SUPERSYMMETRIC PROPERTIES OF BIRKHOFFIAN MECHANICS

A.K.Aringazin
Department of Theoretical Physics
Karaganda State University
Karaganda 470074, Kazakhstan*
and
The Institute for Basic Research
P.O. Box 1577
Palm Harbor, FL 34682, U.S.A.

Received July 3, 1993
Revised version received September 12, 1993
Hadronic J. 16 (1993) 385-409.

Abstract

Recently proposed path integral approach to Hamiltonian mechanics arised naturally to a fundamental supersymmetry lying behind dynamical properties of the Hamiltonian systems. $2n$-ghost sector is proved to be the only sector providing non-trivial physically relevant BRS and anti-BRS invariant state, which is the Gibbs state. By applying the path integral formalism, we study supersymmetric properties of the Birkhoffian generalization of the Hamiltonian mechanics which is characterized by allowing a fundamental 2-form to be dependent on phase space coordinates. It has been shown, in the Birkhoffian mechanics, that among the even-ghost sectors only the $2n$-ghost sector yields relevant invariant state.

*Permanent address.

1 Introduction and summary

Path integral technique is known to be a practical tool for evaluating correlation functions as functional derivatives of the associated generating functional.

Path integral approach in treating stochastic Langevin equations has been developed by Parisi and Sourlas[1, 2]. This formalism naturally arises to involving anticommuting variables so that the associated action appeares to be invariant under a set of supersymmetric transformations. The supersymmetric invariance of the stochastic action is found to have a close relation to the wellknown Onsager principle of microreversibility[3]. Gaussian noise, a sourse of the stochasticity, can be integrated away in the action so that the associated Lagrangian presents, in fact, the effect of the noise, in a "dynamical" way.

Zinn-Justin[4] found that it is no matter has the noise just a Gaussian form or not, - the action is still invariant under a set of supersymmetric BRS-like transformations. This BRS invariance of the general (Gaussian or non-Gaussian) stochastic systems corresponds in effect to the invariance of the stochastic equations under transformations related to a shift in the noise. It should be noted that the latter transformations are essentially nonlinear while the corresponding BRS transformations are linear ones[4] .

Recently, Gozzi[5] proposed a *path integral approach to Hamiltonian mechanics* based on the formalism by Parisi and Sourlas[2] and Zinn-Justin[4]. Gozzi noticed that due to the fact that the stochastic systems reveal the supersymmetric BRS invariance whatever might have been the weight of the noise one can take even a delta function distribution for the noise. This means that there is no noise in this case so that one deals with a non-stochastic, deterministic dynamical system, which inherits the supersymmetric properties. Then, one can apply the functional integral formalism to classical mechanics, dynamics equations of which contain no noise sourse.

In a series of papers, Gozzi, Reuter and Thacker[6, 7, 8, 9, 10] developed the application of path integral technique to Hamiltonian systems. They found[6] that Hamiltonian systems indeed reveal an universal supersymmetry, with the supersymmetric charges forming the algebra of inhomogeneous symplectic $ISp(2)$ group. They established the correspondence of all these conserved supersymmetric charges with the symplectic geometry of phase space[6, 7].

Associated supersymmetric invariant Hamilton function appeared to be a sum of the conventional Liouvillian and a term containing anticommuting variables. So, in the "even" part the formalism meets exactly the Liouvillian (operatorial) formulation of Hamiltonian mechanics constructed earlier by Koopman[11] and von Neumann[12]. This is similar to the case of quantum mechanics where the weight lying behind the Schrodinger operator is $exp(iS_{cl})$, in Feynman's path integral. The states, on which the Hamilton function acts, are defined on the extended phase space, which includes anticommuting variables - ghosts - in addition to the ordinary coordinates of the phase space M_{2n}.

There are also conserved supersymmetric charges of a dynamical origin, that is forming, together with the Hamilton function, a supersymmetric algebra.

