BRST APPROACH TO HAMILTONIAN SYSTEMS

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Abstract

BRST formulation of cohomological Hamiltonian mechanics is presented. In the path integral approach, we use the BRST gauge fixing procedure for the partition function with trivial underlying Lagrangian to fix symplectic diffeomorphism invariance. Resulting Lagrangian is BRST and anti-BRST exact and the Liouvillian of classical mechanics is reproduced in the ghost-free sector. The theory can be thought of as a topological phase of Hamiltonian mechanics and is considered as one-dimensional cohomological field theory with the target space a symplectic manifold. Twisted (anti-)BRST symmetry is related to global $N = 2$ supersymmetry, which is identified with an exterior algebra. Landau-Ginzburg formulation of the associated $d = 1$, $N = 2$ model is presented and Slavnov identity is analyzed. We study deformations and perturbations of the theory. Physical states of the theory and correlation functions of the BRST invariant observables are studied. This approach provides a powerful tool to investigate the properties of Hamiltonian systems.

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1 INTRODUCTION

Recently, path integral approach to classical mechanics has been developed by Gozzi, Reuter and Thacker in a series of papers[1]-[10]. They used a delta function constraint on phase space variables to satisfy Hamilton's equation and a sort of Faddeev-Popov representation. This constraint has been exponentiated with the help of Lagrange multiplier, ghost and anti-ghost fields so that the resulting field theoretic Lagrangian appears to be BRST and anti-BRST invariant.

This is quite analogous to the usual path integral formulation of quantum gauge field theories, in which BRST symmetry of the gauge fixed Lagrangian has been originally found. Due to the standard Faddeev-Popov procedure, one starts with a classical Lagrangian, which is invariant under the action of a gauge group, and the gauge fixing yields additional (gauge fixing and ghost dependent) terms in the Lagrangian. The BRST symmetry of the resulting gauge fixed Lagrangian is well known to be a fundamental property providing, particularly, renormalizability of the theory.

Wellknown alternative method to quantize gauge field theories is just based on the BRST symmetry. Instead of implementing the gauge fixing constraint, one simply insists on the BRST invariance from the beginning, by constructing nilpotent BRST operator and BRST exact Lagrangian. The BRST quantization scheme provides a simple geometrical basis for heuristic Faddeev-Popov method and is known as a powerful tool to deal with not only gauge field theories but also with much more complicated field theories. An important point is to identify the symmetries to be fixed.

Trivial Lagrangians are known to be of much importance in the cohomological quantum field theories[11]-[15]. As it was realized, these theories can be derived by an appropriate BRST gauge fixing of a theory in which the underlying Lagrangian is zero. An extensive literature exists on the topological field theories. Various topological quantum field theories, such as topological Yang-Mills theories, two-dimensional gravity[16]-[22], four-dimensional conformal gravity[23], non-linear sigma model[12], Landau-Ginzburg models[24], two-dimensional BF models[25], WZNW models[26], W-strings[27] are investigated within the BRST quantization scheme; see Ref.[14] for a review. We feel it is worthwhile to broaden this effort and, in this paper, use the BRST procedure to develop a model describing classical dynamical systems.

The model described by the partition function (6) is by construction one-dimensional cohomological field theory, in the sense that the resulting Lagrangian is BRST exact, with trivial underlying Lagrangian. The theory (6) can be thought of as a topological phase of Hamiltonian mechanics. The resulting Lagrangian is in effect the same one obtained by Gozzi, Reuter and Thacker plus additional α dependent terms, which we drop in the subsequent consideration. We should emphasize here that, stating that the theory is, as such, a sort of topological one, they proved[4] that the partition function is proportional to Euler characteristic of the phase space and studied $2n$-ghost ground state sector of the associated supersymmetric model. However, more elaborated analysis is needed. In the present paper, we use the tools of topological field theory to fill this gap.

The most close examples of cohomological field theory to the one considered in this paper are topological nonlinear sigma model[12], in which a basic field is the map from Riemannian surface to a fixed Kahler manifold as the target space, and topological Landau-Ginzburg models[24]. Also, one-dimensional ($d = 1$) sigma model with the target space a compact Kahler manifold having nontrivial homotopy group π_1 has been considered recently by Cecotti and Vafa[28], in connection with the Ray-Singer analytic torsion.

One of the themes underlying this paper is the notion that studying topological field theoretic models has regularly proved useful in developing our understanding of field theories and physical phenomena more generally. For example, studying cohomological content of $d = 2$, $N = 2$ supersymmetric model has enhanced understanding of two-dimensional Ising model[28]. Despite the fact that various topological field theories have been thoroughly studied, we think it is useful to look closely at the specific model, which has its own significance in the context of continued studies of classical dynamical systems. So, we apply various tools and analyze the model in many directions.

Since we need to fix only one symmetry, the problem of giving the BRST formulation itself of the cohomological classical mechanics which we develop in this paper is drastically simpler than that of the forementioned topological field theories, which are characterized by rich field content and a set of symmetries. Nevertheless, we give a systematic representation in order to provide a self-consistent and precise description. Also, the theory is one-dimensional that makes general structure of the theory less complicated than that of the two- or higher dimensional topological field theories; for example, there are no conventional "instanton corrections" and the notion of spin is irrelevant. Furthermore, it should be stressed that the resulting theory is essentially a *classical* one despite the fact that we are using the BRST gauge fixing scheme, which is usually exploited to construct quantum field theories. As the result, sharing many properties with the usual topological quantum field theories, it differs from those by absence of quantum ($\hbar$) corrections. However, we should emphasize that there arises a major distinction from the conventional topological field theories studied in the literature since symplectic structure of the target space rules out quadratic terms from the Lagrangian, leading to a first-order character of the system. Such specific theories, describing Hamiltonian systems, are worth to be studied exclusively.

In view of the above, cohomological classical mechanics represents perhaps the simplest example of topological field theory. So, after constructing the BRST invariant Lagrangian the focus of our paper is to study the implications and novel aspects arising from the BRST approach and associated supersymmetry. The work in this paper enables us to use supersymmetric field theory as a way of deeper understanding of Hamiltonian systems. In general, this approach provides a powerful tool to investigate the fundamental properties and characteristics of Hamiltonian systems such as ergodicity, Gibbs distribution, Kubo-Martin-Schwinger condition[3], integrability, and Lyapunov exponents[8]. Particularly, we think it is illuminating and instructive to map out some identifications one can draw between the topological field theories and Hamiltonian systems. This is, in part, to establish dictionary between the old and modern techniques used in studying classical dynamical systems.

Another motivation of our study is that we consider the BRST formulation of cohomological classical mechanics as providing a basis to give BRST formulation of (cohomological) quantum mechanics and, from then on, apply topological field theory methods to study quantum mechanical systems. The key to making the connection between them lies in treating quantum mechanics as a smooth $\hbar$-deformation of the Hamiltonian one, within the phase space (Weyl-Wigner-Moyal) formulation of quantum mechanics[60]. We hope, in this way, that one might investigate quantum ergodicity and quantum chaos characteristics which are now of striking interest.

In addition, there arises a tempting possibility to give a classification of possible topologies of constant energy submanifolds of the phase space for the case of reduced Hamiltonian systems. Of course, this idea is reminiscent of the one of using BRST symmetry and supersymmetry to obtain various topological results. Indeed, we already know that the instantons and Witten index serve as the tool to obtain such quantities as the Donaldson invariants, Lefschetz number, and Euler characteristic. The idea to combine the tools of topological field theories and classical Morse theory might be productive here as well.

As to examples of field theoretic approach to Hamiltonian systems we notice that the path integral approach to Euler dynamics of ideal incompressible fluid viewed as Hamiltonian system has been developed recently by Migdal[29], to study turbulence phenomenon in terms of the path integral over the phase space configurations of the vortex cells. Hamiltonian dynamics has been used to find an invariant probability distribution which satisfies the Liouville equation, with topological terms in the effective energy being of much importance.

Also, more recently Niemi and Palo[32] considered classical dynamical systems using $d = 2$, $N = 2$ supersymmetric nonlinear sigma models. They followed studies on the Arnold conjecture on the number of T-periodic trajectories[33] by Floer[34], who proved the conjecture for the symplectic manifolds subject to the condition that the integral of symplectic two-form over every two-dimensional sphere is zero. Particularly, they used a generalization of Mathai-Quillen formalism, previously applied in the investigation of Witten's topological sigma model, and studied functional Hamiltonian flow in the space of periodic solutions of Hamilton's equation by breaking the (1,1) supersymmetry with Hamiltonian flow down to a chiral (1,0) supersymmetry to describe properties of the action of the model in terms of (infinite dimensional) Morse theory.

The outline of the paper is as follows.

In Sec. 2, we start with a target space interpretation of Hamiltonian mechanics and explore the BRST gauge fixing scheme to fix diffeomorphism invariance of the trivial underlying Lagrangian (Sec. 2.1). The gauge fixing condition is Hamilton's equation plus some additional α dependent term. When both the BRST and anti-BRST symmetries are incorporated there appears no room for the α dependent terms in the Lagrangian, which exhibits Z_2 symmetry. The Liouvillian of ordinary classical mechanics is reproduced by the associated Hamilton function, in the ghost-free sector. The model reveals symplectic structure represented by using of the cotangent superbundle over phase space naturally supplied by the field content. Then we analyse Slavnov identity to demonstrate that the model is perturbatively trivial and BRST anomaly free (Sec. 2.2).

With this set up, in Sec. 3 we study in some detail the associated supersymmetric model and cohomology. Namely, we use topological twist of the BRST and anti-BRST operators to obtain global $N = 2$ supersymmetry[3] (Sec. 3.1), and relate the supersymmetry to an exterior algebra (Sec. 3.2). In so doing, we are able to identify physical states of the theory. The link between the BRST and anti-BRST symmetries is the supersymmetric ground state sector, *i.e.* the Ramond sector, of the associated $d = 1$, $N = 2$ model. Due to the underlying supersymmetry of the Hamilton function (20), only the Ramond states are of relevance which are found to be in correspondence with cohomology classes of the target manifold. The states can be in general treated as (cohomology classes of) form valued classical probability distributions on the phase space M^{2n}. In the ghost-free sector, they correspond to the probability distribution related[3] to conventional ergodic Hamiltonian systems. Physically relevant (normalizable) solutions of the supersymmetry equations are given specifically by the Gibbs state form coming from the $2n$-ghost sector. We stress that the supersymmetry appears to be a strong constraint on the physics. Criterion for regular/nonregular motion regimes in Hamiltonian systems is related to the Witten index known as a measure for supersymmetry breaking. Partition function evaluates Euler characteristic of the target space and the Witten index is equal to Euler characteristic, too (not surpising result, certainly, obtained earlier in the context of topological $d = 2$, $N = 2$ sigma models). Also, we find that Poincare integral invariants can be naturally identified as homotopically trivial BRST invariant observables. Existence of homotopically nontrivial Poincare invariants is a consequence of the field theoretic approach.

We discuss briefly on the connection of the model to Morse theory (Sec. 3.3) observing that Hamiltonian may serve as a Morse function and then proceed to obtain Landau-Ginzburg formulation of the $d = 1$, $N = 2$ model using a superspace technique (Sec. 3.4). One of the results is that the model admits Landau-Ginzburg desription so that its properties can be largely understood in terms of superpotential. The lowest component of the superpotential has been identified as Hamiltonian. The action appears to be in the form of a D-term. The ring of chiral operators consists of polynomials modulo the relation characterizing critical points of the Hamiltonian flow.

We show that the time reparametrization invariance of the model requires the fundamental homotopy group $\pi_1(M^{2n})$ to be nontrivial.

Valuable information comes from studying possible deformations and perturbations of the action. We analyze deformations of the superpotential and symplectic tensor (Sec. 3.5). What is most interesting is that the supersymmetry preservation condition for the deformation of superpotential (Hamiltonian) by analytic function is explicitly related to integrals of motion. This is another step toward revealing connection between the supersymmetry and integrability properties of the system. We also study a deformation of the coordinate dependent symplectic tensor for which case slight modifications of the BRST structure have been accounted. A remarkable result is that to preserve the supersymmetry Schouten bracket between the deformation tensor and symplectic tensor must be zero. Also, we use a generalized Mathai-Quillen formalism to construct the action in terms of an equivariant exterior derivative in the space of fields (Sec. 3.6) to get a more clear geometrical meaning of the model and to provide a possible set up for studying supersymmetry breaking. This result may serve as a foundation for further work.

In Sec. 4, we study BRST invariant observables and its correlation functions. The BRST invariant observables of interest are closed p-forms on symplectic manifold and correspond to the de Rham cohomology classes (Sec. 4.1). Elaborating connection between the BRST symmetry and supersymmetry, we identify the BRST invariant observables with chiral operators of the $d = 1$, $N = 2$ model. The anti-BRST observables can be treated in the same manner.

Also, there arise naturally homotopy classes of classical periodic orbits (Sec. 4.2) so that coefficients of the p-forms take values in the linear bundles of appropriate representations of $\pi_1(M^{2n})$. This leads to consideration of the loop space consisting of mappings $S^1 \to M^{2n}$ which is a natural object in the topological framework. We remark that periodic orbits are in many ways the key to the classical dynamics. Using the above mapping and an integer valued closed two-form, we construct a term which can be added to the original action and provides a possible mechanism for symmetry breaking.

The correlation functions (Sec. 4.3) are found to be related to the intersection number, with the two-point correlation function, in the homotopically trivial sector, representing the standard intersection form in cohomology identified as the topological metric. As it is in the topolological field theories, there are no "local" degrees of freedom in the theory under consideration that means that the correlation functions are not time dependent.

