q-DEFORMED HARMONIC OSCILLATOR IN PHASE SPACE *

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October 1993

Abstract

Relation between Bopp-Kubo formulation and Weyl-Wigner-Moyal symbol calculus, and non-commutative geometry interpretation of the phase space representation of quantum mechanics are studied. Harmonic oscillator in phase space via creation and annihilation operators, both the usual and q-deformed, is investigated. We found that the Bopp-Kubo formulation is just non-commuting coordinates representation of the symbol calculus. The Wigner operator for the q-deformed harmonic oscillator is shown to be proportional to the 3-axis spherical angular momentum operator of the algebra $su_q(2)$. The relation of the Fock space for the harmonic oscillator and double Hilbert space of the Gelfand-Naimark-Segal construction is established. The quantum extension of the classical ergodiicity condition is proposed.

[*]in: Proc. of Intern. Conf. 'Advances in Fundamental Physics', eds. M.Barone and F.Selleri, (Hadronic Press, Palm Harbor 1995) pp.329-348.

1 Introduction

Phase space formulation of quantum mechanics since pioneering papers by Weyl[1], Wigner[2] and Moyal[3] who were motivated by the obvious reason to realize quantum mechanics as some extended version of the Hamiltonian mechanics rather than somewhat sharp step to the theory of operators acting on Hilbert space, now becomes of special interest at least for two major reasons.

The first reason is that the quantum mechanics in phase space represents an example of theory with non-commutative geometry. Moyal[3] and Bayen *et al.*[4] developed non-commutative algebra of functions on phase space which is aimed to represent non-commutative property of the operators. In turn, the operators are sent to functions on phase space - symbols - due to the symbol map[1, 2], which is well defined one-to-one map. So, the symbol calculus[5, 6] provides a reformulation of whole machinery of quantum mechanics in terms of non-commutative functions on phase space.

Also, Bopp[7] and Kubo[8] extended the phase space and introduced non-commutative variables in terms of which they expressed the Wigner operator[2, 9] and the Wigner density operator[8, 10, 11]. The Bopp-Kubo formulation deals with functions on the extended phase space which are also non-commuting due to the non-commutative character of the variables.

Recent studies of the non-commutative phase space[12]-[14] are much in the spirit of modern non-commutative geometry[15]-[19]. Exterior differential calculus in the quantum mechanics in phase space has been proposed recently by Gozzi and Reuter[20, 21]. They studied in detail algebraic properties of the symbol calculus, and have found[22], particularly, quantum analogue of the classical canonical transformations. Gozzi and Reuter have argued that the quantum mechanics in phase space can be thought of as a smooth deformation of the classical one.

Jannussis, Patargias and Brodimas[23] have constructed creation and annihilation operators in phase space, and studied harmonic oscillator in phase space[24]. Various problems related to the Wigner operator, Wigner distribution function, and the density matrix in phase space have been investigated in a series of papers by Jannussis *et al.*[25]-[30].

The second reason of the importance of the quantum mechanics in phase space is that the resulting formalism is very similar to the Hamiltonian formulation of classical mechanics (not surprise certainly).

An obvious advantage of the phase space formulation of quantum mechanics is that it arises to a tempting possibility to exploit this formal similarity, provided by the smooth deformation, to extend some of the useful notions and tools, such as action-angle variables, ergodicity, mixing, Kolmogorov-Sinai entropy, and chaos, which had been elaborated in Hamiltonian mechanics to quantum mechanics. The only thing that one should keep in mind here is that the phase space quantum mechanics deals with the non-commutative symplectic geometry rather than the usual symplectic geometry. So, one should take care of this, primarily because the usual notion of phase space points is lost in non-commutative case so that one is forced to work mostly in algebraic terms rather than to invoke to geometrical intuition. For example, it is not obvious what is an analogue of the Lyapunov exponents when there are no classical trajectories.

However, as a probe in this direction, we attempt to formulate, in this paper, the extention of the classical ergodicity condition.

We should emphasize here that, clearly, it is highly suitable to have at disposal the phase space formulation before going into details of quantum mechanical analogues of the classical chaos and related phenomena.

As to chaos in dynamical sytems, it should be noted that the evolution equations, both in the classical and quantum mechanics in phase space, are Hamiltonian flows, which are deterministic in the sense that there are no source terms of stochasticity. In view of this, chaos can be still thought of as an extreme sensitivity of the long-time behavior of the probability density and, therefore, of the other observables of interest, to initial state. Another fundamental aspect of this consideration is the process of measurements. However, we shall not discuss this problem here.

As a specific example of quantum mechanical system in phase space, we consider, in this paper, one-dimensional harmonic oscillator.

We study also the *q-deformed* oscillator in phase space which is now of special interest in view of the developments of quantum algebras[31]-[35]. We should note here that the quantum algebras are particular cases of the Lie-admissible algebras[36]-[38]. The algebra underlying the properties of the q-oscillator in phase space appears to be the algebra $su_q(2)$[39]-[44].

The paper is organized as follows.

In Sec 2.1, we briefly recall the Bopp-Kubo formulation of quantum mechanics in phase space.

Sec 2.2 is devoted to Weyl-Wigner-Moyal symbol-calculus approach to quantum mechanics the main results of which are sketched.

In Sec 2.3, we discuss, following Gozzi and Reuter[22], modular conjugation and unitary transformations. We show that the Bopp-Kubo formulation and the symbol calculus are explicitly related to each other. We give an interpretation of the quantum mechanics in phase space in terms of non-commutative geometry. Quantum mechanical extention of the classical ergodicity condition is proposed.

In Sec 2.4, we study translation operators in phase space. Commutation relations of the Bopp-Kubo translation operators, both in Hamiltonian and Birkhoffian cases, are presented.