This $N = 1$ supersymmetry of the Hamilton function has a close relation with the problem of *ergodicity.* Indeed, as it is wellknown[13] ergodic dynamical systems have the only analytic constant of motion - the energy - so that the Liouvillian must have only one eigenstate, for density function, with zero eigenvalue, at fixed energy. It has been shown[7] that such a non-degenerate eigenstate for *2n-ghost* density is just a Gibbs state, *i.e.* the state depending only on the energy. Thus, when the $N = 1$ supersymmetry is exact the system described by $2n$-ghost density is in ergodic phase (unordered motion or deterministic chaos) while when the system is in regular motion phase the supersymmetry is always broken[7, 8, 14]. This gives a new criterion to detect transitions between ordered and unordered motion regimes in Hamiltonian systems[15]. In this regard, it would be interesting to study transient chaos phenomenon exhibited by many dynamical systems which is characterized by signals looking chaotic for a certain time before reaching stationarity, which can be chaos or regular motion alike. Another interesting aspect of this porblem is to find links between the supersymmetric properties of Hamiltonian mechanics and Kolmogorov-Sinai entropy, which is known to be sensative to transitions between regular motion and chaos, in dynamical systems. It has been shown recently[16] that the Lyapunov exponents turn out to be related to the partition functions of the Hamilton function restricted to the spaces of fixed ghost number.

Also, algebraic characterization of the Gibbs form condition of the equilibrium states known as classical KMS condition[17] can be obtained as a consequence of the supersymmetric invariance of the density function.

Conventionally, in Hamiltonian mechanics one deals with a constant fundamental symplectic 2-form ω. However, in general ω can be any closed and non-degenerate 2-form. Then, one can allow the 2-form to be dependent on phase space coordinates provided that the symplectic manifold can be always covered by local charts with constant 2-forms due to Darboux theorem.

It has been shown[18] that supersymmetry is still survived in this case, and the action is invariant under linear as well as nonlinear canonical transformations. However, anti-BRS charge has to be slightly modified by the term proportional to derivative of the 2-form on phase space coordinates, to preserve the algebra of $ISp(2)$[18].

On the other hand, generalization of Hamiltonian mechanics based on generalized 2-form depending on phase space coordinates has been proposed earlier by Santilli[19] and developed in detail in his monograph[20]. This generalization is reffered to as *Birkhoffian mechanics.* Birkhoffian mechanics presents in fact a new mechanics generalizing each and every aspect of the conventional Hamiltonian mechanics.

Consistency of the Birkhoffian mechanics is provided by Lie-isotopic construction[20] assuming derivability of dynamics equations from a variational principle, Lie character of the underlying brackets, and existence of a generalized Hamilton-Jacobi theory.

The Lie-isotopic construction provides in fact both conventional local-differential approach to classical mechanics and non-local (integral) one. Hamiltonian mechanics as well as Birkhoffian one are classified as local-differential formulations with the ordinary symplectic geometry exploited while the Hamilton-Santilli and Birkhoff-Santilli mechanics[23] present non-local (integral) formulations of classical mechanics. The latters are based on symplectic-isotopic structure generalizing the usual symplectic geometry, and defined over isotopically lifted field of reals (isoreals). Also, two dual fields, namely, dual field of reals and dual field of isoreals enables one to construct additional formalisms for (non-local) classical mechanics[23].

In this paper, we show that in the path integral approach to Hamiltonian mechanics *only* $2n$-ghost sector provides non-trivial physically relevant solution for BRS and anti-BRS invariant state, which is characterized by the Gibbs form, $\kappa exp(-\beta H)$, while the other sectors imply either trivial solution or solution characterized by $exp(+\beta H)$. Gozzi *et al.*[14] rised the question what might be an analogue of the Gibbs state obtained in $2n$-ghost sector in the other ghost sectors. The answer is that the other ghost sectors yield only trivial or physically irrelevant solutions (see also[28]). Also, we analyse the BRS and anti-BRS invariance equations in the case of Birkhoffian generalization. We found that among the even-ghost sectors only $2n$-ghost sector leads to non-trivial physically relevant solution, which is again the Gibbs state. Explicit calculations in two-dimensional ($n = 1$) case indicate that all odd-ghost sectors yield only trivial solution.

The paper is organized as follows. In Sec 2, we review the path integral approach to Hamiltonian mechanics[6]. We start with presenting a brief sketch of the main results of the path integral approach to stochastic processes from which the approach to Hamiltonian mechanics arised (Sec 2.1). Conserved supersymmetric charges forming the algebra of $ISp(2)$ (Sec 2.2) reflect symplectic geometry of the phase space (Sec 2.3). There are also supercharges of a dynamical origin forming a genuine $N = 1$ supersymmetry of the Hamiltonian mechanics (Sec 2.4). One of the most interesting implications of the approach is that the supersymmetry is connected to ergodicity of the classical systems. We prove that $2n$-ghost sector is the only sector providing non-trivial physically relevant BRS and anti-BRS invariant state, which is the Gibbs state (Sec 2.5).