A certain kind of correlation functions interwines the BRST and anti-BRST sectors and are known to be related to the Lyapunov exponents, positive values of which are strong indication of chaos in Hamiltonian system. The p-form sectors, Z_p, $p = 0, \ldots, 2n$, of the partition function (6) for the periodic orbits are evaluated via realizing M^{2n} as a covering space and using monodromy.

In Sec. 5, we end the paper with some comments about what questions one might address next.

2 BRST FORMULATION

A Hamiltonian dynamical system can be described geometrically by a phase space manifold M^{2n} equipped by a symplectic form ω and a Hamiltonian H[36]. The evolution of the system in time t is given by a particular set of trajectories on M^{2n}, parametrized by t, such that Hamilton's equation holds.

On the other hand, Hamiltonian dynamical system can be described as one-dimensional field theory, in which dynamical variable is the map, $a^i(t) : M^1 \to M^{2n}$, from one-dimensional space M^1, $t \in M^1$, to $2n$-dimensional symplectic manifold M^{2n}.

The commuting fields $a^i = (p_1, \dots, p_n, x^1, \dots, x^n)$ are local coordinates on the target space, the phase space M^{2n}, endowed with a nondegenerate closed two-form ω; $\omega^n \neq 0$, $d\omega = 0$. In terms of the local coordinates, $\omega = \frac{1}{2}\omega_{ij} da^i \wedge da^j$; $\omega_{ij}\omega^{jk} = \delta_i^k$. We take ω_{ij} to be constant symplectic matrix, *i.e.* use canonical (Darboux) coordinates.

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Our starting point is the partition function

$$Z = \int D[a] \ \exp iI_0, \tag{1}$$

where $I_0 = \int dt\mathcal{L}_0$ and the Lagrangian is trivial, $\mathcal{L}_0 = 0$. This Lagrangian has symmetries more than the usual diffeomorphism invariance.

The BRST gauge fixing scheme[14, 35] assumes fixing of some symmetry of the underlying action by introducing appropriate ghost and anti-ghost fields. The symmetry of the action (1) we are interested in is symplectic diffeomorphism invariance, which leaves the symplectic tensor ω_{ij} form invariant,

$$\delta a^i = \ell_h a^i. \tag{2}$$

Here, $\ell_h = h^i\partial_i$ is a Lie-derivative along the Hamiltonian vector field, $h^i = \omega^{ij}\partial_j H(a)$[36]. The transformations (2) are canonical ones, which preserve the usual Poisson brackets, $\{,\}_\omega$, defined by the symplectic tensor. Note, however, that the symmetry under (2) can be viewed more generally as an invariance under any diffeomorphism, with h treated as a vector field.

By introducing the ghost field $c^i(t)$ and the anti-ghost field $\bar{c}_i(t)$, we write the BRST version of the diffeomorphism (2),

$$sa^i = c^i, \quad sc^i = 0, \quad s\bar{c}_i = iq_i, \quad sq_i = 0, \tag{3}$$

where the BRST operator s is nilpotent, $s^2 = 0$, and q_i is a Lagrange multiplier. The BRST transformations (3) represent a trivial BRST algebra for the BRST doublet (a^i, c^i). By an obvious mirror symmetry to the BRST transformations (3), we demand the following anti-BRST transformations hold:

$$\bar{s}a^i = \omega^{ji}\bar{c}_j, \quad \bar{s}\bar{c}_i = 0, \quad \bar{s}c^i = i\omega^{ij}q_j, \quad \bar{s}q_i = 0, \tag{4}$$

The definition (4) implies $\bar{s}^2 = 0$, and it can be easily checked that the BRST and anti-BRST operators anticommute,

$$s\bar{s} + \bar{s}s = 0. \tag{5}$$

By definition, s and $\bar{s}$ anticommute with $d_t = dt\partial_t$, so that $(d_t + s + \bar{s})^2 = 0$.

To construct BRST invariant Lagrangian one proceeds as follows. The partition function (1) becomes

$$Z = \int D[X] \exp iI, \tag{6}$$

where the measure $D[X]$ represents the path integral over the fields a, q, c, and $\bar{c}$. The total action I is the trivial action I_0 plus s-exact part,

$$I = I_0 + \int dt\, sB. \tag{7}$$

Since s is nilpotent, I is BRST invariant for any choice of B, with sB having ghost number zero. Since ω_{ij} is antisymmetric, all terms quadratic in fields are identically zero, and we choose judiciously the "gauge-fermion" B to be linear in the fields. The form of B is typical, namely, (antighost)$\times$(gauge fixing condition),

$$B = i\bar{c}_i(\partial_t a^i - h^i - \alpha a^i - \gamma\omega^{ij}q_j). \tag{8}$$

where α and γ are parameters. Applying the BRST operator we find that the first two terms in (8) give rise to the term of the form (Lagrange multiplier)$\times$(gauge fixing condition) and the ghost dependent part,

$$sB = q_i(\partial_t a^i - h^i) + i\bar{c}_i(\partial_t c^i - sh^i) - \alpha(q_i a^i - i\bar{c}_i c^i). \tag{9}$$

As a feature of the theory under consideration, the γ dependent term vanishes because ω_{ij} is antisymmetric. To find sh^i, we note that $\delta h^i = (\partial_k h^i)\delta a^k$, and hence $sh^i = c^k\partial_k h^i$. Thus, the resulting Lagrangian becomes $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{gf} + \mathcal{L}_{gh} + \mathcal{L}_\alpha$,

$$\mathcal{L} = \mathcal{L}_0 + q_i(\partial_t a^i - h^i) + i\bar{c}_i(\partial_t\delta^i_k - \partial_k h^i)c^k - \alpha(q_i a^i - i\bar{c}_i c^i). \tag{10}$$

The total Lagrangian $\mathcal{L}$ is BRST invariant by construction, $s\mathcal{L} = 0$.

In the delta function gauge, *i.e.* at $\alpha = 0$, it reproduces exactly, up to $\mathcal{L}_0$, the Lagrangian, which has been derived in [1]-[4], in the path integral approach to

Hamiltonian mechanics by the Faddeev-Popov method. Indeed, by integrating out the fields q, c and $\bar{c}$ we obtain from (6) the partition function in the form

$$Z = \int Da\, \delta(a - a_{cl}) \exp iI_0, \tag{11}$$

where a^i_{cl} denote solutions of Hamilton's equation, $\partial_t a^i = h^i$. The partition function (11), with $I_0 = 0$, has been used in [1] as a starting point of the path integral approach to classical mechanics.

The delta function constraint in (11) corresponds, evidently, to the Faddeev-Popov gauge fixing condition and leads to integration over all paths with a delta function concentrating around the integral trajectories of the Hamiltonian flow. Since the underlying Lagrangian is zero the theory (10) is defined only by the gauge fixing term. Thus, the partition function (6), with appropriate boundary conditions, represents *one-dimensional cohomological field theory* describing Hamiltonian systems[43].

We note that the α dependent terms in the Lagrangian (10) can be absorbed by redefining time derivative by the shift, $\partial_t \to \partial_t - \alpha$. Notice that the latter form is strongly reminiscent of a gauge covariant derivative.

The way to construct explicitly BRST and anti-BRST invariant Lagrangian is to use both s and $\bar{s}$ operators, namely, $\mathcal{L}' = \mathcal{L}_0 + s\bar{s}B'$, with B' being of ghost number zero. With the choice

$$B' = i\omega_{ik} a^i (\partial_t a^k - h^k), \tag{12}$$

we find the Lagrangian in the form

$$\mathcal{L}' = \mathcal{L}_0 + q_i \partial_t a^i - a^i \partial_t q_i + i(\bar{c}_i \partial_t c^i + c_i \partial_t \bar{c}^i) - q_i h^i + i\bar{c}_i \partial_k h^i c^k. \tag{13}$$

We observe that the fields appear in a more symmetric way compared to (10). We will see in Sec. 3 that this form of the Lagrangian arises in a superfield treatment of the theory. It is straightforward to check that the two Lagrangians, $\mathcal{L}$, at $\alpha = 0$, and $\mathcal{L}'$, differ by the derivative term,

$$\mathcal{L}' = \mathcal{L} - \frac{1}{2}\partial_t (a^i q_i - ic^i \bar{c}_i), \tag{14}$$

implying thus the same equations of motion. Also, we conclude that there is no room for the α dependent terms, in this $s\bar{s}$ construction, so that in contrast to (10) the Lagrangian (13) is invariant under the following Z_2 symmetry:

$$t \to -t, \quad a^i \to a^i, \quad q_i \to -q_i, \quad c^i \to c^i, \quad \bar{c}_i \to -\bar{c}_i, \quad h^i \to -h^i. \tag{15}$$

Coupling the system to the "gauge field" α spoils this symmetry.

In Table 1 we collect the fields of the cohomological classical mechanics for the reader convenience (see also [2]).

Field	Meaning	Geometrical meaning	Ghost number
a	map $M^1 \to M^{2n}$	coordinates on M^{2n}	0
c	sdiffeomorphism ghost	differential da^i	+1
$\bar{c}$	anti-ghost of c	$\partial/\partial c^i$	-1
q	Lagrange multiplier	$-i\partial/\partial a^i$	0
h	vector field	symplectic vector field	0

Table 1: The fields of cohomological classical mechanics.

The (anti-)BRST symmetry is, in fact, an inhomogeneous part of larger symmetry of the Lagrangian (13), namely, inhomogeneous symplectic ISp(2) group symmetry, which is generated by the following charges[1]:

$$\begin{gathered} Q = ic^i q_i, \quad \bar{Q} = i\bar{c}_i \omega^{ij} q_j, \quad C = c^i \bar{c}_i, \\ K = \frac{1}{2}\omega_{ij} c^i c^j, \quad \bar{K} = \frac{1}{2}\omega^{ij} \bar{c}_i \bar{c}_j. \end{gathered} \tag{16}$$

Here, Q and $\bar{Q}$ are the BRST and anti-BRST charges respectively; $sb^i = [Q, b^i]$ and $\bar{s}b^i = [\bar{Q}, b^i]$ for a generic field b^i. The generators (16) form the algebra of ISp(2) group,

$$\begin{aligned} \big[Q, Q\big] = \big[\bar{Q}, \bar{Q}\big] = \big[Q, \bar{Q}\big] &= 0, \\ \big[C, Q\big] = Q, \quad \big[C, \bar{Q}\big] &= -\bar{Q}, \\ \big[K, Q\big] = \big[\bar{K}, \bar{Q}\big] &= 0, \\ \big[\bar{K}, Q\big] = \bar{Q}, \quad \big[K, \bar{Q}\big] &= Q, \\ \big[K, \bar{K}\big] &= C, \\ \big[C, K\big] = 2K, \quad \big[C, \bar{K}\big] &= -2\bar{K}. \end{aligned} \tag{17}$$

As it has been found[2]-[6] the ISp(2) algebra (17) reflects the full machinery of the Cartan calculus on symplectic manifold M^{2n}, with the correspondences given in Table 1. The (anti-)BRST operator Q ($\bar{Q}$) is naturally associated with an exterior (co-)derivative d (d^*) on M^{2n}.

The ghosts c^i form a basis for the tangent space TM^{2n} and act by exterior multiplication on the cotangent space T^*M^{2n}, for which the anti-ghosts $\bar{c}_i$ form the basis dual to c^i. They fulfill the Dirac algebra,

$$\{c^i, c^j\} = \{\bar{c}_i, \bar{c}_j\} = 0, \quad \{c^i, \bar{c}_j\} = \delta^i_j, \tag{18}$$

with the equal time anticommutators, and can be treated as the creation and annihilation operators acting on a Fock space. The basic field a^i and Lagrange multiplier q_i satisfy the following commutation relation:

$$[a^i, q_j] = i\delta^i_j, \tag{19}$$

while the other equal time commutators between all the fields are identically zero.

Hamilton function, $\mathcal{H}$, associated with the Lagrangian (10) can be readily derived,

$$\mathcal{H} = q_i h^i + i\bar{c}_i c^k \partial_k h^i + \alpha a^i q_i + i\alpha C, \tag{20}$$

In the ghost-free part, it covers, at $\alpha = 0$, the usual Liouvillian, $L = -h^i \partial_i$, of ordinary classical mechanics derived in the operator formulation of classical mechanics by Koopman and von Neumann[37]. The Hamilton function $\mathcal{H}$ is a generalization of the Liouvillian to describe evolution of the form valued probability distribution, $i\partial_t \rho(a,c) = \mathcal{H}\rho(a,c)$, instead of the usual distribution function (zero-form) governed by the Liouville equation, $\partial_t \rho(a) = -L\rho(a)$. Here, we mean $\rho(a,c)$ is expanded in anticommuting variables c^i giving a set of p-ghost terms corresponding to a set of p-forms, $0 \leq p \leq 2n$, on M^{2n}. From a geometrical point of view, it is highly remarkable that $\mathcal{H}$, at $\alpha = 0$, is proportional to a Lie derivative along the Hamiltonian vector field, $\mathcal{H} = -i\ell_h$, applied this time to p-forms[2, 6]. As we will see in Sec. 3, solving the generalized Liouville equation, for stationary form valued distributions, is equivalent to solving a cohomology problem.

It is worthwhile to note here that the p-form states and observables arise naturally also in the supersymmetric quantum mechanics[38, 39] and in the (Landau-Ginzburg) $N = 2$ supersymmetric models[24, 28, 40], which in various aspects will serve, in Sec. 3, as a guide line for dealing with such an extention. Also, we note that the properties of the ground states in the two-dimensional topological models are studied[40] also via dimensional reduction to one-dimensional models.