The results presented in Sec 2 are used in Sec 3 to study the harmonic ($q-$)oscillator in phase space.

In Sec 3.1, we present the main properties of the one-dimensional oscillator in terms of annihilation and creation operators in phase space. We identify fundamental $2D$-lattice structure of the phase space resulting from the commutation relations of the Bopp-Kubo translation operators. The Fock space for the oscillator is found to be related to the double Hilbert space of the Gelfand-Naimark-Segal construction.

In Sec 3.2, we study q-deformed harmonic oscillator in phase space. The Wigner operator is found to be proportional to the 3-axis spherical angular momentum operator of the algebra $su_q(2)$. Also, the Wigner density operator appeared to be related to the 3-axis hyperbolical angular momentum operator of the algebra $su_q(1,1) \approx sp_q(2,R)$.

2 Phase space formulation of quantum mechanics

2.1 Bopp-Kubo formulation. Non-commutative coordinates

In studying Wigner representation[2] of quantum mechanics, Bopp[7] and Kubo[8] started from classical Hamiltonian $H(p,q)$ and used the variables (see also [27, 29])

$$P = p - \frac{i\hbar}{2}\frac{\partial}{\partial q}, \quad Q = q + \frac{i\hbar}{2}\frac{\partial}{\partial p} \tag{1}$$

$$P^* = p + \frac{i\hbar}{2}\frac{\partial}{\partial q}, \quad Q^* = q - \frac{i\hbar}{2}\frac{\partial}{\partial p} \tag{2}$$

instead of the usual (p, q) and obtained the Wigner operator, W_-, and the Wigner density operator, W_+, in the following form:

$$W_\pm = H(P, Q) \pm H(P^*, Q^*) \tag{3}$$

These operators enter respectively the Wigner equation

$$i\hbar\partial_t \rho = W_- \rho \tag{4}$$

and the Bloch-Wigner equation[8]

$$\partial_\beta F + \frac{1}{2} W_+ F = 0 \qquad\qquad \beta = \frac{1}{kT} \tag{5}$$

Here, $\rho = \rho(p, q)$ is the Wigner distribution function[2, 30] and
$F = F(p, q, p', q'; \beta)$ is the Wigner density matrix[11].

As it is wellknown, the Wigner equation (4) is a phase space counterpart of the usual von Neumann equation of quantum mechanics while the Bloch-Wigner equation (5) describes quantum statistics in phase space[10, 25].

In view of the definitions (1)-(2) of the variables, it is quite natural to treat the above formulation in terms of non-commutative geometry[18].

With the usual notation, $\phi^i = (p_1, \ldots, p_n, q^1, \ldots, q^n)$, $\phi^i \in M_{2n}$, the first step is to extend the phase space M_{2n} to the (co-)tangent phase space TM_{2n} and define the complex coordinates,

$$\Phi^i_\pm = \phi^i \pm \frac{i\hbar}{2}\omega^{ij}\frac{\partial}{\partial \phi^j} \tag{6}$$

where ω^{ij} is a fundamental symplectic tensor, $\omega^{ij} = -\omega^{ji}$; $\omega_{ij}\omega^{jk} = \delta^k_i$.

We observe immediately that these coordinates are non-commutative,

$$[\Phi^i_\pm, \Phi^j_\pm] = \pm i\hbar\omega^{ij} \qquad [\Phi^i_\pm, \Phi^j_\mp] = 0 \tag{7}$$

and do not mix under time evolution. The natural projection $TM_{2n} \rightarrow M_{2n}$ comes with the classical limit $\hbar \rightarrow 0$.

Commutation relations (7) imply that the "holomorphic" functions, $f(\Phi_-)$, and "anti-holomorphic" functions, $f(\Phi_+)$, form two mutually commuting closed algebras on space of functions $C(TM_{2n})$.

Thus, the holomorphic, $H(\Phi_-)$, and anti-holomorphic, $H(\Phi_+)$, Hamiltonians define two separate dynamics, which are not mixed. Wigner operators (3) are simply sum and difference between these two Hamiltonians, respectively,

$$W_{\pm} = H(\Phi_-) \pm H(\Phi_+) \tag{8}$$

So, physical dynamics comes with the combinations of these two Hamiltonians. In the classical limit, the Wigner operators cover the Liouvillian L and the Hamiltonian,

$$W_- = -i\hbar L + O(\hbar^2) \qquad W_+ = 2H(p,q) + O(\hbar^2) \tag{9}$$

where $L \equiv \ell_h = -h^i \partial_i$ is a Lie derivative along the Hamiltonian vector field $h^i = \omega^{ij}\partial_j H$[45, 46].

According to complex character of the variables (6), one can define the involution $\mathcal{J}$ acting simply as complex conjugation

$$\mathcal{J}: \ \Phi^i_{\pm} \rightarrow \Phi^i_{\mp} \tag{10}$$

This involution may be thought of as a conjugation interchanging the two pieces of the physical dynamics.

2.2 Weyl-Wigner-Moyal formulation. Symbol calculus

In order to achieve phase space formulation of quantum mechanics, Weyl[1] and Wigner[2] introduced symbol map associating with each operator $\hat{A}$, acting on Hilbert space, a symbol $A(\phi)$, function on phase space, $A(\phi) = symb(\hat{A})$, due to

$$A(\phi) = \int \frac{d^{2n}\phi_0}{(2\pi\hbar)^n} exp\Big[\frac{i}{\hbar}\phi_0^i \omega_{ij}\phi^j\Big] Tr(\hat{T}(\phi_0)\hat{A}) \tag{11}$$

with

$$\hat{T}(\phi_0) = exp\Big[\frac{i}{\hbar}\phi_0^i \omega_{ij}\hat{\phi}^j\Big] \tag{12}$$

The symbol map is well defined invertible one-to-one map from space of operators, $\mathcal{O}$, to space of functions depending on phase space coordinates[5, 21], $\mathcal{O} \rightarrow C(M_{2n})$. Particularly, Hermitean operators are mapped to real functions, and vice versa.