In Sec 3, we outline basic elements of the Birkhoffian mechanics[20] including the Lie-isotopic construction (Sec 3.1) and Birkhoff's equations (Sec 3.2).

In Sec 4, we review the path integral analysis on the generalized phase space[18] which is in fact the path integral approach to Birkhoffian mechanics. The supersymmetric Birkhoff function as well as the Hamilton function of Sec 2 find their geometrical meaning according to the Liouvillian formulation (Sec 4.1). In Birkhoffian mechanics, the anti-BRS charge has to be modified to preserve the structure of the algebra of $ISp(2)$ (Sec 4.2). The equation for anti-BRS invariant state is also modified. Analysis shows that among the even-ghost sectors only $2n$-ghost sector provides non-trivial physically satisfactory solution for the BRS and anti-BRS invariant state, which appears to have again the Gibbs form. The KMS condition remaines unchanged (Sec 4.3).

2 Path Integral Approach to Hamiltonian Mechanics

Path integral approach to Hamiltonian mechanics arised from the path integral approach to stochastic systems. We start with a brief sketch of basic results of the functional integral formulation of the stochastic equations.

Zinn-Justin[4] proved that for a general distribution of the noise η

$$d\rho(\eta) = exp(-\sigma(\eta))d\eta \tag{1}$$

the associated generating functional

$$Z_F(J) = \int D\eta Da\ \delta[F_i(a) - \eta_i]detM\ exp(-\sigma(\eta))exp[\int dx J_i a_i(x)] \tag{2}$$

for a general stochastic diffusion equation, $F_i(a(x)) = \eta_i(x)$, where a is a field and $M_{ij}(x,y) = \delta F_i(a(x))/\delta a_j(y)$, can be rewritten, by exponentiating the δ-function and determinant, as

$$Z_F(J) = \int DaD\bar{c}DcDq\ \exp[-S(a,c,\bar{c},q) + \int dx J_i(x)a_i(x)] \tag{3}$$

Here, the associated action S is

$$S = -W(q) + \int dx\ q_i(x)F_i(a) - \int dxdy\ \bar{c}_i(x)M_{ij}(x,y)c_j(y) \tag{4}$$

and c and $\bar{c}$ are auxiliary anticommuting (Grassmannian) fields - ghosts - appeared due to the exponentiation of the determinant. This action is found to be invariant under the following transformations that resemble the BRS ones of gauge theory:

$$\begin{aligned} \delta a_i(x) &= \epsilon c_i(x) \\ \delta c_i(x) &= 0 \\ \delta \bar{c}_i(x) &= \epsilon q_i(x) \\ \delta q_i(x) &= 0 \end{aligned} \tag{5}$$

It is remarkable that the associated anti-BRS transformations $\bar{\delta}$ leave the action S invariant too only if the following potential conditions hold:

$$\frac{\delta F_i}{\delta a_j} = \frac{\delta F_j}{\delta a_i} \tag{6}$$

The origin of these conditions is known to be Onsager principle of microreversibility so that the supersymmetry of the stochastic systems is gained owing to Onsager principle.

Correlation functions can be obtained from the generating functional (2) in a usual way

$$\langle a_i(0)a_j(t_1)\cdots a_k(t_m)\rangle = \frac{\partial^m Z_F(J)}{\partial J_i(0)\cdots\partial J_k(t_m)|_{J=0}} \tag{7}$$

2.1 Path integral for Hamiltonian mechanics

Gozzi[3] proposed this formalism to use in a classical mechanical system putting the weight of the noise to be the delta function,

$$d\rho(\eta) = \delta(\eta)d\eta \tag{8}$$

Indeed, it is reasonable to find analogue of the supersymmetric BRS symmetry (5) in classical mechanics provided that the stochastic action (4) is supersymmetric invariant for general noise distribution, a particular case of which is the delta function, or "no noise", distribution (8).

Due to Hamiltonian formulation of classical machanics, Hamilton's equations read

$$\dot{a}^i(t) = \omega^{ij}\partial_j H(a(t)) \tag{9}$$

where $a^i = (q^1,\ldots,q^n,p_1,\ldots,p_n), i,j,... = 1,...,2n$, are coordinates on the phase space M_{2n}. H is a Hamiltonian and $\omega^{ij} = -\omega^{ji}$ is a symplectic tensor

$$(\omega^{ij}) = \begin{pmatrix} O & I \\ -I & O \end{pmatrix} \tag{10}$$

Fundamental Poisson brackets have the form $\{a^i, a^j\} = \omega^{ij}$.