Besides symplectic structure of the target space, the model (13) reveals symplectic structure provided by the field content. Recall first that the phase space M^{2n} is usually considered as a cotangent bundle over configuration space M^n, $x^\alpha \in M^n$. Analogously, let us consider cotangent bundle M^{4n} over M^{2n} endowed by symplectic two-form $\tilde{\Omega}$ with local cooordinates $\tilde{y}^c = (q_i, a^j)$; $c, d = 1, \ldots, 4n$. Here, q_i and a^j are canonical conjugates which is indicated by (19). M^{4n} can be thus viewed as the second generation phase space with a base space M^{2n} and natural projection $p : M^{4n} \to M^{2n}$ provided by $(q_i, a^j) \mapsto (0, a^j)$. Enlarging the bundle M^{4n} by a Grassmannian part with coordinates $(c^i, \bar{c}_j)$, we define the cotangent superbundle $M^{4n|4n}$ equipped by block diagonal supersymplectic matrix $||\Omega_{ab}|| = \mathrm{diag}(||\tilde{\Omega}_{cd}||, ||E_{cd}||)$, where E is $4n \times 4n$ unit matrix, with local coordinates $y^a = (\tilde{y}^d, c^i, \bar{c}_j)$; $a, b = 1, \ldots, 8n$. The graded Poisson brackets in $M^{4n|4n}$ has been introduced in [2, 60], and can be defined by using Ω in a standard way:

$$\{F, G\}_\Omega = (\partial_a F)\Omega^{ab}(\partial_b G), \tag{21}$$

where F and G are functions on $M^{4n|4n}$ and $\partial_a = \partial/\partial y^a$; $\Omega_{aa'}\Omega^{a'b} = \delta^b_a$.

Thus, the Hamiltonian (20) and charges in (16) can be treated as functions on $M^{4n|4n}$ acting by taking the graded Poisson brackets. It is a matter of straightforward calculations to verify that the ISp(2) algebra relations (17) hold, with the graded commutators replaced by the graded Poisson brackets; for example, $\{\bar{K}, Q\}_\Omega = \bar{Q}$. We note that this is, in fact, a nontrivial result because of emerging of the supersymplectic structure Ω having no counterpart in the Cartan calculus. The probability distribution forms ρ on M^{2n} can be viewed in general as functions on $M^{4n|4n}$, with the graded Poisson bracket algebra being the algebra of classical observables. The Hamilton function (20), at $\alpha = 0$, is ad_H operator acting on functions on $M^{4n|4n}$ and represents horizontal vector field in the fiber bundle. The Schrodinger-like equation for evolution of the distribution form can be rewritten in a Hamiltonian form, $i\partial_t \rho = \{\mathcal{H}, \rho\}_\Omega$. In a sense, we can say that the model is twice symplectic.

Due to the structure of the Hamilton function (20) the partition function (6) can be factorized into three different sectors: The Liouvillian sector, the ghost sector, and the α dependent sector.

In the following, we use the delta function gauge omitting the α dependent terms in (20), one of which is the ghost number operator C.

2.2 SLAVNOV IDENTITY

In order to draw further parallels with the topological quantum field theories, the point of an immediate interest is to translate the BRST invariance of the theory under consideration into Slavnov identity. Particularly, the Slavnov identity technique is used[53] to study anomalies and renormalizability of a theory and to incorporate all the symmetries and constraints of a model (BRST invariance, vector supersymmetry, ghost equations, etc.). Since the theory under consideration is linear and one-dimensional, its perturbative properties and anomalies can be reliably derived from general arguments. It is instructive, however, to prove explicitly that it is indeed perturbatively trivial and symplectic diffeomorphism anomaly free.

In order to write down the Slavnov identity, we introduce a set of invariant external sources $(J^a, J^q, J^c, J^{\bar{c}})$ coupled to the BRST variations of the fields,

$$I_{ext} = \int dt (J^a sa + J^q sq + J^c sc + J^{\bar{c}} s\bar{c}). \tag{22}$$

According to (3) the total action,

$$\Sigma = I + I_{ext} = \int dt (q_i(\partial_t a^i - h^i) + \bar{c}_i \partial_t c^i + \bar{c}_i \partial_j h^i c^j + J^a_i c^i + i J^i_{\bar{c}} q_i), \tag{23}$$

does not depend on J^c and J^q since $sc = sq = 0$, while the other BRST transformations in (22) are linear. This linearity implies that there are no "radiative corrections" to these transformations so that linear dependence of the action (23) on the BRST sources is radiatively preserved.

It is straightforward to check that the extended action (23) satisfies the following Slavnov identity:

$$\mathcal{S}(\Sigma) = 0, \tag{24}$$

where

$$\mathcal{S}(\Sigma) = \int dt \Big(\frac{\delta \Sigma}{\delta J_i^a} \frac{\delta \Sigma}{\delta a^i} + \frac{\delta \Sigma}{\delta J_{\bar{c}}^i} \frac{\delta \Sigma}{\delta \bar{c}_i} \Big). \tag{25}$$

The corresponding extended BRST operator B_Σ is linear,

$$B_\Sigma = \int dt \Big(c^i \frac{\delta}{\delta a^i} + i q_i \frac{\delta}{\delta \bar{c}_i} \Big), \tag{26}$$

and provides no extension to the BRST sources. It is easy to verify that B_Σ is nilpotent, $B_\Sigma^2 = 0$. With the absence of the radiative corrections to this equality we arrive at the conclusion that there are no "quantum deformations" (no surprise certainly).

In topological Yang-Mills field theories, nontrivial cohomology of an extended BRST operator in the space of integrated polynomials in fields and BRST sources is refered to as a gauge anomaly[53]. So, anomaly may come from nontrivial cohomology of the extended BRST operator B_Σ in such a space. However, it is easy to verify that its cohomology is trivial since the fields in (26) appear only in BRST doublets. Note that one should take into account all symmetries of the theory to write down extended BRST operator. Since in our case there are no additional symmetries to be incorporated, this completes the prove that the Slavnov identity is symplectic diffeomorphism anomaly free.

3 PHYSICAL STATES AND TOPOLOGICAL TWIST

Due to the BRST and anti-BRST invariance of the theory we will study in this Section the BRST and anti-BRST invariant states. The possible physical states, $\rho \equiv |\text{phys}\rangle$, are then found as solutions of the system of equations consisting of the BRST and anti-BRST cohomology equations[7, 41, 42],

$$Q\rho = 0, \quad \bar{Q}\rho = 0. \tag{27}$$

They are equivalence classes of appropriate Q and $\bar{Q}$ cohomologies, $\rho \sim \rho + Q\rho' + \bar{Q}\rho''$.

3.1 $d = 1$, $N = 2$ SUPERSYMMETRIC MODEL

To study the physical states, we exploit the identification of the twisted (anti-)BRST operator algebra with $N = 2$ supersymmetry which is usually perfomed in topological quantum field theories[38]. Conventionally, the topological twist is used to obtain BRST theory from a supersymmetric one[11, 38, 44, 45]. Below we use the twist to obtain, conversely, supersymmetry from the BRST and anti-BRST symmetries.

Using the twisted BRST and anti-BRST operators

$$Q_\beta = e^{\beta H} Q e^{-\beta H} = Q - \beta c^i \partial_i H, \quad \bar{Q}_\beta = e^{-\beta H} \bar{Q} e^{\beta H} = \bar{Q} + \beta \bar{c}_i \omega^{ij} \partial_j H, \tag{28}$$

where $\beta \geq 0$ is a real parameter, one can easily find that they are conserved nilpotent supercharges and their anticommutator closes on the Hamilton function,

$$\begin{aligned} \{Q_\beta, Q_\beta\} = \{\bar{Q}_\beta, \bar{Q}_\beta\} = 0, \\ \{Q_\beta, \mathcal{H}\} = \{\bar{Q}_\beta, \mathcal{H}\} = 0, \\ \{Q_\beta, \bar{Q}_\beta\} = 2i\beta\mathcal{H}. \end{aligned} \tag{29}$$

Consequently, these supercharges, together with the Hamilton function $\mathcal{H}$, build up global $N = 2$ supersymmetry. The supersymmetry transformations leaving the Hamilton function $\mathcal{H}$ invariant are

$$\delta_s a^i = c^i, \quad \delta_s c^i = 0, \quad \delta_s \bar{c}_i = i q_i - \beta \partial_i H, \quad \delta_s q_i = -i\beta c^k \partial_i \partial_k H, \tag{30}$$

$$\bar{\delta}_s a^i = \omega^{ki} \bar{c}_k, \quad \bar{\delta}_s c^i = i\omega^{ik} q_k + \beta \omega^{ik} \partial_k H, \quad \bar{\delta}_s \bar{c}_i = 0, \quad \bar{\delta}_s q_i = i\beta \omega^{jk} \bar{c}_j \partial_i \partial_k H. \tag{31}$$

It is important to note that the topological twist (28) does not change the Lagrangian of the theory. It is well known that in the case of $d \geq 2$ topological field theories a crucial property of the twist is that it changes statistics of some fields. Apart from the case of the $d \geq 2$ theories, we are dealing with the fields which have no spin because it makes no sense in $d = 1$ case. So, the problem of changing of statistics is irrelevant.

The supersymmetry (29) has been originally found by Gozzi and Reuter[3], who stressed that it is of fundamental character in Hamiltonian systems. Particularly, this supersymmetry has been used[3] to derive the classical Kubo-Martin-Schwinger condition justifying algebraically the preference of the Gibbs distribution and has been related[5, 6, 7] to the regular/nonregular motion regimes in Hamiltonian systems with Hamiltonian, which does not explicitly depend on time.

In general, the supersymmetry garantees that there will be a set of exactly degenerate ground states. More specifically, if the supersymmetry is exact the Hamiltonian system is in the nonregular motion regime since, in this case, there is only one conserved entity (energy) while if the system is in regular motion regime, *i.e.* there is at least one additional nontrivial integral of motion, the supersymmetry is always broken. So, the condition of the supersymmetry breaking is of much importance and it can be thought of as a criterion to distinguish between the regimes. We will see shortly that it is naturally related to the Witten index. However, there is a subtlety to make one-to-one correspondence. Namely, broken supersymmetry does not necessary imply regular motion regime and, also, nonregular motion regime does not rule out broken supersymmetry. It seems that the problem lies precisely in possible degeneracy of the ground state, that is, $\ker \mathcal{H}$ modulo cohomological equivalence may consist of several elements[1]. The indication of this is that an ergodic Hamiltonian system is characterized just by *nondegenerate* zero eigenvalue solution of the Liouville equation. So, one is led to study physics coming from the degenerate vacuum. In the following, we assume there is a discrete set of supersymmetric ground states. This corresponds to the case of an elliptic operator on compact manifold.

The relation of the BRST and anti-BRST symmetries of the original theory to the $N = 2$ supersymmetry (29) is as follows. Usually, BRST exact theory is refered to as a topological theory. Anti-BRST exact theory is viewed as its conjugate, anti-topological theory. The crucial link between these two theories is the supersymmetric ground state sector, –refered to as *Ramond sector*– of the associated $d = 1$, $N = 2$ supersymmetric model, in accordance with the topological-antitopological fusion by Cecotti and Vafa[40]. Namely, the physical states of both the topological theories are in one-to-one correspondence with the Ramond vacua, as it can be seen in Sec. 3.2.

It is instructive to note here that as it has been argued recently by Perry and Teo[22], in the context of topological Yang-Mills theory, both the BRST symmetry and anti-BRST symmetry should be taken into account on an equal footing to get a clear geometrical meaning of the topological theory. It is worth stressing that this argument is supported by the cohomological classical mechanics, in which all the symmetries and fields have clear geometrical meaning, with both the BRST and anti-BRST symmetries being incorporated; see Table 1 and (17). In fact, this reflects canonical isomorphism between the tangent and cotangent spaces, TM^{2n} and T^*M^{2n}, provided by the symplectic structure.

3.2 COHOMOLOGY

Our next step in studying the physical states is identification of the $N = 2$ supersymmetry (29) with an exterior algebra, in analogy with the identification made in Witten's supersymmetric quantum mechanics[38]. This allows us to relate the supersymmetric properties of the model to topology of the target space M^{2n}.

To begin with, we mention that it has been argued[6] that cohomology of Q_β is isomorphic to de Rham cohomology. In a strict consideration, to which we are turning now, one should associate, in a standard way, an elliptic complex to it. Namely, we make the following identifications:

$$d_\beta \leftrightarrow Q_\beta, \quad \bar{d}_\beta \leftrightarrow \bar{Q}_\beta,$$

[1] The space $\ker \mathcal{H}$ is finite dimensional since $\mathcal{H}$ is identified with an elliptic operator.

$$\Delta_\beta \equiv d_\beta \bar{d}_\beta + \bar{d}_\beta d_\beta \leftrightarrow \{Q_\beta, \bar{Q}_\beta\} = 2i\beta\mathcal{H}, \tag{32}$$
$$(-1)^p \leftrightarrow (-1)^C,$$

where the exterior (co-)derivative $d_\beta = d + \beta c^i \partial_i H$ ($\bar{d}_\beta = d^* - \beta \bar{c}_i \omega^{ij} \partial_j H$) acts on p-forms, $\rho \in \Lambda^p$, and C is the ghost number.