The key property of the symbol calculus is that the ordinary pointwise product of the functions is appropriately generalized to reproduce the non-commutative product of the operators. The product on $C(M_{2n})$, making the symbol map an algebraic homomorphism, is the Moyal product[3, 6],

$$\begin{aligned} (A * B)(\phi) &= symb(\hat{A}\hat{B}) \\ &= A(\phi) exp[\frac{i}{2\hbar}\bar{\partial}_i \omega_{ij} \vec{\partial}_j] B(\phi) \\ &= A(\phi)B(\phi) + O(\hbar) \end{aligned} \tag{13}$$

The Moyal product is associative but apparently non-commutative, and represents, in $C(M_{2n})$, non-commutative property of the algebra of operators, and non-local character of quantum mechanics.

The Moyal bracket[3]

$$\begin{aligned} \{A, B\}_{mb} &= symb(\frac{1}{i\hbar}[A, B]), \\ &= \frac{1}{i\hbar}(A * B - B * A) \\ &= \{A, B\}_{pb} + O(\hbar^2) \end{aligned} \tag{14}$$

is a symbol of commutator between two operators, and reduces to the usual Poisson bracket $\{.,.\}_{pb}$ in the classical limit. Thus, the algebra $(C(M_{2n}), \{,\}_{mb})$ is an algebra of quantum observables, and it can be continuously reduced to the algebra $(C(M_{2n}), \{,\}_{pb})$ of classical observables.

Symbol map of the von Neumann's equation is written as[21]

$$\begin{aligned} \partial_t \rho(\phi, t) &= -\{\rho, H\}_{mb} \\ &= -\ell_h \rho + O(\hbar^2) \end{aligned} \tag{15}$$

In the classical limit, this equation covers the Liouville equation of classical mechanics[47, 48].

To summarize, the symbol calculus can be treated as a smooth deformation of classical mechanics linking non-associative Poisson-bracket algebra of classical observables, $A(\phi), \ldots$, and an associative commutator algebra of quantum observables, $\hat{A}, \ldots$. Full details of the symbol calculus may be found in [21, 22] and references therein.

2.3 Chiral symmetry and unitary transformations

Gozzi and Reuter[22] have investigated recently the algebraic properties of the quantum counterpart of the classical canonical transformations using the symbol-calculus approach to quantum mechanics.

They found, particularly, that the operators L_f and R_f acting as the left and right multiplication with symbol f, respectively,

$$L_f g = f * g \qquad R_f g = g * f \tag{16}$$

form two mutually commuting closed algebras, $\mathcal{A}_L$ and $\mathcal{A}_R$, (cf. [26])

$$[L_{f_1}, L_{f_2}] = i\hbar L_{\{f_1 f_2\}_{mb}} \qquad [R_{f_1}, R_{f_2}] = -i\hbar R_{\{f_1 f_2\}_{mb}} \qquad [L_{f_1}, R_{f_2}] = 0 \tag{17}$$

which are explicitly isomorphic to the original Moyal-bracket algebra on $C(M_{2n})$.

Also, L_f and R_f can be presented by virtue of the Moyal product (13) as[22]

$$L_f =: f(\Phi^i_+) : \qquad R_f =: f(\Phi^i_-) : \tag{18}$$

where $\Phi^i_\pm$ are defined by (6), and $: \ldots :$ means normal ordering symbol (all derivatives ∂_i should be placed to the right of all Φ's).

It has been shown[22] that the linear combinations of the above operators,

$$V^\pm_f = \frac{1}{i\hbar}(L_f \pm R_f) \tag{19}$$

for real f, generate non-unitary, $\hat{g} \to \hat{U}\hat{g}\hat{U}$, and unitary, $\hat{g} \to \hat{U}\hat{g}\hat{U^{-1}}$, transformations, respectively ($\hat{U}$ is an unitary operator).

We see that the Wigner operators, $W_\pm$, given in the Bopp-Kubo formulation by (8), are just

$$W_\pm = L_H \pm R_H = i\hbar V^\pm_H \tag{20}$$

so that the Wigner equation (4) can be written as

$$\partial_t \rho = V^-_H \rho \tag{21}$$

where H is the Hamiltonian. So, we arrive at the conclusion that the representations (18) provide the relation between the Bopp-Kubo and Weyl-Wigner-Moyal formulations.

Various algebraic properties of the generators V_f^- have been found by Gozzi and Reuter[22]. Particularly, they found that in two dimensional phase space the generators V_f^-, in the basis $V_{\vec{m}} = -exp(i\vec{m}\vec{\phi})$, $\vec{m} = (m_1, m_2) \in Z^2$, on torus $M_2 = S^1 \times S^1$, satisfy a kind of the W_∞-algebra commutation relations,

$$[V_{\vec{m}}^-, V_{\vec{n}}^-] = \frac{2}{\hbar} sin(\frac{\hbar}{2} m_i \omega^{ij} m_j) V_{\vec{m}+\vec{n}}^- \tag{22}$$

which are deformed version of the w_∞-algebra of the classical $sdiff(T^2)$, area preserving diffeimorphisms on the torus.