In the operatorial approach to classical mechanics[11, 12], one has a probability density function $\rho(a,t)$ on phase space the time evolution of which is given by

$$\frac{\partial}{\partial t}\rho = -\{\rho, H\} = -L\rho(a,t) \tag{11}$$

where the Liouville operator

$$L = -\partial_i H\omega^{ij}\partial_j \tag{12}$$

It is wellknown in quantum field theory that if one has an operatorial formulation one can also construct a corresponding path integral formulation. Thus, having the Liouvillian (operatorial) approach to classical mechanics together with path integral approach for general stochastic (and non-stochastic) dynamical systems, one can try to formulate a path integral approach to Hamiltonian mechanics.

Gozzi, Reuter and Thacker[6] suggested to write down the generating functional for Hamiltonian mechanics in the following simple form:

$$Z = \int Da\ \delta(a^i - a^i_{cl}) \tag{13}$$

where a^i_{cl} are solutions of the Hamilton's equations (9). Thus, the measure is related to a space of classical solutions. The delta function forces the system to be the classical one, and can be rewritten, due to (9), as

$$\delta(a^i - a^i_{cl}) = \delta[\dot{a}^i - \omega^{ij}\partial_j H] det(\partial_t \delta^i_j - \omega^{ik}\partial_k\partial_j H) \tag{14}$$

Then, exponentiating the delta function and the determinant using auxiliary commuting variables q_i and anticommuting variables c_i and $\bar{c}_i$, respectively, we have for the generating functional (13)

$$Z = \int DaDqDcD\bar{c}\ exp[i \int dt\mathcal{L}] \tag{15}$$

where the Lagrangian

$$\mathcal{L} = q_i(\dot{a}^i - \omega^{ij}\partial_j H) + i\bar{c}_i(\partial_t \delta^i_j - \omega^{ik}\partial_k\partial_j H)c^j \tag{16}$$

Associated *Hamilton function* can be then immediately derived from the Lagrangian (16)

$$\mathcal{H} = q_i\omega^{ij}\partial_j H + i\bar{c}_i\omega^{ik}\partial_k\partial_j H c^j \tag{17}$$

By the use of $\mathcal{H}$, ordinary way to calculate the equal-time (anti-)commutators between the fields then yields

$$[a^i, q_j] = i\delta^i_j, \quad [\bar{c}_i, c^j] = \delta^i_j \tag{18}$$
$$[a^i, a_j] = [q^i, q_j] = [c^i, c^j] = [\bar{c}_i, \bar{c}_j] = 0$$

We see that the commutator between canonical variables a^i vanish identically. This means obviously that we are within a classical theory.

2.2 Supersymmetric charges

Due to the BRS invariance of the action (4), it is not surprise that the classical mechanical Hamilton function (17) is invariant under some BRS transformations like (5). Indeed, it is easy to check that the Hamilton function $\mathcal{H}$ is invariant under the following set of BRS transformations:

$$\delta a^i = \epsilon c^i, \quad \delta c^i = 0, \quad \delta \bar{c}_i = \epsilon q_i, \quad \delta q_i = 0 \tag{19}$$

where ϵ is a Grassmannian parameter. Note that the BRS transformations have the property $\delta^2 = 0$. This supersymmetry (19) is generated by the following five supersymmetric conserved charges:

$$Q = i c_i q_i \tag{20}$$

$$\bar{Q} = i \bar{c}_i \omega^{ij} q_j \tag{21}$$

$$C = c^i \bar{c}_i \tag{22}$$

$$K = \frac{1}{2} \omega_{ij} c^i c^j \tag{23}$$

$$\bar{K} = \frac{1}{2} \omega^{ij} \bar{c}_i \bar{c}_j \tag{24}$$

Here, Q and $\bar{Q}$ are nilpotent BRS and anti-BRS operators respectively,

$$Q^2 = 0, \quad \bar{Q}^2 = 0 \tag{25}$$

The generators (20)-(24) form the algebra of $ISp(2)$ group

$$\begin{aligned}
[Q, Q] = [\bar{Q}, \bar{Q}] = [Q, \bar{Q}] = 0 \\
[C, Q] = Q \\
[C, \bar{Q}] = -\bar{Q} \\
[K, Q] = [\bar{K}, \bar{Q}] = 0 \\
[\bar{K}, Q] = \bar{Q} \\
[K, \bar{Q}] = Q \\
[K, \bar{K}] = C \\
[C, K] = 2K, \quad [C, \bar{K}] = -2\bar{K}
\end{aligned} \tag{26}$$