The d_β cohomology groups,

$$H^p(M^{2n}) = \{\ker d_\beta / \mathrm{im}\, d_\beta \cap \Lambda^p\}, \tag{33}$$

are finite when M^{2n} is compact. According to Hodge theorem, canonical representatives of the cohomology classes $H^p(M^{2n})$ are harmonic p-forms,

$$\Delta_\beta \rho = 0. \tag{34}$$

They are closed p-forms,

$$d_\beta \rho = 0, \quad \bar{d}_\beta \rho = 0. \tag{35}$$

One can then define B_p as the number of independent harmonic forms, *i.e.*

$$B_p(\beta) = \dim\{\ker \Delta_\beta \cap \Lambda^p\}. \tag{36}$$

Formally, B_p continuosly varies with β but, being a discrete function, it is, in fact, independent on β so that one can find B_p by studying the vacua of the Hamilton function (20),

$$\mathcal{H}\rho = 0. \tag{37}$$

This equation is, in fact, the only equation we need to study. Due to (29), it can be rewritten as

$$Q_\beta \rho = 0, \quad \bar{Q}_\beta \rho = 0, \tag{38}$$

and defines the Ramond sector. The equivalence between (37) and (38) needs a comment. While it is obvious that (38) implies (37), vice versa may appear to be problematic for spaces with a lack of a positive-definite scalar product. For example, when proving that if an external differential form is harmonic then it is closed and coclosed, one uses the fact that scalar product of forms is positive definite. So, we define a scalar product in the space of p-forms in a standard way, $\langle \rho, \rho' \rangle = \int \rho \wedge *\rho'$, which is positive definite, to ensure that (38) follows from the harmonic condition (37).

One would expect that the spectrum of the theory is defined by the whole set of the eigenvalues κ_i and eigenfunctions ρ_i of the Hamilton function, $\mathcal{H}\rho_i = \kappa_i \rho_i$. However, only the states with zero eigenvalue of $\mathcal{H}$ are nontrivial in the BRST and anti-BRST cohomology since $\mathcal{H}$ commutes with both the BRST and anti-BRST charges. Indeed, by a standard argument all the states except for the ground states are of no relevance due to the supersymmetry (the superpartners' states give a net zero contribution). To be more specific, if we consider deformations δa^i along the solution of Hamilton's equation, then in order for $a^i + \delta a^i$ to still be a solution it has to satisfy the deformation equation $\partial_t \delta a^i = \delta h^i$.

This equation is the equation for the conventional Jacobi fields, first variations, which are tangent to the target manifold, $\delta a^i \in TM^{2n}$, and can be thought of as the "bosonic zero modes". As it is common in topological field theories, these modes are just compensated by anti-commuting zero modes through the ghost dependent term, in the Lagrangian (13). Indeed, the associated equation of motion for ghosts, $\partial_t c^i = \partial_k h^i c^k$, represents BSRT variation of Hamilton's equation.

Thus, the physical states are those in the Ramond sector, *i.e.* satisfying (38), which interwines corresponding BRST and anti-BRST symmetries. Topologically, the relation of the supersymmetry equations (38) to the cohomology equations (27) can be readily understood by taking into account the fact that the twist (28) is a homotopy operation.

According to the identification (32) with the exterior algebra, the physical states are harmonic p-forms on the target space M^{2n}. In the standard de Rham complex, the B_p's are simply Betti numbers, with the alternating sum, $\chi = \sum^{2n}(-1)^p B_p$, being the Euler characteristic of the symplectic manifold M^{2n}. Due to the identifications (32), it is then straightforward to show that the Witten index[39], $\mathrm{Tr}(-1)^C$, is just the Euler characteristic of M^{2n} (cf.[10]). Here, the conserved charge C is identified with the Fermi number operator, $F = C$. This is a natural result due to the fact that the Witten index is completely independent of finite perturbations of the theory for $N \geq 1$ supersymmetric theories in any dimensions. So, we conclude that the criterion for the regular/nonregular regimes in Hamiltonian systems which is related to the supersymmetry breaking is the Witten index. To break supersymmetry the Witten index needs to be zero. We arrive at the conclusion that the motion regimes are related to topology of M^{2n}.

Equation (37) can be thought of as a generalization of the *ergodicity* condition equation[46], $L\rho(a) = 0$, of the usual Hamiltonian mechanics which is now extended to the p-ghost (p-form valued) distributions, $\rho = \rho(a, c)$. We recall that nondegenerate solution of the latter equation characterizes an ergodic Hamiltonian system. It is wellknown that the general solution of this equation is a function of Hamiltonian, $\rho = \rho(H(a))$, The supersymmetry strengthen this statement by fixing dependence on H. Recent studies[7, 41, 42] of the physical states (38) showed that physically relevant (normalizable) solutions to the generalized ergodicity equation (37) come from the $2n$-ghost sector and have specifically the form of the Gibbs state characterizing thermodynamical equilibrium,

$$\rho = \kappa K^n \exp[-\beta H] \leftrightarrow \kappa \exp[-\beta H] da^1 \wedge \cdots \wedge da^{2n}, \tag{39}$$

where κ is a constant. It is important to note that under the field redefinition this state transforms as $2n$-form rather than as a scalar. The reason of this lies in the cohomology. The other ghost sectors yield solutions of the form $\rho = \kappa K^p \exp[+\beta H]$, for the even-ghost sectors, $p = 2, 4, \ldots, 2n-2$, and those are either trivial or not depending on β, for the odd-ghost sectors. It is assumed that

$\rho(a,c)$ must be normalizable in each p-ghost sector, *i.e.* $\int \rho(a,da)da^1\wedge\cdots\wedge da^p = 1,\ p = 0,\ldots,2n$. While the result for the even-ghost sectors is reliable and quite clear, the odd-ghost sector solutions are a bit cumbersome. In Appendix, we sketch analysis on solutions in simplest two-dimensional case[43], both to illustrate emerging of the Gibbs distribution and to clarify the meaning of the odd-ghost sector.

To summarize, we observe that the supersymmetry is helpful in obtaining some important results on Hamiltonian systems so that it is worthwhile to study features of the $d=1$, $N=2$ model more closely. We postpone this to Secs. 3.3-3.6.

Our next observation is that, in the limit $\beta\to 0$, we recover the classical Poincare integral invariants[36] corresponding to K^p, $p=1,\ldots n$, as the solutions of (38), which are indeed invariants under Hamiltonian flow, $\mathcal{H}K^p=0$. In particular, the $2n$-ghost K^n, which we are viewing as cohomological representative of the unit, corresponds to the volume form ω^n of the phase space conservation of which is statement of the Liuoville theorem. They are fundamental BRST invariant (topological) observables of the theory, $\{Q,K^p\}=0$, and form the classical cohomology ring, $K^{n+1}=0$. Indeed, in the untwisting limit, $\beta\to 0$, the supersymmetry generators Q_β and $\bar{Q}_\beta$ become the original BRST and anti-BRST operators, respectively. Geometrically, this follows from $dK^p=0$ since $K^p\leftrightarrow\omega^p$ and $d\omega=0$. Similarly, the conjugates of the Poincare invariants, $\bar{K}^p$, are the anti-BRST invariant (anti-topological) observables, $\{\bar{Q},\bar{K}^p\}=0$, $p=1,\ldots n$, which are powers of the Poisson bivector $\bar{K}$.

The following comments are in order.

(i) We recall that the physical states considered above are defined as the BRST *and* anti-BRST invariant ones. We see that this requirement, which is equivalent to unbroken supersymmetry, put strong limits on the possible physical states restricting it in effect to the (highest) $2n$-ghost sector (39). This is, in fact, a physically acceptable result leading to non-zero expectation values of scalar (ghost-free) observables, $\langle A(a)\rangle = \int A\rho$, whereas for the other p-ghost (p-form valued) observables, $1<p\le 2n$, we have that their averages are identically zero. However, we should note that one can consider only the BRST invariant theory, as the topological sector of the $d=1$, $N=2$ supersymmetric model.

(ii) Since the vacuum distribution forms, ρ, are annihilated by the supersymmetry charges (28) the modified forms $\lambda=\exp[-\beta H]\rho$ and $\tilde{\lambda}=\exp[\beta H]*\rho$ are d-closed. Here, $*$ denotes Hodge duality operator and according to even-dimensionality of M^{2n} we have $d=*d^**$ and $d^*=-*d*$. The forms $\tilde{\lambda}$ can be viewed as representatives of the *relative* de Rham classes (see, for example, Ref.[40]), $H^p(M^{2n},D)$, with $D\subset M^{2n}$ being the region where βH is greater than a certain large value. The forms λ correspond to the dual cohomology of the associated cycles, which form an integral basis for the Ramond vacua. A remarkable feature of the relative de Rham cohomology is that it can be non-trivial even if the usual de Rham classes of M^{2n} are trivial; for example, when $M^{2n}=R^{2n}$. However, we will not discuss further on the relative cohomology here restricting consideration on compact M^{2n}, for which case the usual de Rham classes are nontrivial.

3.3 CONNECTION TO MORSE THEORY

Let us to note that there appears to be no relation of the $d = 1$, $N = 2$ model (29) to Morse theory[47, 48, 52] quite analogous to that found in Witten's supersymmetric quantum mechanics[38], because of the absence of the term quadratic in h^i, in the Hamilton function (20), which would play a role of the potential energy.

An immediate reason is that the symplectic two-form ω is *closed* so that this does not allow us to construct, or obtain by the BRST procedure, non-vanishing terms in the Hamilton function quadratic in fields, except for the ghost-antighost term, which contains the Hessian, $\partial_i\partial_j H(a)$. Put differently, this is due to the symplectic structure of the target space M^{2n} which has been used as the *only* differential geometry structure to construct the cohomological classical mechanics. On the other hand, quadratic term, which is natural in (topological) quantum field theories when one uses Riemannian (or Kahler) structure of the target space, would produce stochastic contribution (Gaussian noise) to the equations of motion[49] that would, clearly, spoil the deterministic character of the Hamiltonian mechanics. As a consequence, we can think of linearity of the Lagrangian (13) in the *commuting* fields as a condition of the classical deterministic behavior of the system[2]. It should be emphasized here that the path integral approach to classical mechanics[2] relied basically on the work by Parisi and Sourlas[49] who studied classical stochastic equations.

Due to the absence of a term quadratic in h^i in the Hamilton function (20), there are no localized states and solitons similar to that of supersymmetric quantum mechanics which could be used to find a deeper connection between the (twisted) $d = 1$, $N = 2$ model and Morse theory. Although the Hamilton function (20) is linear in the commuting fields, it contains the Hessian of the Hamiltonian $H(a)$, which can serve as Morse function, the number of isolated critical points of which are known to be related to Euler characteristic. This link has been analyzed in detail in [4]. The critical points here are simply stationary points of the Hamiltonian flow, $h^i = 0$, with the number of critical points

$$\sum_{dH=0} sign(\det ||\partial_i\partial_j H(a)||). \tag{40}$$

We will mention a bit more on the connection to Morse theory in Sec. 4, in the context of partition function.

[2] Liouvillian L is a linear differential operator.

3.4 LANDAU-GINZBURG FORMULATION

As a preliminary observation, we note that the form of the definition (28) suggests that the Hamiltonian H plays the role analogous to that of superpotential in supersymmetric quantum mechanics (one-dimensional version of the Landau-Ginzburg model). Namely, action of the supercharge Q_β on forms can be casted in the form $Q_\beta \rho = \partial\rho + dH \wedge \rho$, where we have rescaled H by β for a moment.

Due to the underlying $N = 2$ supersymmetry (29), it is instructive to give a superfield representation of the cohomological classical mechanics which is usualy used in the $N = 2$ supersymmetric models as well as in the topological Yang-Mills theory[22], to write down the basic settings in a simple closed form. Advantage of this formulation is that one could readily see whether the cohomological classical mechanics admits a kind of Landau-Ginzburg description[12, 24] so that it could be largely understood in terms of superpotential. Since the superpotential is, in effect, the ordinary classical Hamiltonian $H(a)$, this arises to possibility to classify integrable Hamiltonian systems, within the Landau-Ginzburg framework.

Collection of fields composing a Landau-Ginzburg type system is the following: real field a^i, two anticommuting real fields c^i and $\bar{c}_i$, real field q^i, and superpotential W solely responsible for the "interaction" terms. We choose a single superfield as follows (cf.[2]):

$$X^i(t,\theta_1,\theta_2) = a^i(t) - i\theta_1 c^i(t) + i\theta_2 \omega^{ij}\bar{c}_j(t) + i\theta_1\theta_2\omega^{ij}q_j(t), \qquad (41)$$

where θ_I, $I = 1, 2$, are real anticommuting parameters, and the component fields are nothing but a collection of the BRST doublets. Geometrically, the components of the superfield form local coordinates of the tangent and cotangent fiber bundles over the phase space M^{2n} (see Table 1).

A manifestly covariant Lagrangian

$$\mathcal{L} = \int d\theta_1 d\theta_2 \left\{ \frac{1}{2}\omega_{ij}X^i D_1 D_2 X^j + iW(X) \right\}, \qquad (42)$$

can be written with the help of the covariant derivatives in the superspace with local coordinates (t, θ_1, θ_2),

$$D_1 = \frac{\partial}{\partial\theta_1} + i\theta_2\frac{\partial}{\partial t}, \quad D_2 = \frac{\partial}{\partial\theta_2} + i\theta_1\frac{\partial}{\partial t}, \qquad (43)$$

$D_1^2 = D_2^2 = 0$, $\{D_1, D_2\} = 2i\partial_t$, and the superpotential $W(X)$, which is a real analytic function of X. In terms of the component fields, we find the Lagrangian (42) in the form

$$\begin{aligned} \mathcal{L} = {} & q_i\partial_t a^i - a^i\partial_t q_i + i(\bar{c}_i\partial_t c^i + c^i\partial_t\bar{c}_i) \\ & + \frac{\partial W(a)}{\partial a^i}\omega^{ij}q_j - i\omega^{ij}\bar{c}_i\frac{\partial^2 W(a)}{\partial a^k\partial a^j}c^k, \end{aligned} \qquad (44)$$

where $W(a)$ is the lowest component of the superpotential. With the identification $W(a) = H(a)$, the Lagrangian (44) covers the original Lagrangian (13). So, we conclude that one can start with the one-dimensional Landau-Ginzburg $N = 2$ model (42) and obtain via topological twist the cohomological theory (13) with already gauge fixing. Here, topological twist provides transition from supercharges to BRST charges. Note that the Lagrangian (42) provides the action to be of the form of a D-term.