Also, an important result shown in [22] is that V_f^- is invariant under the modular conjugation operator defined on symbols by

$$\mathcal{J} f = f^* \qquad \mathcal{J}(f * g) = \mathcal{J}(g) * \mathcal{J}(f) \tag{23}$$

Namely,

$$\mathcal{J} L_f \mathcal{J} = R_f \qquad \mathcal{J} R_f \mathcal{J} = L_f \qquad \mathcal{J} V_f^- \mathcal{J} = V_f^- \qquad \mathcal{J} V_f^+ \mathcal{J} = -V_f^+ \tag{24}$$

This symmetry resembles the *chiral* symmetry and seems to be broken in the classical mechanics. This argument is supported by the fact that the Moyal product (13) becomes commutative in the classical limit. Indeed, in the classical case, the difference between the left and right multiplications on $C(M_{2n})$ disappears so that there is no room for the modular conjugation operator $\mathcal{J}$, and the original algebra $\mathcal{A}_L \otimes \mathcal{A}_R$ is contracted to its diagonal subalgebra[22].

The operator V_f^+ seems to have no analogue in the geometry of phase space of classical mechanics since V_f^+ blows up at $\hbar \to 0$ due to the factor $1/i\hbar$ in the definition (19). However, $i\hbar V_f^+$ is $\mathcal{J}$-invariant and has the classical limit $i\hbar V_f^+ = 2f + O(\hbar^2)$ so that $i\hbar V_H^+ = W_+ = 2H + O(\hbar^2)$ is simply two times Hamiltonian. In the Bloch-Wigner equation (5), $i\hbar V_f^+$ plays the role of Hamiltonian defining the density matrix in quantum statistics[10].

The operator V_f^- has an explicit interpretation[22] as a *quantum deformed Lie derivative along the hamiltonian vector field* in accordance with (21). Furthermore, in quantum mechanics the $\mathcal{J}$-invariance of V_f^- provides *unitary* time evolution due to the Wigner equation (21).

The structure of the Weyl-Wigner-Moyal calculus, which deals with non-commutative algebra, may be seen in a more refined way from the non-commutative geometry[18, 19] point of view as follows.

First, recall that usual definition of topological space M is equivalent to definition of commutative algebra $\mathcal{A}$ due to the identification $\mathcal{A} = C(M)$, with the algebra $C(M)$ of continuous complex valued functions on M (Gelfand correspondence). Conversely, M can be understood as the spectrum of algebra $\mathcal{A}$, *i.e.* points $x \in M$ are irreducible representations owing to the relation $x[f] = f(x)$ when $f \in \mathcal{A}$. Next step is that one is free to assume that the algebra $\mathcal{A}$ is non-commutative in general, and then think about a non-commutative version of the space M. Particularly, classical notion of point $x \in M$ is modified, in non-commutative geometry, due to the basic relation mentioned above.

Specific example we will consider for our aims is a non-commutative vector bundle. In classical geometry, sections of a vector bundle E above a manifold M play, in physical context, the role of matter fields. Here, an important point to be noted is that the space $\mathcal{E}$ of the sections is a bimodule over the algebra $\mathcal{A} = C(M)$ of the functions on M. In the non-commutative case, there are *left* and *right* modules over non-commutative algebra $\mathcal{A}$ instead of the bimodule. That is, for $\sigma \in \mathcal{E}$ and $f \in \mathcal{A}$, $f\sigma$ and σf are not both made sense as elements of $\mathcal{E}$. One may choose, for convenience, the right module, and then characterize the non-commutative vector bundle as a quotient of free module A^m, *i.e.* as the (right) projective module over the algebra $\mathcal{A}$, $\mathcal{E} = P\mathcal{A}^m$, for some projector P, $P^2 = P$, and some m.

In the symbol calculus, we have, obviously, $\mathcal{E} = \mathcal{A}$ itself, where $\mathcal{A} = (C(M_{2n}), *)$ is the non-commutative algebra endowed with the Moyal product. The sections are functions on M_{2n} acting by the left and right multiplications and forming, respectively, left and right $\mathcal{A}$-modules. The modular conjugation acts due to

$$\mathcal{J} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$$

$$(L, R) \mapsto (R, L)$$

and $\mathcal{E}$ is the quotient, $\mathcal{E} = \mathcal{A} \otimes \mathcal{A}/\mathcal{J}$.

The $\mathcal{A}$-modules to be *unital* one has to put $I_L * f = f$ and $f * I_R = f$, $\forall f \in C(M_{2n})$, with $I_{L,R}$ being the left and right "identity" elements of $\mathcal{E}$.

Because $\mathcal{E} = \mathcal{A}$, we have actually $I_L = I_R = I \in C(M_{2n})$ so that the above conditions imply

$$f * I - I * f = 0 \qquad f * I + I * f = 2f$$

According to definitions (16) and (20), these equations can be rewritten as

$$V_f^- I = 0 \qquad i\hbar V_f^+ I = 2f \qquad \forall f \in C(M_{2n}) \tag{25}$$

The question arises as to existence of such unique function I that the both equations (25) are satisfied for any function f. We observe that in the classical case the last two equations have correct limits at $\hbar \to 0$, and are satisfied for any f identically only if $I(\phi) = 1$, as it was expected ($1f = f1 = f$).

The Bopp-Kubo representation provides a realization of representation space of the algebra $\mathcal{A}$, with the variables $\Phi^i_\pm$, which extends the usual M_{2n} for the non-commutative case.

In the remainder of this section, we will consider the extention of the classical *ergodicity* condition[49]. Quantum mechanical analogue of the classical condition of ergodicity can be written as

$$V_H^- \rho = 0 \tag{26}$$

due to comparison of (15) and (21), with the solution ρ being non-degenerate, at least at the classical level.