2.3 Geometrical interpretation of supersymmetric charges

The algebra (26) can be realized in a usual differential operator form. It is straightforward to verify that with[6, 7]

$$\bar{c}_i = \frac{\partial}{\partial c^i}, \quad q_i = -i\frac{\partial}{\partial a^i} \tag{27}$$

for canonical conjugates the (anti-)commutation relations (18) are satisfied. So, one has the following differential operator representations for the charges (20)-(24)

$$Q = c^i \frac{\partial}{\partial a^i} \tag{28}$$

$$\bar{Q} = \omega^{ij} \frac{\partial}{\partial c^i} \frac{\partial}{\partial a^i} \tag{29}$$

$$C = c^i \frac{\partial}{\partial c^i} \tag{30}$$

$$K = \frac{1}{2}\omega_{ij} c^i c^j \tag{31}$$

$$\bar{K} = \frac{1}{2}\omega^{ij} \frac{\partial}{\partial c^i} \frac{\partial}{\partial c^i} \tag{32}$$

Ghost variables c^i play in fact the role of usual 1-forms da^i because their equation stemmed from $\mathcal{L}$ is just the equation for Jacobi fields, the first variations δa^i. It is then easy to interpret these charges in symplectic geometrical terms. The nilpotent BRS charge Q and anti-BRS charge $\bar{Q}$ act as an exterior derivative and exterior co-derivative on phase space, respectively. The ghost number charge C counts 1-form and vector number (+1 for 1-form c^i and -1 for tangent vector $\bar{c}_i$). It has integer eigenvalues running from 0 to $2n$, for ghost-dependent states. K and $\bar{K}$ are symplectic 2-form and simplectic bivector, respectively, and their conservation corresponds to the Liuoville theorem of classical mechanics. In general, one may state that $ISp(2)$ reflects geometry of the phase space.

2.4 Dynamical supersymmetric charges

In addition to the five charges (20)-(24), there are also two conserved charges, for conserved Hamiltonian H,

$$Q_H = e^{\beta H} Q e^{-\beta H} = Q - \beta N \tag{33}$$

$$\bar{Q}_H = e^{-\beta H}\bar{Q}e^{\beta H} = \bar{Q} + \beta\bar{N} \tag{34}$$

where

$$N = c^i\partial_i H \tag{35}$$

$$\bar{N} = \bar{c}_i\omega^{ij}\partial_j H \tag{36}$$

and β is a real parameter. They are *dynamical* supersymmetric quantities. Indeed, their anticommutator yields the Hamilton function $\mathcal{H}$[8, 9]

$$[Q_H, \bar{Q}_H] = 2i\beta\mathcal{H} \tag{37}$$

$$[Q_H, Q_H] = [\bar{Q}_H, \bar{Q}_H] = 0$$

They are nilpotent operators due to the property (25),

$$Q_H^2 = 0, \quad \bar{Q}_H^2 = 0 \tag{38}$$

The supersymmetry (37) is a genuine, dynamical $N = 1$ supersymmetry of Hamiltonian mechanics including one (anti-)supercharge and the Hamilton function $\mathcal{H}$ as its generators.

2.5 Gibbs state and KMS condition

Inserting differential operators (27) into the Hamilton function (17) one obtaines[14]

$$\mathcal{H} = -i\omega^{ij}\partial_j H\partial_j + i\omega^{ik}\partial_k\partial_j H c^j\frac{\partial}{\partial c^i} \tag{39}$$

Thus, in the case the density $\rho = \rho(a, c)$ does not contain ghost variables c^i, *i.e.* for scalar density $\rho(a)$, we reproduce the Liouvillian (12)

$$\mathcal{H}_{|c=0} = -i\omega^{ij}\partial_j H\partial_j = -iL \tag{40}$$

It is wellknown that the system is ergodic if the Liouvillian has a non-degenerate eigenstate ρ_0 with zero eigenvalue,

$$L\rho_0 = 0 \tag{41}$$

for a given energy E, $\rho_0 = \rho_0(E)$.