For completeness, let us to note that the whole set of BRST and anti-BRST transformations (3) and (4) takes the form of the following constraint (see also [2]):

$$(s_I - \frac{\partial}{\partial \theta_I}) X^i = 0, \tag{45}$$

where we have denoted $s_1 = s$, $s_2 = \bar{s}$, saying that the BRST and anti-BRST operators can be treated as derivatives in the odd coordinates of superspace. Therefore, the topological invariance of the action is obvious in superspace because of supertranslation invariance, $X^i(t, \theta_1, \theta_2) \to X^i(t, \theta_1 + \theta_1', \theta_2 + \theta_2')$, of the Berezin integration.

It is simple but important consequence of supersymmetry algebra that the action with the Lagrangian (42), like any D-term since it is the highest component, can be written *both* as $I = \{Q_\beta, \xi\}$ and $I = \{\bar{Q}_\beta, \bar{\xi}\}$, where ξ is some odd field integrated over time. Note also that we were not forced to use a F-term, which is defined as an integral over only half of superspace, to reproduce the original action.

The above are the essential ingredients necessary for arguments in analyzing implications of the supersymmetric structure of cohomological Hamiltonian mechanics.

For example, it is wellknown that the symplectic two-form ω can not be in general defined to be constant globally on compact M^{2n}, so the question arises as to cohomology classes of ω in M^{2n}. Two choices of the Lagrangian, for which ω are in different cohomology classes, differ by F-term. On the other hand, since the action is of the form of a D-term we have no topological effect of changing the classes which could be in principle considered by perturbing the action by a F-term.

From such a general point of view, it may seem that the problem on cohomological class $[\omega]$ of ω in the theory is extrinsic. However, in fact it has a direct link to time reparametrization invariance of the model.

Consider the time reparametrization $t \to e^{\phi} t$, where ϕ is a parameter ($d = 1$ Lorentz transformation). In the Lagrangian (42), the only effect it produces is the scaling $\omega \to e^{-\phi}\omega$. This implies scaling of the volume of compact phase space, $V = \int_{M^{2n}} \omega^n \to e^{-n\phi} V$, so that with the factor $1/V^{1/n}$ the Lagrangian (42) becomes time reparametrization invariant. Without loss of generality, assume that ω is an exact two-form in some region $U \subset M^{2n}$, that is, $\omega = d\vartheta$, where $\vartheta = \vartheta_i da^i$ is a symplectic one-form. The general consistency requirement is then that the Wilson loop integral $\exp[2\pi i \oint_{\partial D} \vartheta]$ should not depend on the disk D, for which ∂D is its boundary. Hence, we must have $\int_{S^2} \omega = k$, where k is an integer number. The scaling of ω demands $k = 0$, so we arrive at the conclusion that unless the condition

$$\int_{S^2} \omega = 0, \qquad \forall S^2 \subset M^{2n}, \tag{46}$$

is satisfied, the time reparametrization invariance of the model is broken.

The condition (46) is known in mathematical literature[32, 34] in another aspect, and essentially implies that the fundamental homotopy group must be nontrivial, $\pi_1(M^{2n}) \neq 0$. This can be seen as follows. As the symplectic two-form is closed but in general is not exact, the condition (46) means that all cycles $S^2 \subset M^{2n}$ are contractible, $[\omega]\pi_2(M^{2n}) = 0$. The class $[\omega]$ is nontrivial in $H^2(M^{2n})$, i.e. $[\omega] \neq 0$, since cohomology class of the volume form ω^n is n times the class $[\omega]$ and it is nontrivial since $V \neq 0$. If we let $\pi_1(M^{2n}) = 0$, we have the isomorphism $\pi_2(M^{2n}) \simeq H_2(M^{2n}, Z)$ according to the Gurevich theorem. Therefore, according to the de Rham theorem the condition (46) leads to $[\omega] = 0$, that contradicts to $V \neq 0$.

In other words, necessary condition for unbroken time reparametrization invariance of the model is that there should be non-contractible loops in M^{2n} for which, particularly, the Wilson loop integrals build up a representation. In Sec. 4.3, we show how one can construct a term leading to broken time reparametrization invariance for compact M^{2n} with nontrivial $\pi_1(M^{2n})$.

Below, we turn to some basic notions of the supersymmetric model relevant to subsequent consideration.

The most basic elements of the given $d = 1$, $N = 2$ theory are analogues of the chiral and anti-chiral rings of the (topologically twisted) two-dimensional $N = 2$ models[40]. Since we are originally interested in the topologically twisted $N = 2$ model only the ground states are kept, so it is simple to make identification of these with the operators. Namely, the chiral operators, ϕ_i, are defined as the ones satisfying $[Q_\beta, \phi_i] = 0$, and the anti-chiral operators, $\bar{\phi}_i$, satisfy $[\bar{Q}_\beta, \bar{\phi}_i] = 0$. They are irreducible representations of the supersymmetry algebra. Then, we can make a correspondence between the Ramond ground states defined by (38) and chiral fields by choosing a canonical ground state $|0\rangle$, with the identification $\phi_i|0\rangle = |i\rangle + Q_\beta|\lambda\rangle$. Similarly, there is a natural isomorphism between the anti-chiral fields and the adjoint states $|\bar{i}\rangle$. In terms of the Landau-Ginzburg formulation, the chiral ring consists of the polynomials modulo the relation $dW = 0$, which defines critical points of the Hamiltonian vector field.

The inner product on the space of the ground states corresponding to the fields ϕ_i and $\phi_{\bar{j}}$ is $g_{i\bar{j}} = \langle \bar{j}|i\rangle$, and geometrically plays the role of a metric in the associated Hilbert space while $\langle i|j\rangle$ gives rise to the topological metric, $\eta_{ij} = \langle \phi_i \phi_j \rangle_{top}$, which will be discussed in Sec. 4, in the context of correlation functions of BRST observables. The real structure matrix, $M = g\eta^{-1}$, relates the ground states with its adjoints, $\langle \bar{k}| = \langle j| M^{j}_{\bar{k}}$.

3.5 DEFORMATIONS AND PERTURBATIONS

Within the Batalin-Vilkovisky formalism (see for a review Ref.[55]), recent general analysis by Anselmi[54] of the predictivity and renormalizability of (reducible and irreducible) topological field theories which are known to be entirely determined by the gauge fixing (the classical action is either zero or topological invariant), shows that any topological field theory is predictive. The central point for that theories is thus the gauge fixing; for example, two gauge fixings which can not be continuosly deformed one into the other give rise to inequivalent theories.

In the case under study, the gauge fixing condition is Hamilton's equation whose deformations should be studied in order to garantee correctness of the definition of observables of the theory. Also, it would be interesting to study symmetry preserving perturbations of the action (7) in order to find metric of the supersymmetric ground state space (see discussion in Sec. 5) and possible deformations of the cohomology ring $\mathcal{R}$ of observables. However, there seems to be no nontrivial deformations of the cohomology ring since at least there is neither "quantum" nor conventional instanton corrections to the linear $d = 1$ theory.

In general, the supersymmetry preserving F-term perturbation of the action can be written using the chiral and anti-chiral fields,

$$\delta I = \sum_k \delta t^k \int dt \{\bar{Q}_\beta, [\bar{Q}_\beta, \phi_k]\} + \sum_k \delta \bar{t}^k \int dt \{Q_\beta, [Q_\beta, \bar{\phi}_k]\}, \tag{47}$$

where δt^k and $\delta \bar{t}^k$ are coupling parameters. Since we are interested in Hamiltonian systems, we leave the form of Hamilton's equation unchanged, and it is suffice for our purpose to look at the deformations of (i) Hamiltonian and (ii) symplectic tensor entering Hamilton's equation, to identify which type of them preserves the symmetries of the theory.

(i) Let us consider deformation of the Hamiltonian,

$$H(a) \to H(a) - \delta t_P P(a), \tag{48}$$

where $P(a)$ is a local polynomial, and δt_P is a coupling constant parametrizing the deformation. This leads immediately to the following perturbation of the action (7):

$$\int dt \mathcal{L} \to \int dt \mathcal{L} + \delta t_P \int dt \mathcal{O}_P, \tag{49}$$

where

$$\mathcal{O}_P = i q_i \omega^{ij} \partial_j P(a) - \bar{c}_i \omega^{ij} \partial_k \partial_j P(a) c^k. \tag{50}$$

Direct calculations show that $\mathcal{O}_P$ is BRST and anti-BRST closed, $s\mathcal{O}_P = \bar{s}\mathcal{O}_P = 0$; see (52) below, with $\beta = 0$.

The matter of an immediate interest is whether the possible deformations of the superpotential are in one-to-one correspondence with the possible topological perturbations of the theory, as it is, for example, in the Landau-Ginzburg formulation of $d = 2$, $N = 2$ superconformal field theories. Non-trivial topological perturbation may have place only if the deformation term is not a BRST exact cocycle. It can be readily checked that, in our case,

$$\mathcal{O}_P = s\{\bar{c}_i \omega^{ij} \partial_j P(a)\}, \tag{51}$$

so that there are no *nontrivial* topological perturbations coming from the deformation of the superpotential. In other words, nothing is changed in the topological sector when one deformes the superpotential by local polynomial. This result confirms our remark concerning the homotopical character of the topological twist (28).

However, supersymmetry appears to be sensitive to the deformation. We now examine the condition for the deformation $\mathcal{O}_P$ to be supersymmetry preserving. Using the definitions (28) and (50) we find directly

$$[Q_\beta, \mathcal{O}_P] = -i\beta c^k \partial_k(\omega^{ij}\partial_i H \partial_j P), \quad [\bar{Q}_\beta, \mathcal{O}_P] = i\beta\omega^{mn}\bar{c}_m\partial_n(\omega^{ij}\partial_i H \partial_j P). \tag{52}$$

Sufficient condition to both the commutators in (52) vanish is that the Poisson bracket

$$(\partial_i H)\omega^{ij}(\partial_j P) \equiv \{H, P\}_\omega = \text{const}, \tag{53}$$

or, more precisely, is equal to a locally constant function. For the case of linearly connected M^{2n}, equation (53) is necessary and sufficient condition for the Hamiltonian flows defined by the functions P and H to commute, with P viewed as another Hamiltonian[3]. We note that, in general, when one knows a Hamiltonian flow commuting with the flow under study it is possible to construct an integral of motion[36]. Hence, the equations (52) represent a link between the supersymmetry and integrability.

When examining the formal evolution of P, we see that P linearly changes with time, $dP/dt = \text{const}$. For compact connected M^{2n} polynomial $P(a)$ is bounded so that the constant must be zero, *i.e.* P is an integral of motion, $\{H, P\}_\omega = 0$. It has been argued[3, 7] that the existence of an additional *nontrivial* integral of motion leads to *broken* supersymmetry. This argument is based on analysis made on the form of the supersymmetric ground state (39). In view of this, polynomial P should be a trivial integral of motion to preserve the supersymmetry.

For noncompact M^{2n}, the polynomial P is not necessarily bounded so that we are left with the general condition (53). The same is true for M^{2n} with

[3] For the phase space with nontrivial π_1 one should use here local Hamiltonian flows.

nontrivial π_1 and also when P is an analytic function. However, in general, if (polynomial or analytic function) P is in involution with H it must be a *trivial* first integral.

From the above analysis we conclude that the symmetries do not fix the Lagrangian uniquely, with nontrivial supersymmetry preserving perturbation term (50), where P satisfies (53) with *non-zero* constant, can be added to the action. However, in the case of compact connected M^{2n} there are no nontrivial supersymmetry preserving perturbations supplied by the deformation with polynomial.

(iia) Let us turn to considering of deformation of the constant symplectic tensor. Under an infinitesimal change $\omega_{ij} \to \omega_{ij} + \epsilon_{ij}$, one sees from (28) that Q_β is invariant whereas $\bar{Q}_\beta$ changes by

$$\delta\bar{Q}_\beta = [Q_\beta, \bar{K}_\epsilon] = \bar{c}_i \epsilon^{ij}(iq_j + \beta\partial_j H), \tag{54}$$

with

$$\bar{K}_\epsilon = \frac{1}{2}\epsilon^{ij}\bar{c}_i\bar{c}_j, \tag{55}$$

where $\epsilon_{ij}\epsilon^{jk} = \delta_i^k$. Using (29) and (54) one finds that the Hamilton function changes by

$$\delta\mathcal{H} = \frac{1}{2i\beta}\{Q_\beta, [Q_\beta, \bar{K}_\epsilon]\} = q_i\epsilon^{ij}\partial_j H + i\bar{c}_i\partial_k(\epsilon^{ik}\partial_j H)c^j. \tag{56}$$

In order to preserve the $N = 2$ supersymmetry algebra, $\bar{K}_\epsilon$ must commute with $\bar{Q}_\beta$,

$$[\bar{Q}_\beta, \bar{K}_\epsilon] = \frac{1}{2}\omega^{kl}\partial_l\epsilon^{ij}\bar{c}_i\bar{c}_j\bar{c}_k, \tag{57}$$

where we have used $q_l = -i\partial_l$. The *r.h.s.* of (57) vanishes if and only if ϵ_{ij} are antisymmetric and constant so that it appears to be the case of a variation of the symplectic structure. As a consequence, this variation preserves also the BRST and anti-BRST symmetries. We notice that the form of Eqs.(55) and (56) is very suggestive to represent the Hamilton function in the form

$$\mathcal{H} = \frac{1}{2i\beta}\{Q_\beta, [Q_\beta, \bar{K}]\}. \tag{58}$$

This representation stemms naturally from combination of supersymmetry algebra (29) and ISp(2) algebra (17), and thus is specific to the model.