In the classical limit, this equation covers the usual equation, $L\rho = 0$, where L denotes the Liouvillian, whose non-degenerate eigenfunctions with zero eigenvalues describe ergodic Hamiltonian systems[49], which are characterized by the only constant of motion, energy H. As to solutions, recent studies[50]-[54] of the classical ergodicity condition within the path integral approach to classical mechanics show that the solution is given specifically by the Gibbs state form.

The condition (26) can be rewritten in the Bopp-Kubo representation as

$$H(\Phi_+)\rho(\phi) = H(\Phi_-)\rho(\phi) \tag{27}$$

where we have used the relations (8) and (20), that means that the holomorphic and anti-holomorphic Hamiltonians have the same spectrum. Also, it is remarkable to note that the equation (26) is similar to the first equation of (25), with $f(\phi) = H(\phi)$ and $I(\phi) = \rho(\phi)$.

We pause here with the further discussion stating that more analysis is needed to verify the proposed extention of the ergodicity condition (26) which may be made elsewhere.

2.4 Translation operators

The operator $T(\phi_0)$ defined by (12) and used to represent the Weyl symbol map (11) has a meaning of the operator of translations in phase space. Bopp[7] has introduced such an operator in $\Phi^i_\pm$-variables representation and Jannussis *et al.* [23] have studied their properties.

Let us define the translation operators, in the Bopp-Kubo formulation,

$$T_\pm(\phi_0) = exp[\pm\frac{i}{\hbar}\phi_0^i\omega_{ij}\Phi^j_\pm] \tag{28}$$

where $\Phi^i_\pm$ are defined by (6). It is easy to verify that due to the fundamental commutation relations (7) they build up two mutually commuting algebras,

$$[T_\pm(\phi_1), T_\pm(\phi_2)] = \pm 2i\ sin(\frac{1}{\hbar}\phi_1^i\omega_{ij}\phi_2^j)T_\pm(\phi_1+\phi_2) \tag{29}$$

$$[T_\pm(\phi_1), T_\mp(\phi_2)] = 0 \tag{30}$$

In the case of Birkhoffian generalization of Hamiltonian mechanics[53]-[58] one supposes that the symplectic 2-form ω depends on phase space coordinates, $\omega = \omega(\phi)$, but it is still non-degenerate and closed, $d\omega = 0$. Consistency of the Birkhoffian mechanics is provided by the Lie-isotopic construction[59]-[61]. In this case, the fundamental commutation relations (7) are essentially modified,

$$\begin{aligned} [\Phi^i_\pm, \Phi^j_\pm] &= \pm i\hbar\omega^{ij} + (\frac{i\hbar}{2})^2\omega^{mn}\omega^{ij}{}_{,m}\partial_n \\ [\Phi^i_\pm, \Phi^j_\mp] &= \mp(\frac{i\hbar}{2})^2\omega^{mn}\omega^{ij}{}_{,m}\partial_n \end{aligned} \tag{31}$$

Consequently, the commutation relations (29) for the translation operators are also changed. Tedious calculations show that

$$[T_\pm(\phi_1), T_\pm(\phi_2)] = \pm 2i\ sin\Big(\frac{1}{\hbar}\phi_1^i\phi_2^j(\omega_{ij} + \frac{1}{2}\omega_{ij,m}\phi^m)\Big)T_\pm(\phi_1+\phi_2) \tag{32}$$

$$\begin{aligned} &[T_\pm(\phi_1), T_\mp(\phi_2)] = \\ &\pm 2i\ sin\Big(\frac{1}{\hbar}\phi_1^i\phi_2^j(\omega_{im,j} - \frac{1}{2}\omega_{ij,m})\phi^m\Big)exp[\pm\frac{i}{\hbar}(\phi_1^i\omega_{ij}\Phi^j_\pm - \phi_2^i\omega_{ij}\Phi^j_\mp)] \end{aligned} \tag{33}$$

Here, we have used the identity $\omega^{im}\omega^{jk}{}_{,m} + \omega^{jm}\omega^{ki}{}_{,m} + \omega^{km}\omega^{ij}{}_{,m} = 0$, and denote $\omega_{ij,m} = \partial_m\omega_{ij}$. We see that in the Birkhoffian case the holomorphic and anti-holomorphic functions do not form two mutually commuting algebras, in contrast to the Hamiltonian case characterized by $\omega_{ij,m} = 0$. Evidently, the Birkhoffian generalization is important for the case when the symplectic manifold can not be covered by *global* chart with constant symplectic tensor ω_{ij}. This is, for example, the case of M_{2n} with a non-trivial topology. However, it should be noted that the symplectic manifold can be always covered by local charts with constant ω_{ij} due to Darboux theorem.

3 q-deformed harmonic oscillator in phase space

3.1 Harmonic oscillator in the Bopp-Kubo phase space representation

Instead of studying the harmonic oscillator in phase space via the Wigner equation (4) it is more convenient to exploit corresponding creation and annihilation operators in the phase space.

