \neq \{\widehat{S}\}$. □

Analogously to the isoresolvable case, using the previous result (1), we arrive at the sum of all the isonilpotent isoideals of $\widehat{U}$ being another isonilpotent isoideal, which we will denote the *isonilradical* of $\widehat{U}$, to distinguish it from the nil-radical of $\widehat{U}$, the sum of its radical ideals. We will represent it by isonil-rad $\widehat{U}$, to distinguish it from nil-rad $\widehat{U}$, satisfying, in addition, that isonil-rad $\widehat{U} \subset$ nil-rad $\widehat{U} \cap$ isorad $\widehat{U} \subset$ nil-rad $\widehat{U} \subset$ rad $\widehat{U}$.

On the other hand, we can relate an irresolvable Lie isotopic isoalgebra with its isoalgebra derived by means of the following:

Proposition 5.3.14 *Under the assumptions of Proposition 5.2.5, an Lie isotopic isoalgebra is isoresolvable if and only if its derivative isoalgebra is isonilpotent.*

Proof

Let $\widehat{U}$ be an isoresolvable Lie isotopic isoalgebra, which will be associated with a resolvable Lie algebra U. We know, by conventional Lie theory, that this is equivalent to the algebra derived from U being nilpotent. Then, Proposition 5.3.12 ensures that the isoalgebra derived from $\widehat{U}$ is isonilpotent, to coincide by construction with the isotopic lifting of the algebra derived from U, with $\widehat{U}\hat{\cdot}\widehat{U} = \widehat{U \cdot U}$.

Then using this last equality, we have that if the isoalgebra derived from $\widehat{U}$ is isonilpotent, the algebra derived from U, which is an isotopic lifting, should be nilpotent, whereby U should be resolvable and therefore, according to Proposition 5.3.8, $\widehat{U}$ will be isoresolvable. □

We will finish this chapter and thus this text by studying the isotopic lifting of filiform Lie algebras and some notable observations:

Definition 5.3.15 *An isonilpotent Lie isotopic isoalgebra* $(\widehat{U}, \hat{\circ}, \hat{\bullet}, \hat{\cdot})$ *is called* isofiliform *if, being an isotopy of a filiform Lie algebra, it satisfies that* $\dim \widehat{U}^2 = n-2, \ldots, \dim \widehat{U}^i = n-i, \ldots, \dim \widehat{U}^n = 0$, *with* $\dim \widehat{U} = n$.

We note that any theory concerning a filiform Lie algebra U is founded in terms of the basis of the algebra. Thus, departing from a certain basis $\{e_1, \ldots, e_n\}$ of U, preferably an *adapted basis*, we can study the dimensions of U and some of the elements of the nilpotency sequence, the invariants i and j of U and in general the rest of properties originating from the structure coefficients, which are in fact the key elements in the study of filiform Lie algebras.

Now, in an isotopic lifting we have already seen that in general they do not have to retain the elements of the departing basis nor its dimension. Therefore, the study of relations between an isofiliform isoalgebra and a filiform algebra, of which the first was obtained by isotopy, cannot be made in the same way as has been done so far.

On the other hand, we know that if we follow the isotopic construction model of Example 4.1.3, the conservation of the departing basis will be preserved, in the sense already seen at the time regarding the conservation of the structure constants. The latter implies in particular

that filiform algebras of the same dimension cannot be isotopically related by means of this model, but with different structure coefficients. Therefore, if a filiform Lie algebra is fixed, we realize an isotopy conveniently following the model of Example 4.1.3, the isofiliform isoalgebra that is reached will behave at all times in the same way as the initial one, having its same properties.

It would be interesting to relate filiform algebras of different or equal dimensions but with different structure coefficients. The latter can, however, be achieved from the more general standpoint of isotopy theory. It would suffice for this, given two filiform Lie algebras U and U' of respective bases $\{e_1, \ldots, e_n\}$ and $\{e'_1, \ldots, e'_n\}$, to consider U' as the isotopic lifted U, taking $\widehat{U} = U'$, with basis $\{\widehat{e_1} = e'_1, \ldots, \widehat{e_n} = e'_n\}$. This procedure would be an isotopy in the more broadly defined sense, since it is both a mathematical lifting of an initial structure and a filiform Lie algebra resulting in a new structure that verifies the same axioms as the initial one; namely, we arrive at a new filiform Lie algebra.

With this, a particular case of a topic already mentioned for other structures would be demonstrated: that we can consider that at the axiomatic level, all filiform Lie algebras of the same dimension can be isotopically identified, with which we would have in particular that isotopically there is only one type of isofiliform Lie isotopic isoalgebra for every possible dimension.

Finally, this particular case of filiform Lie algebras is due to their constituting the central motive of study of the research group in which this text is presented. However, a more specific development of the lifting of this type of algebra would be outside the framework that we have raised at the beginning of our study, regarding the introduction of Lie-Santilli isotheory. For this reason, what we have just seen are patent improvements that the Lie-Santilli isotheory achieves and the possibility of realizing these improvements more.

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