(iib) When one attempts change by *non-constant* tensor, $\epsilon_{ij} = \epsilon_{ij}(a)$, the previous arguments break down because the second equality in (54) does not hold, and, even more, the anti-BRST operator receives non-nilpotent contribution. This case, however, is important since, as it was mentioned above, constant symplectic tensor can not be in general globally defined on a symplectic manifold. For instance, on a compact one, for which case one uses a covering by local chartes with constant ω_{ij} owing to Darboux theorem telling us that in some neighborhood of any point one can find local coordinates such that $\omega = dx^\alpha \wedge dp_\alpha$. Also, a reasonable expectation is that this might yield a mechanism for supersymmetry breaking.

So, we are led to consideration of the model with a coordinate dependent symplectic structure, $\omega_{ij} = \omega_{ij}(a)$, so called Birkhoffian mechanics, in which one does not use the Darboux coordinates and attempts to treat symplectic structure in a full generality. Analysis made on this generalized model[6, 41, 42] has shown that with the following modification of the anti-BRST operator,

$$\bar{Q} = i\bar{c}_i \omega^{ij}(a) q_j - \frac{1}{2}\partial_k \omega^{ij}(a) c^k \bar{c}_i \bar{c}_j, \tag{59}$$

obtained by virtue of $[\bar{K}, Q] = \bar{Q}$, the ISp(2) algebra (17) is regained, with all the basic results of the constant symplectic structure case being reproduced.

The BRST approach to this generalized model can be readily developed in the same fashion as it for the case of Darboux coordinates. Besides slight modifications, which do not influence the algebraic structure of the original model, we encounter the following remarkable difference. According to the modification (59) the supercharge in (28) can be brought to the form

$$\bar{Q}_\beta = \bar{c}_i D^i - \frac{1}{2} f^{kl}{}_m c^m \bar{c}_k \bar{c}_l, \tag{60}$$

where

$$D^i = \omega^{ij}(a)(\partial_j + \beta \partial_j H) \tag{61}$$

and

$$f^{ijk} = \omega^{im}(a)\partial_m \omega^{jk}(a) - \omega^{jm}(a)\partial_m \omega^{ik}(a). \tag{62}$$

Our observation is that, in the BRST approach to gauge field theories, the operators placed similarly as D^i in (60) play the role of generators of Lie group characterizing gauge symmetry of the theory, and $f^{kl}{}_m$ are the structure constants. It is easy to check that D^i fulfills the commution rule

$$[D^i, D^j] = f^{ijk} D_k, \tag{63}$$

and, owing to Jacobi identity of the Poisson bracket algebra, $\omega^{im}\partial_m \omega^{jk} + \omega^{km}\partial_m \omega^{ij} + \omega^{jm}\partial_m \omega^{ki} = 0$, f^{ijk} satisfies

$$f^{ijk} + f^{kij} + f^{jki} = 0, \tag{64}$$

so that the operators D^i constitute a Lie algebra[4]. Note that the algebra defined by (63) is *degenerate* in Darboux coordinates of Hamiltonian mechanics, in which case we have identically $f^{ijk} = 0$.

[4]We assume that f^{ijk}'s are local constants.

Now, with the infinitesimal change, $\omega_{ij}(a) \to \omega_{ij}(a) + \epsilon_{ij}(a)$, the first equalities in Eqs. (54) and (56), where $\mathcal{H}$ and $\bar{Q}_\beta$ are defined by (58) and (59) respectively, are still valid while (57) becomes

$$[\bar{Q}_\beta, \bar{K}_\epsilon] = \frac{1}{2}\{\omega^{kl}(a)\partial_l \epsilon^{ij}(a) + \epsilon^{kl}(a)\partial_l \omega^{ij}(a)\}\bar{c}_i \bar{c}_j \bar{c}_k, \tag{65}$$

with the result is that, again, closeness of the two-form ϵ is sufficient for the *r.h.s.* of (65) to be zero, and thus the $N = 2$ supersymmetry to be preserved.

It is highly remarkable, however, that the above condition is equivalent to the one that the following Schouten bracket[30] is zero,

$$[[\omega, \epsilon]]^{kij} \equiv \sum_{(kij)} (\omega^{kl}\partial_l \epsilon^{ij} + \epsilon^{kl}\partial_l \omega^{ij}) = 0, \tag{66}$$

which is necessary and sufficient condition for ω and ϵ to be a Poisson pair[31], *i.e.* for $k_1\omega + k_2\epsilon$ to be a two-parameter family of tensors defining a Poisson bracket on M^{2n}.

So, the general result both for (iia) and (iib) is that the $N = 2$ supersymmetry is preserved under the deformation when Schouten bracket between ω and ϵ is zero.

The following comments are in order.

(i) We see that the supersymmetry imposes nontrivial condition (66) for deformation ϵ of the original Poisson bracket. Indeed, locally or globally, there may be both trivial and nontrivial deformations. Clearly, the class of global nontrivial deformations is related to topology of M^{2n} and, thus, is most interesting to investigate.

(ii) Also, one can study the anomalies,

$$[[\omega, \epsilon]] = \Gamma, \tag{67}$$

where antisymmetric rank-three tensor Γ measures supersymmetry breaking. We emphasize that, in general, this provides very attractive mechanism for supersymmetry breaking.

(iii) In some cases such anomalies may come naturally. Namely, it is known that some of nonlinear Poisson brackets describing dynamics of physical systems can be made linear by appropriate deforming original ω. Generally, it looks like one attempts a deformation inside the usual Poisson bracket so that we have not to extend our study for nonlinear Poisson bracket case. Note that such deformations are not trivial, at least locally. Particularly, in some cases they are parametrized by a set of parameters, and, as the supersymmetry is related to the motion regimes, one can use criterion $\Gamma = 0$ to find critical values of the parameters distinguishing between the regular and nonregular regimes. We will not discuss here specific examples which can be made elsewhere.

3.6 EQUIVARIANT EXTERIOR DERIVATIVE

In this Section, we briefly present construction of the model under study by the use of a generalized Mathai-Quillen formalism[32].

Clear geometrical meaning of the model suggests that its constructing can be refined using an *equivariant* exterior derivative. The generalized Mathai-Quillen formalism appeared to be useful[32] in analyzing supersymmetry properties of models describing classical dynamical systems, in an exterior calculus framework. Particularly, this technique can be used to construct the supersymmetric models for Hamiltonian systems which are not necessarily of cohomological type. Also, it provides a relevant basis to attempt breaking of supersymmetry, which appeared to be concerned to motion regimes discussed in Sec. 3.5. However, we will not try to use it for this purpose here, restricting our investigation on setting up the formulation.

Since the fields a^i and $\bar{c}_i$ can be viewed as local coordinates of the cotangent bundle T^*M^{2n} the corresponding basic one-forms can be identified with c^i and q_i respectively; see Table 1. The nilpotent exterior derivative on the exterior algebra in the space of mappings from circle S^1 to T^*M^{2n} is thus given by

$$d = \int dt \Big(c^i \frac{\partial}{\partial a^i} + q_i \frac{\partial}{\partial \bar{c}_i} \Big). \tag{68}$$

Comparing (68) with (3) we see that the exterior derivative d and the BRST operator $\int dt\ s$ are equivalent to each other (herebelow, we omit i factors for simplicity).

By introducing the interior multiplication operator along the vector field v,

$$v = (\partial_t a^i - h^i,\ \partial_t \bar{c}_k + \bar{c}_j \partial_k h^j), \tag{69}$$

namely,

$$i_v = \int dt \Big((\partial_t a^i - h^i)\tau_i + (\partial_t \bar{c}_i + \bar{c}_j \partial_k h^j)\pi^i \Big). \tag{70}$$

where τ_i and π^i form the basis of contractions dual to c^i and q_i respectively, *i.e.*

$$\tau_i c^k = \delta_i^k \delta(t-t'), \quad \pi^i q_k = \delta_k^i \delta(t-t'), \tag{71}$$

we define the following equivariant exterior derivative

$$Q_v \equiv d + i_v = \int dt \Big(c^i \frac{\partial}{\partial a^i} + q_i \frac{\partial}{\partial \bar{c}_i} + (\partial_t a^i - h^i)\tau_i + (\partial_t \bar{c}_i + \bar{c}_j \partial_k h^j)\pi^i \Big). \tag{72}$$

Note that the second component of the vector field v is Jacobi variation of the first one. The corresponding Lie derivative is given by

$$\ell = Q_v^2, \tag{73}$$

so that according to (72)

$$\ell = \int dt\Big(\partial_t a^i \frac{\partial}{\partial a^i} + \partial_t \bar{c}_i \frac{\partial}{\partial \bar{c}_i} + \partial_t c^i \tau_i + \partial_t q_i \pi^i + \ell_h\Big), \tag{74}$$

and hence $\ell = \int dt\ (\partial_t + \ell_h)$, where ℓ_h is a Lie derivative along h^i, and it is obvious that

$$[\ell, Q_v] = 0. \tag{75}$$

Eqs.(73) and (75) constitute a superalgebra. We see that (74) is the operator corresponding to the Liouville equation of classical mechanics. Action of the equivariant exterior derivative (72) on contraction of the basic one-forms,

$$B'' = c^i q_i, \tag{76}$$

yields the action,

$$I = Q_v B'' = \int dt\Big(q_i(\partial_t a^i - h^i) + \bar{c}_i \partial_t c^i + \bar{c}_i \partial_j h^i c^j - \partial_t(\bar{c}_i c^i)\Big), \tag{77}$$

where (71) has been used. It is equivalent to the original action with the Lagrangian (13). Zeroth of the τ_i component of the vector field in (72) are solutions to $\partial_t a^i = h^i$ so that the action (77) describes these field configurations. Comparing the derivation of (77) with the one of the BRST scheme, we see that the trick provided by this technique is that the gauge fixing condition, $\partial_t a^i - h^i = 0$, is encoded in the equivariant exterior derivative Q_v rather than it is described by the "gauge fermion" B'' and thus, in contrast to s, the operator Q_v itself carries information on the dynamical system.

4 BRST OBSERVABLES AND CORRELATION FUNCTIONS

4.1 BRST OBSERVABLES

In general, observables of interest are of the form

$$\mathcal{O}_A = A_{i_1 \cdots i_p}(a) c^{i_1} \cdots c^{i_p}, \tag{78}$$

which are p-forms on M^{2n}, $A \in \Lambda^p$.

In general, the space of p-forms on a manifold equiped by Poisson bracket has a structure of Lie superalgebra in respect to the following Karasev bracket (supercommutator) between the forms[31]:

$$[A, B]_K = d\omega(A, B) + \omega(dA, B) + (-1)^{deg(A)+1}\omega(A, dB), \tag{79}$$

where

$$\omega(A,B)_{i_1\cdots i_{m+n-2}} = \sum_{(i'_1,\cdots,i'_{m+n-2})} (-1)^{e(i'_1,\cdots,i'_{m+n-2})} A_{ri'_1\cdots i'_{m-1}} \omega^{rs} B_{si'_m\cdots i'_{m+n-2}}, \tag{80}$$

the sum is over all cyclic permutations, and $e(\cdots)$ denotes index of permutation. This is an algebra of observables in our case[5].

However, one can easily find that for the BRST invariant observables $\{Q, \mathcal{O}_A\} = 0$ if and only if A is closed since $\{Q, \mathcal{O}_A\} = \mathcal{O}_{dA}$. Consequently, the BRST observables correspond to the de Rham cohomology and form a classical cohomology ring $\mathcal{R}$ of M^{2n} which corresponds to the ring of chiral operators ϕ_i. So, for the BRST observables on a symplectic manifold the Lie superalgebra defined by (79) is *trivial* since for closed ω, A, and B we have identically $[A,B]_K = 0$.

The BRST observables are related immediately to the BRST invariant states, via construction analogous to that of relating the chiral fields to the supersymmetric ground states made in Sec. 3.4. The distribution forms are identified with differential forms as

$$A_{i_1\cdots i_p}(a) c^{i_1}\cdots c^{i_p}|0\rangle \leftrightarrow A_{i_1\cdots i_p}(a) da^{i_1}\wedge\cdots\wedge da^{i_p}. \tag{81}$$

In terms of the vacuum distribution forms, the isomorphism between the chiral fields and states in the Ramond sector becomes more explicit. Namely, the Hilbert space of the model consists of all square summable p-forms, $|A_i\rangle = |i\rangle$, with coefficients taking value in some linear bundle E on which the operators ϕ_i corresponding to the cohomology classes act by wedge product.

We note that since the flow equation is real the complex conjugate of the vacuum distribution form A_i should be again a vacuum distribution form, and thus can be expressed as a linear combination of the vacuum distribution forms.

4.2 HOMOTOPY CLASSES OF THE FIELDS

The basic field $a^i(t)$ is characterized by homotopy classes of the map $M^1 \to M^{2n}$. Clearly, these are in general classes of the map $S^1 \to M^{2n}$, that is the classes of conjugated elements of the fundamental homotopy group. These classes can be weighted with different phases and controlled by some coupling parameter. Namely, let us consider the space $\mathcal{E}(a_0, a_1)$ of the fields $a^i(t)$ coinciding with a^i_0 at $t = t_0$ and with a^i_1 at $t = t_1$. The functional integral (6) is performed over histories $\mathcal{E}(a_0, a_1)$, and can be presented as

$$Z = \sum_\alpha e^{ig_\alpha} \int_{\mathcal{E}(a_0,a_1)} D[X] \exp\ iI, \tag{82}$$

[5]Note that Karasev bracket (79) for forms corresponds to Schouten bracket for associated antisymmetric tensors.

where g_α is the phase and α runs over components of $\mathcal{E}(a_0, a_1)$. The case $a_0 \neq a_1$ can be reduced to the case $a_0 = a_1$ since $\mathcal{E}(a_0, a_1)$ is either homotopically empty, or homotopically equivalent to $\mathcal{E} = \mathcal{E}(a_0, a_0)$. Consequently, the components of $\mathcal{E}(a_0, a_1)$ are in one-to-one correspondence with elements of $\pi_1(M^{2n}, a_0)$. Furthermore, since the spaces of fields $a^i(t)$ at different fixed t are trivially equivalent to each other, the groups $\pi_1(M^{2n}, a_0)$ at different a_0^i are isomorphic to each other. So, we are led to consider closed paths, $a^i(t_0) = a^i(t_1)$, which are elements of the loop space $\mathcal{E}$ or, equivalently, fields on a circle, $t \in S^1$, with the index α in (82) running over $\pi_1(M^{2n})$[6]. Physically, as the energy of the system is finite, different homotopical classes of the fields can be thought of as they are separated by infinitely high energy barriers.