Jannussis, Patargias and Brodimas[23] have defined the following two pairs of the creation and annihilation operators following the Bopp-Kubo formulation:

$$a_- = \frac{1}{\sqrt{2}}(\sqrt{\frac{m\omega}{\hbar}}Q + i\sqrt{\frac{1}{m\omega\hbar}}P) \tag{34}$$

$$a_-^+ = \frac{1}{\sqrt{2}}(\sqrt{\frac{m\omega}{\hbar}}Q - i\sqrt{\frac{1}{m\omega\hbar}}P) \tag{35}$$

$$a_+ = \frac{1}{\sqrt{2}}(\sqrt{\frac{m\omega}{\hbar}}Q^* + i\sqrt{\frac{1}{m\omega\hbar}}P^*) \tag{36}$$

$$a_+^+ = \frac{1}{\sqrt{2}}(\sqrt{\frac{m\omega}{\hbar}}Q^* - i\sqrt{\frac{1}{m\omega\hbar}}P^*) \tag{37}$$

These operators obey the following usual commutation relations:

$$[a_\pm, a_\pm^+] = 1 \quad [a_\pm, a_\mp^+] = [a_\pm, a_\mp] = [a_\pm^+, a_\mp^+] = 0 \tag{38}$$

The Bopp-Kubo holomorphic and anti-holomorphic Hamiltonians for the harmonic oscillator then read

$$H(P,Q) = \frac{P^2}{2m} + \frac{m}{2}\omega^2 Q^2 = \hbar\omega(a_-^+ a_- + \frac{1}{2}) \tag{39}$$

$$H(P^*, Q^*) = \frac{P^{*2}}{2m} + \frac{m}{2}\omega^2 Q^{*2} = \hbar\omega(a_+^+ a_+ + \frac{1}{2}) \tag{40}$$

$$\tag{41}$$

and the Wigner operator due to (3) takes the form

$$W_- = \hbar\omega(a_+^+ a_+ - a_-^+ a_-) \tag{42}$$

$$\equiv \hbar\omega(\hat{n_1} - \hat{n_2}) \tag{43}$$

In the two-particle Fock space $\mathcal{F}_1 \otimes \mathcal{F}_2$ with the basis $|n_1\, n_2\rangle$, the pairs of operators (34)-(37) act due to

$$a_-^+|n_1\, n_2\rangle = \sqrt{n_1+1}|n_1+1\, n_2\rangle \tag{44}$$

$$a_-|n_1\, n_2\rangle = \sqrt{n_1}|n_1-1\, n_2\rangle \tag{45}$$

$$a_+^+|n_1\, n_2\rangle = \sqrt{n_2+1}|n_1\, n_2+1\rangle \tag{46}$$

$$a_+|n_1\, n_2\rangle = \sqrt{n_2}|n_1\, n_2-1\rangle \tag{47}$$

Then, the Wigner operator (43) has the following eigenvalues

$$W_-|n_1\, n_2\rangle = (n_1 - n_2)|n_1\, n_2\rangle \tag{48}$$

The eigenfunctions of the Wigner operator (43) have the following form[23, 24]:

$$\varphi_{n_1 n_2}(p,q) = \int dp_0 dq_0\; T_+(p_0,q_0)\varphi_{0n_2}(p,q)\varphi_{n_1 0}(p,q) \tag{49}$$

where

$$\varphi_{n_1 0} = \frac{1}{\pi\sqrt{\hbar}}\frac{1}{\sqrt{n_1!}}\Big(\frac{2m\omega}{\hbar}\Big)^{n_1/2}\Big(q - i\frac{p}{m\omega}\Big)^{n_1} exp\Big(-\frac{2H(p,q)}{\hbar\omega}\Big) \tag{50}$$

and the same for φ_{0n_2} with the replacement $n_1 \to n_2$ in the r.h.s. of (50).

The vacuum is characterized by the Gibbs state form

$$\varphi_{00} = \frac{1}{\pi\sqrt{\hbar}} exp\Big(-\frac{2H(p,q)}{\hbar\omega}\Big) \tag{51}$$

The action of the Bopp translation operators on the functions (50) can be easily determined, and the result is

$$T_\pm(p_0,q_0)\varphi_{n_1 0}(p,q) = exp\Big(\pm\frac{i}{\hbar}(p_0 q - q_0 p)\Big)\varphi_{n_1 0}(p+p_0, q+q_0) \tag{52}$$

The commutators (29) take the form

$$[T_\pm(p_1,q_1),T_\pm(p_2,q_2)] = \pm 2i\ sin\frac{1}{\hbar}(p_1q_2-q_1p_2)T_\pm(p_1+p_2,q_1+q_2) \qquad (53)$$

The translation operators in (53) commute when

$$\frac{1}{\hbar}(p_1q_2-q_1p_2)=\pi l \qquad l\in Z \qquad (54)$$

This condition is similar to the one of quantization of magnetic flux for $2D$-electron gas in uniform magnetic field[23].

This means that the phase space asquires $2D$-lattice structure with the basic unit-cell vectors $\vec{\phi}_1=(p_1,q_1)$ and $\vec{\phi}_2=(p_2,q_2)$ obeying (54), i.e.

$$\vec{n}\cdot\vec{\phi}_1\times\vec{\phi}_2 = l\Psi_0 \qquad \Psi_0=\pi\hbar \qquad (55)$$

The degeneracy of the energy levels of the harmonic oscillator in the phase space is then related to the lattice structure. Namely, the representation (49) of $\varphi_{n_1n_2}$ means that one "smears" the product $\varphi_{n_10}\varphi_{0n_2}$ (a "composite state" of two identical systems) over all the phase space. So, $\varphi_{n_1n_2}$ remain to be eigenfunctions with the same eigenvalues under the translations of the form $\vec{R}=N_1\vec{\phi}_2+N_2\vec{\phi}_2,\ N_{1,2}\in Z$, leaving the lattice invariant. This is a kind of the magnetic group periodicity[62]-[64].