Due to the dynamical supersymmetry (37) it is important to find BRS and anti-BRS invariant state, *i.e.* the state $\rho(a,c)$ annihilated by both Q_H and $\bar{Q}_H$,

$$Q_H \rho(a,c) = 0 \tag{42}$$

$$\bar{Q}_H \rho(a,c) = 0 \tag{43}$$

General representation of the density $\rho(a,c)$ has the following form:

$$\rho(a,c) = \sum_{k=0}^{2n} \rho_{i_1 \ldots i_k}(a) c^{i_1} \cdots c^{i_k} \tag{44}$$

Here, $\rho_{i_1 \ldots i_k}(a)$ are totally antisymmetric functions. For the mean value of the observable $A(a,c)$

$$A(a,c) = \sum_{k=0}^{2n} A_{i_1 \ldots i_k}(a) c^{i_1} \cdots c^{i_k} \tag{45}$$

where $A_{i_1 \ldots i_k}(a)$ are totally antisymmetric functions, we have

$$\langle A \rangle = \int d^{2n}a d^{2n}c \; A(a,c)\rho(a,c) = 2n! \int d^{2n}a \sum_{s=0}^{2n} A_{1 \cdots s}(a) \rho_{s+1 \cdots 2n}(a) \tag{46}$$

With the use of the definitions (33) and (34) for the charges and the expansion (44), the equations (42) and (43) take the form [1]

$$Q_H \rho(a,c) = \sum_{k=0}^{2n} c^j c^{i_1} \cdots c^{i_k} D_j^- \rho_{i_1 \ldots i_k}(a) = 0 \tag{47}$$

$$\bar{Q}_H \rho(a,c) = \sum_{p=0}^{2n} \sum_{k=0}^{2n} \alpha(p) c^{i_1} \cdots \delta_j^{i_p} \cdots c^{i_k} D_+^j \rho_{i_1 \ldots i_k}(a) = 0 \tag{48}$$

Here, we have denoted

$$D_j^- = \partial_j - \beta \partial_j H \tag{49}$$

$$D_+^i = \omega^{ij}(\partial_j + \beta \partial_j H) \tag{50}$$

and $\alpha(p) = +1$ (-1) for odd (even) p. Consistency of these two equations requires that to get *non-trivial* solutions one of the above equations should

[1] See Appendix A

be satisfied identically [2].

Let us consider the even-ghost sector (even k) and the odd-ghost sector (odd k) separately. This can be done always due to the fact that the Grassmannian algebra can be presented as a direct sum of the even and odd subalgebras.

(a) Even-ghost sector (k is an even number, $k = 0, \dots, 2n$). There are *only* two possibilities to satisfy one of the equations (47)-(48) identically: (i) For $k = 0$ the equation (48) is satisfied identically so that the solution of the first equation (47) is $\rho(a,c) = \kappa exp(+\beta H)$. This solution is evidently not physically reasonable (β is assumed to be a positive parameter). (ii) For $k = 2n$ the equation (47) is satisfied identically so that the solution of the second equation (48) is

$$\rho(a,c) = \kappa e^{-\beta H} c^1 \cdots c^{2n} \tag{51}$$

This solution has a *Gibbs state* form (cf.[6, 14]).

In the even-ghost sector, it appears to be instructive to present the density $\rho(a,c)$ in the following form:

$$\rho = \rho_m(a) K^m, \quad m = 0, \dots, n \tag{52}$$

where K is the generator (23) and, particularly,

$$K^n = n! \Big(det[\omega_{ij}(a)] \Big)^{\frac{1}{2}} c^1 \cdots c^{2n} \tag{53}$$

Note that the functions $\rho_m(a)$ are commuting functions for any m.