The fields are thus characterized by appropriate representations σ of the group $\pi_1(M^{2n})$, which we assume to be nontrivial, partially for the reason mentioned in Sec. 3.4. Therefore, the coefficients of the cohomology ring $\mathcal{R}$ take values in the linear bundles E_σ associated to the representations σ.

Thus, we should study specifically periodic orbits in M^{2n} characterized by period $T = |t_1 - t_0|$. The T-periodic solutions to Hamilton's equation are elements of the loop space of Hamiltonian system which is a subject of recent studies[32, 33, 34] on infinite dimensional version of Morse theory. We remark that periodic orbits are presumably dense in phase space and at finite time scale may mimic typical dynamics arbitrary well. Moreover, the families of periodic orbits have the unique property that they continue smoothly across the fractal boundary between the regular region and the chaotic region, with stable and unstable character in these regions respectively, being thus the only unifying agents between these two disparate regions.

When necessary one can replace circle by the real line by taking the limit $T \to \infty$. This procedure is useful from a general point of view, and, particularly, it allows one to extract[8] Lyapunov exponents, positive values of which are wellknown to be a strong indication of chaos in Hamiltonian systems, from correlation functions.

4.3 CORRELATION FUNCTIONS

Let us now turn to consideration of the correlation functions of the BRST invariant observables (78). If N is a closed submanifold of (compact) M^{2n} representing some homology class of codimension m ($2n$-m cycle), then, by Poincare duality, we have m-dimensional cohomology class A (m-cocycle), which can be taken to have delta function support on N[50]. Thus, any closed form A is cohomologious to a linear combination of the Poincare duals of appropriate N's. The general correlation function is then of the form,

$$\langle \mathcal{O}_{A_1}(t_1) \cdots \mathcal{O}_{A_m}(t_m) \rangle, \tag{83}$$

[6] For a complete definition, appropriate boundary condition for ghosts should be specified as well.

where A_k are the Poincare duals of the N's. Our aim is to find the contribution to this correlation function on S^1 coming from a given homotopy class of the map $S^1 \to M^{2n}$. The conventional techniques with the moduli space $\mathcal{M}$[11] consisting of the fields $a^i(t)$ of the above topological type can be used here owing to the BRST symmetry. Namely, the non-vanishing contribution to (83) can only come from the intersection of the submanifolds $L_k \in \mathcal{M}$ consisting of a's such that $a^i(t) \in N_k$, and we obtain familiar formula[51],

$$\langle \mathcal{O}_{A_1}(t_1) \cdots \mathcal{O}_{A_m}(t_m) \rangle_{S^1} = \#\Big(\sum^{m} \cap L_k \Big), \tag{84}$$

relating the correlation function to the number of intersections. As it was expected, the correlation functions do not depend on time but only on the indeces of the BRST observables. These results are typical for all topological field theories. Now we turn to some specific results following from this consideration.

Homotopically trivial sector.

This sector is characterized by a's which are homotopically constant maps, $[a^i(t)] = [a^i(t_0)]$, and, therefore, the correlation function (84) can be presented as

$$\int A_1 \wedge \cdots \wedge A_m. \tag{85}$$

The standard Poincare integral invariants, K^p, identified in Sec. 3.2 as fundamental BRST observables correspond to this homotopically trivial sector since they have been originally formulated in the Darboux coordinates, in which ω_{ij} are constant coefficients. Also, two-point correlation function can be used to define the topological metric, $\eta_{ij} \equiv \langle A_i | A_j \rangle = \int A_i \wedge A_j$, which is just the intersection form in the cohomology (cf. Ref.[28]). Particularly, it is easy to check that in canonical basis of the forms, the action of the real structure matrix M reads $*A_j^* = g_{i\bar{j}} A_i$, and the topological metric is $\eta_{ij} = \delta_{ij}$.

Note that circle S^1 is mapped by $a^i(t)$ to some one-dimensional cycle, $C(a) \subset M^{2n}$, associated to the field. Using this cycle and a closed two-form, ψ, one can construct a multivalued term which can be added to the action of the model. Namely,

$$I_\psi = 2\pi \int\limits_{\gamma(a)} \psi, \tag{86}$$

where $\gamma(a)$ is an arbitrary two-dimensional surface, for which $C(a)$ is its border, $\partial\gamma(a) = C(a)$. In general, such a surface may not exist since the cycle $C(a)$ may not be homological to zero. So, we restrict our consideration to the case when such a surface exists. This may be done either by imposing topological restriction $\pi_1(M^{2n}) = 0$ on the phase space that we still avoid to accept, or by considering homotopically trivial class of fields $a^i(t)$, for which $C(a)$ is homologically zero.

Clearly, the value of I_ψ may depend on the choice of $\gamma(a)$. However, when ψ is an *integer valued* two-form, *i.e.*

$$\oint_\gamma \psi = k, \qquad \gamma \subset H_2(M^{2n}, Z), \tag{87}$$

where γ is an arbitrary two-dimensional cycle and k is an integer number, then any two choices of $\gamma(a)$ in (86) differ by $2\pi k$ so that exp iI_ψ is univalued.

Thus, from the point of view of a functional formulation of field theory, for integer valued closed two-form ψ the term (86) is well defined. Such a term can be added to the original action, and may play important role when analyzing symmetries of the model. Particularly, we expect that with an appropriate choice of the cocycle ψ it can be used to break some of the symmetries. For example, in the case of low-energy limit of QCD inclusion of such a topological term provides breaking of excessive symmetry of Goldstone fields to meet experimental data.

One of the candidates for ψ is properly symplectic two-form ω. One can show that in this case time reparametrization invariance of the model for compact M^{2n} is necessarily broken. Indeed, I_ψ with $\psi = \omega$ breaks the time reparametrization invariance unless the condition $\oint_\gamma \omega = 0$ is satisfied because ω should obey both of (87) and (46). The latter condition entails that the cohomological class of ω is zero. However, for compact M^{2n} this class is necessarily nonzero.

Homotopically nontrivial sectors.

Existence of homotopically nontrivial Poincare invariants, K^p, follows from the fact that, globally, $\omega_{ij}(a)$ may not be chosen constant, and there are nontrivial homotopy classes of a^i. Nontrivial character of these invariants comes from the fact that cohomology class of ω on compact M^{2n} is necessarily nontrivial.

Let us turn to a particular kind of observables interwinning the BRST and anti-BRST ones. The BRST observable (78) can be naturally understood as $(p,0)$-form corresponding to the general (p,q)-form,

$$\mathcal{U}_A = A^{j_1\cdots j_q}_{i_1\cdots i_p}(a)c^{i_1}\cdots c^{i_p}\bar{c}_{j_1}\cdots\bar{c}_{j_q}|0\rangle \leftrightarrow A^{j_1\cdots j_q}_{i_1\cdots i_p}(a)da^{i_1}\wedge\cdots\wedge da^{i_p}\partial_{j_1}\wedge\cdots\wedge\partial_{j_q}, \tag{88}$$

which can be viewed as a function on $M^{4n|4n}$, where the anti-ghosts represent the anti-BRST sector. Accordingly, we associate the $(0,q)$-forms to the anti-BRST observables, which can be treated in the same manner as the BRST ones.

A particular kind of the observable (88) interwining the BRST and anti-BRST sectors has been studied recently by Gozzi and Reuter[8],

$$\mathcal{U}_A = \delta(a(t_0)-a_0)c^{i_1}(t)\cdots c^{i_p}(t)\bar{c}_{i_1}(t_0)\cdots\bar{c}_{i_p}(t_0)|0\rangle. \tag{89}$$

After normal ordering, the observable (89) can be thought of as the operator creating p ghosts from the Fock vacuum (p-volume form in TM^{2n}) at some time t_0 and point $a_0\in M^{2n}$, and then annihilating them at some later time t. Certainly, we should arrange also time-ordering to define this operator correctly. However, we have not to specify the time in the associated correlation function since we dealing with $t\in S^1$. Indeed, the correlation function for (89),

$$\langle\mathcal{U}_A\rangle_{S^1} = \langle\bar{A}(t)A(t_0)\rangle_{top} \equiv \Gamma_p(T,a_0), \tag{90}$$

does not depend on specific time, and Γ_p depends only on the period T.

A nice result[8] is that the higher order ($p\geq 1$) largest Lyapunov exponents can be extracted from this correlation function, namely,

$$l_p(a_0) = \sum_{m=1}^{p}\lim_{T\to\infty}\sup\frac{1}{T}ln\,\Gamma_m(T,a_0). \tag{91}$$

The p-form sector of the partition function (6) with appropriate periodic boundary conditions and the fields defined on circle,

$$Z_p(T) = \mathrm{Tr}_{S^1} \exp[-i\mathcal{H}_p t], \tag{92}$$

can be expressed in terms of Γ_p in the following normalized form:

$$Z_p(T) = \mathrm{Tr}_{S^1}\Gamma_p(t,a)/\mathrm{Tr}_{S^1}1, \tag{93}$$

where Tr_{S^1} denotes the path integral over all the a's, which are T-periodic solutions to Hamilton's equation.

The problem in computing $Z_p(T)$ given by (92) arises due to the fact that the integral over all the a's does not converge. Finite value can be evaluated when realizing that the integral receives contributions from the homotopy classes of the a's mentioned above so that we need to subtract excessive degrees of freedom due to (93).

Below, we perform explicit computation by realizing M^{2n} as a covering space, $f : M^{2n} \to Y^{2n}$, of a suitable linearly connected manifold Y^{2n} having the same fundamental group as it of S^1, $\pi_1(Y^{2n}, t_0) \simeq \pi_1(S^1, t_0)$. Since $\pi_0(Y^{2n}, t_0) = 0$ the set of preimages is *discrete*,

$$f^{-1}(t_0) = \{a_0, a_1, \ldots\}. \tag{94}$$

The number of elements of $f^{-1}(t_0)$ and of the monodromy group given by the factorization,

$$G = \pi_1(Y^{2n}, t_0)/f_*\pi_1(M^{2n}, a_0), \tag{95}$$

coincides due to canonical one-to-one correspondence between the set $f^{-1}(t_0)$ and the monodromy group G, and does not depend on t_0 due to $\pi_0(Y^{2n}, t_0) = 0$. Hence,

$$Z_p(T) = \sum_{g\in G} \Gamma_p(T, a_g), \tag{96}$$

which is finite if G is a finite group.

Clearly, the Z_p's are topological entities, which can be used to define *topological* entropy[8] of Hamiltonian systems. Also, in terms of Morse theory it has been shown[4, 10] that the partition function $Z(T)$ at $T \to 0$ localizes to critical points of H the number of which is given by (40) so that the sum $\sum(-1)^p Z_p$ is the number of T-periodic solutions to Hamilton's equation and is equal to Euler characteristic of M^{2n}. Moreover, recent studies[32] showed that if Hamiltonian H is a *perfect* Morse function on M^{2n}, for which case one has to have $H^{2k+1}(M^{2n}) = 0$, it saturates the lower bound in the Arnold conjecture, which states that the number of nondegenerate contractible T-periodic solutions of Hamilton's equation is greater or equal to sum of Betti numbers of M^{2n}.

In regard to the case of the symplectic tensor ω_{ij} depending on phase space coordinates considered in Sec. 3.5, we have every reason to believe that the BRST observables and the correlation functions will remain essentially of the same form because they are intimately related to topology, which is insensitive to the implemented differential (symplectic) structure.

5 DISCUSSION

After having analyzed the main ingredients of the construction we make comments on the obtained results and discuss briefly on the open problems and further developments.

As we have already mentioned in Sec. 2, the symmetries of the resulting Hamilton function (20) appeared to be even more than the BRST and anti-BRST symmetries we have demanded upon. Indeed, the action I is invariant under the ISp(2) symmetry generated by the charges (16). So, the reason of the occurance of the additional symmetries, K, $\bar{K}$, and C, should be clarified, in the context of the BRST approach. For sure, these symmetries are natural and establish the Poincare integral invariants, as the fundamental topological observables, its conjugates, and the ghost-number conservation. Supersymmetry and also some of the above symmetries might be broken by the term (86), the role of which should be investigated in a more detail.

There are many directions worth pursuing. Probably the most interesting are the following.

(i) It is interesting to develop BRST approach to the theory with explicit accounting for conservation of Hamiltonian, $\dot{H} = 0$. Due to the fact that the $(2n-1)$-dimensional submanifold, $M^{2n-1} \subset M^{2n}$, of constant energy, $H(a) = E$, is invariant under the Hamiltonian flow, and the p-forms evolve to p-forms on M^{2n-1}[8], one can treat this as a "reducible" action of the symplectic diffeomorphisms, for which case more refined Batalin-Vilkovisky gauge fixing formalism (see for a review Ref.[55]) can be applied, instead of the usual BRST one used in this paper. In general, the problem is to construct cohomological theory for *reduced* phase space of the model.