To implement the lattice structure of the phase space explicitly one may start with the vacuum state (51), which is characterized by zero angular momentum, to define four sets of functions

$$\varphi^{(\alpha)}_{\vec{k}}(\vec{\phi}) = \sum_{\vec{R}^\alpha} exp(i\vec{k}\vec{R}^\alpha)T_+(\vec{R}^\alpha)\varphi_{00}(\vec{\phi}) \qquad (56)$$

where

$$\begin{array}{cccc} \vec{R}^\alpha=\vec{R}_0+I^\alpha_i & \vec{R}_0=N_1\vec{\phi}_2+N_2\vec{\phi}_2 & \alpha=0,1,2,3 & \\ I^0_i=(0,0) & I^1_i=(1,0) & I^2_i=(0,1) & I^3_i=(1,1) \end{array} \qquad (57)$$

and the sum is over all four-sets of the $2D$-lattice points. The unit cell in the definition of each $\vec{R}^\alpha$ has $4l$ flux quanta Ψ_0 passing through it.

Gozzi and Reuter[22] have argued that there is a close relation between the symbol-calculus formalism and the Gelfand-Naimark-Segal construction[65].

In general, the GNS construction is specifically aimed to define non-commutative measure and topology[19].

The GNS construction provides bra-ket-type averaging, instead of the usual trace averaging, in the thermo field theory[66] when one deals with *mixed* states. This construction assumes a double Hilbert space representation of states, $||\hat{A}\rangle\rangle = \sum A_{\alpha\beta}|\alpha\rangle \otimes |\beta\rangle \in \mathcal{H} \otimes \mathcal{H}$. So, particularly, the average of $\hat{A}$ is given by $\langle \hat{A} \rangle = \langle\langle \hat{\rho}^{1/2} || \hat{A} \otimes \hat{I} || \hat{\rho}^{1/2} \rangle\rangle$, with $\hat{I}$ being identity operator. The modular conjugation operator $\mathcal{J}$ acts on the double Hilbert space by interchanging the two Hilbert spaces[22].

Time evolution of the GNS density is given by $i\hbar\partial_t ||\hat{\rho}^{1/2}\rangle\rangle = H^- ||\hat{\rho}^{1/2}\rangle\rangle$, Here, the GNS Hamiltonian $H^- = \hat{H} \otimes I - \mathcal{J}(\hat{H} \otimes I)\mathcal{J}$ can be evidently associated with the Wigner operator $W_- = i\hbar V_H^-$.

In view of the analysis of the oscillator in phase space, the eigenfunctions $\varphi_{n_1 0}$ and $\varphi_{0 n_2}$ can be ascribed to the two pieces of the GNS double Hilbert space. Also, the GNS double Hilbert space is associated to the double Fock space $\mathcal{F}_1 \otimes \mathcal{F}_2$, with the modular conjugation operator $\mathcal{J}$ acting on $\mathcal{F}_1 \otimes \mathcal{F}_2$ by interchanging the two Fock spaces.

3.2 q-deformed harmonic oscillator in phase space

The q-deformation of the commutation relations (38) for the Bopp-Kubo creation and annihilation operators (34)-(37) reads

$$b_- b_-^+ - \frac{1}{q} b_-^+ b_- = q^{\hat{n}_1} \qquad b_+ b_+^+ - \frac{1}{q} b_+^+ b_+ = q^{\hat{n}_2} \tag{58}$$

The bozonization procedure of the above operators according to Jannussis *et al.*[23] yields the following expressions for the q-deformed operators (q-bosons):

$$b_- = \sqrt{\frac{[\hat{n}_1 + 1]}{\hat{n}_1 + 1}}\, a_- \qquad b_-^+ = a_-^+ \sqrt{\frac{[\hat{n}_1 + 1]}{\hat{n}_1 + 1}} \tag{59}$$

$$b_+ = \sqrt{\frac{[\hat{n}_2 + 1]}{\hat{n}_2 + 1}}\, a_+ \qquad b_+^+ = a_+^+ \sqrt{\frac{[\hat{n}_2 + 1]}{\hat{n}_2 + 1}} \tag{60}$$

where $[x] = (q^x - q^{-x})/(q - q^{-1})$ and $a_\pm$ and $a_\pm^+$ are given by (34)-(37). Due to these definitions, we can directly find that

$$b_- b_-^+ = [\hat{n}_1 + 1] \qquad b_-^+ b_- = [\hat{n}_1] \tag{61}$$

$$b_+ b_+^+ = [\hat{n}_2 + 1] \qquad b_+^+ b_+ = [\hat{n}_2] \tag{62}$$

Clearly, $b_\pm^+ = (b_\pm)^\dagger$ if $q \in R$ or $q \in S^1$. The actions of the q-boson operators on the Fock space $\mathcal{F}_1 \otimes \mathcal{F}_2$ with the basis

$$|n_1\, n_2\rangle = \frac{(b_-^+)^{n_1}(b_+^+)^{n_2}}{\sqrt{n_1!}\sqrt{n_2!}}|0\,0\rangle \tag{63}$$

have the form

$$b_-|n_1\, n_2\rangle = \sqrt{[n_1]}|n_1 - 1\, n_2\rangle \qquad b_-^+|n_1\, n_2\rangle = \sqrt{[n_1+1]}|n_1 + 1\, n_2\rangle \tag{64}$$

$$b_+|n_1\, n_2\rangle = \sqrt{[n_2]}|n_1\, n_2 - 1\rangle \qquad b_+^+|n_1\, n_2\rangle = \sqrt{[n_1+1]}|n_1\, n_2 + 1\rangle \tag{65}$$

In the following we consider the algebra implied by the generators

$$J_+ = b_- b_+^+ \qquad J_- = b_+ b_-^+ \tag{66}$$

It is a matter of straightforward calculations to find that

$$[J_+, J_-] = [2J_3] \qquad [J_3, J_\pm] = \pm J_\pm \tag{67}$$

where

$$2J_3 = \hat{n}_1 - \hat{n}_2 \tag{68}$$

One can recognize that the above relations are standard quantum algebra $su_q(2)$ commutation relations, in the Kulish-Reshetikhin-Drinfeld-Jimbo realization[39]-[43] according to which $su_q(2)$ can be realized by two commuting sets of q-bosons ($q-$deformed version of the Jordan-Schwinger approach to angular momentum). Hereafter, we write $su_q(2)$ to denote the quantum algebra which is in fact $U_q(su(2))$.