The submanifolds M^{2n-1}, forming a one-parameter family, have a rich set of possible topologies, depending on the value of E, so that more refined analysis can be made on generic Hamiltonian systems, for example, on bifurcations (of invariant Liouville tori) in the system. Indeed, this approach having a great deal of cohomology might yield information regarding topology and topology changes of the submanifolds. Studies on classifying topologies of the constant energy submanifolds are known in the mathematical literature. Particularly, in four-dimensional case ($n = 2$), the submanifolds for Hamiltonian systems having Bott integral of motion defined on the submanifold has been studied by Fomenko *et al.*[57], who succeeded in complete classification of possible topologies of the three dimensional submanifolds for this case.

Particular way to construct the effective theory is that one can start with the theory with M^{2n} as a target space. Next, implement the energy conservation constraint $\delta(H(a) - E)$ to the path integral, that gives rise to additional Lagrange multiplier, and introduce an auxiliary field to constrain the flow to be tangent to the appropriately embedded hypersurface M^{2n-1}, together with accompanying ghost and anti-ghost fields. More geometrical way to account for the energy conservation might be to extend the phase space by local coordinates $x^{n+1} = t$ and $p^{n+1} = E$, and define the extended symplectic two-form $\omega' = \omega - dt \wedge dE$. Then, the associated Hamilton's equation for the extended Hamiltonian $H' = H - E$, on the hypersurface defined by the equation $H' = 0$, is equivalent to the ordinary Hamilton's equation on M^{2n} plus two equations, $\dot{t} = -\partial H'/\partial E$ and $\dot{E} = \partial_t H$. Having such an extended formulation one might rerun appropriately modified BRST procedure made in Sec. 2.

One of the remarkable points we would like to note here is the following. The constraint in the form $H(a) - \lambda E = 0$ with λ viewed as a gauge parameter, can be accounted for to obtain the theory with gauge fixed symmetry in respect to transformation $\delta\lambda$. This corresponds exactly to the so called scaling systems, *e.g.* billiards, that have the same dynamics at all energies and have received most of the attention so far because they are easier to analyze and in many cases display hard chaos.

(ii) More general approach to the above problem is to develop a general functional scheme for *Poisson manifold* instead of symplectic one by relaxing the closeness and nondegeneracy conditions for two-form ω. The reason is that Poisson bracket, which is a central point of consideration in this case, may be degenerate, for example, for constrained systems, so one is led to study symplectic shelves of Poisson manifold[31] on which Hamiltonian dynamics is well defined and easier to treat. Classical Lie-Cartan reduction of the phase space and celebrated theorem of non-commutative integrability (KAM theory) are specific examples of such an approach. Also, notice that Dirac bracket formalism is used to restore Poisson brackets from known symplectic shelves defined by integrals of motion.

In general, this leads to consideration of nonlinear and/or degenerate Poisson brackets, which are in fact most worth to study since many systems reveals such a Poisson structure (after or even without reduction of their phase space); for example, oscillator, pendulum, Euler rotations of rigid body, spin dynamics of B-phase of superfluid ^3He, and systems described by classical Yang-Baxter equation. Also, we note that due to (iib) of Sec. 3.5, in the degenerate case, even small deformations of Poisson bracket may cause global changes in topology of symplectic shelves (bifurcations).

The second reason of importance to develop functional approach to dynamics on Poisson manifold is that just Poisson bracket is a subject for usual and deformational quantization.

(iii) We note that there is a tempting possibility to start with a nontrivial topologically invariant underlying action I_0, if it exist, instead of the trivial one. The problem is to construct an appropriate nontrivial topological underlying Lagrangian $\mathcal{L}_0$, if any, for which the action I_0 will not be dependent on the metric on M^1, a positive definite function of time, *einbein* $g = g(t)$, that is, $\delta g =$ *arbitrary*, $\delta I_0 = 0$. Such an invariance of the total action would be a kind of time reparametrization invariance, $t \to \exp[\phi(t)]t$, and, in fact, means coupling of the model to one-dimensional gravity (see, for example, Ref.[56]).

We note in this regard that one may be led to *localize* the BRST symmetry

(3). To do this, one can introduce gauge field η with ghost number -1, associated ghost b with ghost number zero, and define local version of the BRST operator, s_l, with the closed BRST algebra being of the form $s_l a^i = bc^i$, $s_l c^i = 0$, $s_l \bar{c}_i = bq_i$, $s_l q_i = 0$, $s_l \eta = -\partial_t b$, $s_l b = 0$. Then, replacing ∂_t in the gauge fermion (8) by the BRST covariant operator, $\partial_t - \eta s$, one arrives at the locally BRST invariant Lagrangian, $\mathcal{L} = s_l B$. Simple calculations shows that the ghost b appears as an overall factor and thus can be got rid of by rescaling B, with the resulting Lagrangian being of the same form as (10) minus $iq_i\eta c^i$. The superspace interpretation of the gauge field η is quite clear, namely, it is a mixed component of the superspace metric, $dx^2 = g^2dt^2 + \eta dtd\theta_1 + d\theta_1 d\theta_2$. So, when requiring the metric independence of the total action, one may insist on the independence on the gauge field η as well. It is highly remarkable to note that the latter may impose nontrivial constraints on the form of Hamiltonian vector field since it is the only auxiliary field in the theory outside of the supermultiplet.

(iv) Ultimately, of course, one would like go further in the analysis of the $d = 1$, $N = 2$ supersymmetric model. One of the interesting problems, which escaped consideration in this paper, and is presumably of much importance is geometry of supersymmetric ground states, forming a space on the coupling parameters entering (47). The metric of the Ramond ground states, $g_{i\bar{j}}$, is used to extract interesting information on the physics, and satisfies the topological-antitopological ($t\bar{t}$) equations[40]. In many cases they reduce to a familiar equation of mathematical physics. It seems that valuable information can be obtained when analyzing $t\bar{t}$ equations for the $d = 1$, $N = 2$ model under consideration, for which we have shown that it admits Landau-Ginzburg description. An example of the type of questions that we might want to understand in the context of classical dynamical systems is, what is the model where the same equations as the $t\bar{t}$ ones, for this case, appear naturally. The difference from the known analysis of the $t\bar{t}$ equations, both in $d = 1$ and $d = 2$ cases[28], may arise because not every symplectic manifold admits a Kahler structure.

(v) Further development can be made along the line of the phase space formulation of ordinary quantum mechanics originated by Weyl, Wigner and Moyal[58]. The key point one could exploit here is that it is treated as a *smooth $\hbar$-deformation*[60] of the classical mechanics (see also [59]-[61]). Indeed, there is an attractive possibility to give an explicit geometrical BRST formulation of the model describing quantum mechanics in phase space, following the lines of the present paper. The resulting theory could be thought of as a topological phase of quantum mechanics in phase space. The crucial part of the work has been done[60] in the path integral formulation, where the associated extended phase space and quantum $\hbar$-deformed exterior differential calculus in quantum mechanics has been proposed. The core of this formulation is in the deforming of the Poisson bracket algebra of classical observables.

The central point we would like to use here is that the extended phase space can be naturally treated as the cotangent superbundle $M^{4n|4n}$ over M^{2n} endowed with the second symplectic structure Ω and graded Poisson brackets (21).

Besides clarifying the meaning of the ISp(2) algebra, appeared as a symmetry of the field theoretic model, it allows one, particularly, to combine symplectic geometry and techniques of fiber bundles. The underlying reason of our interest in elaborating the fiber bundle construction is that one can settle down Moyal's $\hbar$-deformation in a *consistent* way by using both of the Poisson brackets, $\{,\}_\omega$ and $\{,\}_\Omega$. Namely, the two symplectic structures and Hamiltonian vector fields coexisting in the single fiber bundle are related to each other[62]. Note that this relation is not direct since $\{a^i, a^j\}_\omega = \omega^{ij}$ while for the projection of coordinates in the fundamental Poisson bracket $\{\lambda^a, \lambda^b\}_\Omega = \Omega^{ab}$ to the base M^{2n} we have $\{a^i, a^j\}_\Omega = 0$. Also, Z_2 symmetry (15) of the undeformed Lagrangian (13) can be used as a further important requirement for the deformed extension. Naively, the problem is to construct $\hbar$-deformed BRST exact Lagrangian, identify BRST invariant observables, and study BRST cohomology equation and corresponding correlation functions. Also, having the conclusion that the $d = 1$, $N = 2$ supersymmetry plays so remarkable role in the classical case it would be interesting to investigate its role in the quantum mechanical case.

In this way, one might formulate, particularly, quantum analogues of the Lyapunov exponents (91) in terms of correlation functions rather than to invoke to nearby trajectories, which make no sense in quantum mechanical case. The case of compact classical phase space corresponds to a finite number of quantum states. Also, we note that for chaotic systems expansion on the periodic orbits constitutes the only semiclassical quantization scheme known. Perhaps, this is a most interesting problem, in view of the recent studies of quantum chaos.

However, we should emphasize here that the geometrical BRST analogy with the classical case is not straightforward, as it may seem at first glance, since one deals with non-commutative geometry[63] of the phase space in quantum mechanical case (see Ref.[61] and references therein). Particularly, quantum mechanical observables of interest are supposed to be analogues of the closed p-forms on M^{2n}, with noncommuting coefficients arising to nonabelian cohomology.

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A APPENDIX

In Appendix[43], we obtain explicitly general solutions of the supersymmetric ground state equations, in the case of two-dimensional phase space, $n = 1$, to illustrate emerging of the Gibbs distribution.

In the phase space with Darboux coordinates $a^1 = p$ and $a^2 = x$ the non-vanishing coefficients of the symplectic tensor are given by $\omega^{12} = -\omega^{21} = 1$. The general expansion of the ghost dependent distribution reads

$$\rho(a,c) = \rho_0(a) + \rho_1(a)c^1 + \rho_2(a)c^2 + \rho_{12}(a)c^1c^2. \tag{97}$$

In general, each ghost sector in $\rho(a,c)$ can be used to define some ordinary type of distribution. The ground state equations (38) then read

$$c^1(\partial_1 - \beta h_1)\rho_0 = 0, \quad c^2(\partial_2 - \beta h_2)\rho_0 = 0, \tag{98}$$

$$c^1(\partial_1 + \beta h_1)\rho_{12} = 0, \quad c^2(\partial_2 + \beta h_2)\rho_{12} = 0, \tag{99}$$

$$c^1c^2\big[(\partial_1 - \beta h_1)\rho_2 - (\partial_2 - \beta h_2)\rho_1)\big] = 0, \tag{100}$$

$$(\partial_1 + \beta h_1)\rho_2 - (\partial_2 + \beta h_2)\rho_1 = 0. \tag{101}$$

Here, $\partial_i \equiv \partial/\partial a^i$ and $h_i \equiv \partial H(a^1,a^2)/\partial a^i$ $(i = 1,2)$. For the ghost-free sector (98) and the two-ghost sector (99) we have immediately

$$\rho_0 = \kappa_0 \exp[+\beta H], \quad \rho_{12} = \kappa \exp[-\beta H], \tag{102}$$

where κ_0 and κ are constants. These ghost sectors define scalar and pseudoscalar distributions, $\rho_s = \rho_0$ and $\rho_{ps} = \rho_{12} da^1 \wedge da^2$, respectively.

The equations (100) and (101) can be rewritten as

$$\partial_1\rho_2 - \partial_2\rho_1 = 0, \quad 2\beta(h_1\rho_2 - h_2\rho_1) = 0, \tag{103}$$

or, taking $\beta > 0$,

$$\frac{h_2}{h_1}\partial_1\rho_1 - \partial_2\rho_1 = -\rho_1\partial_1\frac{h_2}{h_1}, \quad \rho_2 = \frac{h_2}{h_1}\rho_1. \tag{104}$$

To solve the nonhomogeneous first-order partial differential equation (104) for ρ_1, we write down, by a standard technique, its characteristic equations,

$$\frac{da^1}{dr} = \frac{h_2}{h_1}, \quad \frac{da^2}{dr} = -1, \quad \frac{d\rho_1}{dr} = -\rho_1\partial_1\frac{h_2}{h_1}, \tag{105}$$

where r is a parameter, from which the first and the second integrals follow,

$$U_1 = \int (h_1 da^1 + h_2 da^2), \quad U_2 = \frac{h_2}{h_1}\rho_1. \tag{106}$$

The general solution is then of the form $\Phi(U_1, U_2) = 0$, where Φ is a function, that is, we can write

$$\rho_1 = \frac{h_1}{h_2} f(U_1), \tag{107}$$

and hence $\rho_2 = f(U_1)$, where f is an arbitrary function. Symmetrically, one can arrive at the solutions in the form $\rho_1 = f(U_1)$ and $\rho_2 = (h_2/h_1)f(U_1)$. Geometrically, the odd-ghost sectors ρ_1 and ρ_2 constitute the vector distribution, $\rho_v = \vec{\rho}(a)d\vec{a}$.

To clarify the possible meaning of such a distribution we make some comments. It is clear that this distribution is not of a Gibbs form, does not depend on the parameter $\beta > 0$, and singular at the critical points of the gradient vector field $\vec{h} \equiv (h_1, h_2)$. The latter implies that ρ_v is not in general normalizable. However, in the region that does not include critical points of $\vec{h}$ the distribution ρ_v is well defined. When, additionally, we specify the integrating in U_1 to be over a closed path ∂D we have $U_1 = 0$ identically since $U_1 = \int_{\partial D} \vec{h}\, d\vec{a} = \int_D \mathrm{rot}\vec{h}\, d\vec{\sigma} = \int_D \mathrm{rot}\,\mathrm{grad} H\, d\vec{\sigma} \equiv 0$. Hence, the solution (107) reduces to $\rho_1 = h_1/h_2$ and $\rho_2 = \mathrm{const}$.

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