Comparing (68) with (43) we see that the Wigner operator for harmonic oscillator is just proportional to the 3-axis projection of the (q-deformed) spherical angular momentum operator,

$$W_- = 2\hbar\omega J_3 \tag{69}$$

Indeed, in the $su(2)$ notations[40] for basis vector $|n_1 n_2\rangle$,

$$|j\, m\rangle = |n_1\, n_2\rangle \qquad j = \frac{1}{2}(n_1 + n_2) \qquad m = \frac{1}{2}(n_1 - n_2) \tag{70}$$

the operators $J_\pm$ and J_3 act on $\mathcal{F}_1 \otimes \mathcal{F}_2$ according to

$$\begin{aligned} J_-|j\,m\rangle &= \sqrt{[j+m][j-m+1]}|j\,m-1\rangle \\ J_+|j\,m\rangle &= \sqrt{[j-m][j+m-1]}|j\,m+1\rangle \\ J_3|j\,m\rangle &= m|j\,m\rangle \end{aligned} \tag{71}$$

For a fixed value $2j \in Z$, vector $|j\ m\rangle$ span the irrep (j) of the quantum algebra $su_q(2)$. We assume that q is not root of unity. Acordingly, the charge operator $J = \frac{1}{2}(\hat{n_1} + \hat{n_2})$ commutes with $J_{\pm,3}$, and $J|j\,m\rangle = j|j\,m\rangle$.

The indication of the Wigner density operator W_+ may be seen from the following. The basic fact[43] is that the vector $|n_1 n_2\rangle \equiv |j\ m\rangle$ can be represented also as a basis vector $|k\ l\rangle$ for the irrep beloning to the positive discrete series of $su_q(1,1) \approx sp_q(2,R)$ with the hyperbolic angular momentum, $k = \frac{1}{2}(n_1 - n_2 - 1) = m - \frac{1}{2}$, and 3-axis projection, $l = \frac{1}{2}(n_1 + n_2 + 1) = j + \frac{1}{2}$. The generators of $su_q(1,1)$ are

$$K_+ = b_+^+ b_-^+ \qquad K_- = b_+ b_- \qquad K_3 = J + \frac{1}{2} \tag{72}$$

Particularly, the 3-axis hyperbolic angular momentum operator K_3 acts due to

$$K_3|k\ l\rangle = l|k\ l\rangle \tag{73}$$

Thus, the Wigner density operator $W_+ = H(\Phi_-) + H(\Phi_+) = \hbar\omega(\hat{n}_1 + \hat{n}_2 + 1)$ can be immediately identified with K_3,

$$W_+ = 2\hbar\omega K_3 \tag{74}$$

To summarize, we note that the harmonic (q-)oscillator in phase space naturally arises to the Jordan-Schwinger approach to (q-deformed) angular momentum, with the Wigner operator W_- (W_+) being identified with the 3-axis spherical (hyperbolical) angular momentum operator.

As a final remark, we notice that there are ways to give geometrical interpretation of the quantum algebras and its representations. Namely, one may follow the line of reasoning by Fiore[44] and construct a realization of the quantum algebra within $Diff(M_q)$, where M_q is a q-deformed version of the ordinary manifold. For example, in the context of the q-oscillator in phase space it is highly interesting to find such a realization for the algebra $su_q(1,1) \approx sp_q(2,R)$, which is concerned the q-deformed phase space.

Also, there is a possibility[67] to give a geometric interpretation of the representations of $su_q(2)$ following the lines of the standard Borel-Weyl-Bott theory[68, 69].

4 Conclusions

We studied the relation between the Bopp-Kubo formulation and the Weyl-Wigner-Moyal calculus of quantum mechanics in phase space which is found to arise from the fact that the Moyal product of functions on phase space, $f(\phi) * g(\phi)$, can be rewritten equivalently as the product of functions defined on the extended phase space, $f(\Phi)g(\phi)$.

From the non-commutative geometry point of view, the phase-space formulation of quantum mechanics is an example of the theory with non-commutative geometry. The non-commutative algebra $\mathcal{A}$ is the algebra of functions on phase space endowed with the Moyal product. The right and left $\mathcal{A}$-modules are interchanged by the modular conjugation $\mathcal{J}$ so that the space of sections $\mathcal{E} = \mathcal{A} \otimes \mathcal{A}/\mathcal{J}$, and there is a kind of chiral symmetry due to the non-commutativity.

Due to a similarity between the phase-space formulation of quantum mechanics and Hamiltonian formulation of classical mechanics, there is an attractive possibility to extend useful classical notions and tools to quantum mechanics. An attempt is made to formulate the quantum extension of the classical ergodicity condition.

We studied one-dimensional harmonic (q-)oscillator in phase space. The phase space has a $2D$-lattice structure, similar to the one of the magnetic group periodicity for the $2D$-electron gas in magnetic field.

The Fock space for the oscillator is related to the double Hilbert space of the Gelfand-Naimark-Segal construction in accordance with the relation between the symbol calculus and the GNS construction.

For the q-oscillator, the Wigner operator $W_-(W_+)$ is found to be proportional to the 3-axis spherical (hyperbolical) angular momentum operator of the q-deformed algebra $su_q(2)$ $(su_q(1,1) \approx sp_q(2,R))$.

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