It is straightforward to verify that K^m is invariant under Hamiltonian flow that is

$$\mathcal{H} K^m = 0, \quad m = 0, \dots, n \tag{54}$$

The equations (42) and (43) now read

$$Q_H K^m \rho_m(a) = 0 \tag{55}$$

$$\bar{Q}_H K^m \rho_m(a) = 0 \tag{56}$$

[2]See Appendix B

With the aid of some algebra it is straightforward to calculate the commutators

$$
\begin{aligned}
[K, \bar{N}] = N, \quad [K, N] = 0 \\
[K^m, Q_H] = 0 \\
[K^m, \bar{Q}_H] = mK^{m-1}(Q + \beta N)
\end{aligned}
\tag{57}
$$

so that the system (42)-(43) takes the form

$$
K^m(Q - \beta N)\rho_m(a) = 0 \tag{58}
$$

$$
mK^{m-1}(Q + \beta N)\rho_m(a) = 0 \tag{59}
$$

where we have used $\bar{Q}_H \rho_m(a) = 0$. Again, consistency of these equations requires that one of the equations should be satisfied identically to get a *non-trivial* solution of the system (58)-(59): (i) For $m = 0$ (ghost-free sector), the equation (59) is satisfied identically so that solution has the form $\rho(a, c) = \kappa exp(+\beta H)$, which is evidently non-physical solution. (ii) For $m = n$ ($2n$-ghost sector), K^n contains maximal number of ghosts so that $K^n Q = 0$, and hence the equation (58) is satisfied identically. Therefore, the solution is of the Gibbs state form

$$
\rho(a, c) = \kappa e^{-\beta H} K^n \tag{60}
$$

Thus, in the even-ghost sector the only non-trivial physical solution has the Gibbs form (51) provided by $2n$-ghost sector.

(b) Odd-ghost sector. (k is an odd number, $k = 1, \ldots, 2n-1$). There is no any possibility to satisfy identically one of the equations (47)-(48) in this case. Therefore, we are leaved only with *trivial* solution $\rho(a, c) = 0$.

As the result of the analysis of the equations (42)-(43), we may conclude that *only 2n-ghost* sector yields non-trivial physically reasonable solution for BRS and anti-BRS invariant state which appears to be the Gibbs state (51). The other sectors imply only trivial or physically unsatisfactory solutions.

Thus, for exact supersymmetry (37) there are no any other analytic constants of motion besides the energy, and therefore the system is in ergodic phase (unordered motion). Futhermore, it has been proven that if the system is in ordered motion phase, *i.e.* there are more constants of motion, the supersymmetry (37) is broken. This gives a new criterion to detect transitions of the system between ordered and unordered motion phases[15].

It is remarkable that the classical KMS condition[17]

$$\langle \dot{A}_1(0) A_2(t) \rangle_\rho = \frac{1}{\beta} \langle \{A_1(0), A_2(t)\} \rangle_\rho \tag{61}$$

which is known as the origin lying behind the Gibbs state can be derived directly from the supersymmetry (37) without any reference to the explicit form of the ground state[14].

3 Elements of Birkhoffian mechanics

Generalization of Hamiltonian mechanics based on a general 2-form ω depending on phase space coordinates a^i has been developed by Santilli[19, 20]. This generalization is reffered to as Birkhoffian mechanics. Birkhoffian mechanics presents in fact a new mechanics generalizing each and every aspect of the conventional Hamiltonian mechanics.

Consistency of the Birkhoffian mechanics is provided by Lie-isotopic construction[20, 21] assuming derivability of dynamics equations from a variational proncipie, Lie character of the underlying brackets, and existence of a generalized Hamilton-Jacobi theory.

3.1 Lie-isotopy

Algebraically, Lie-isotopy is defined as a lifting of an algebra $\mathcal{A}$ with the product ab, $a, b, \ldots \in \mathcal{A}$, into an algebra $\hat{\mathcal{A}}$ which is the same linear space as $\mathcal{A}$ equipped with a generalized product $a * b$, which preserves the structure of the original algebra $\mathcal{A}$.

For example, a commutator algebra $[\mathcal{A}]$ may be Lie-isotopically lifted into a commutator algebra $[\hat{\mathcal{A}}]$ by the rule

$$\begin{aligned} & [\mathcal{A}] \to [\hat{\mathcal{A}}] \\ [a, b]_{\mathcal{A}} = ab - ba \to [a, b]_{\hat{\mathcal{A}}} = a * b - b * a = & aTb - bTa \end{aligned} \tag{62}$$

where $T \in \mathcal{A}$. It is straightforward to check that the last brackets are antisymmetric and Jacobi identity is satisfied in $[\hat{\mathcal{A}}]$.

3.2 Lie-isotopic lifting of Hamiltonian mechanics

With Pfaffian variational principle[20]

$$\delta S = \delta \int dt [R_i(a)\dot{a}^i - H(a,t)] = 0 \tag{63}$$

we have the following generalization of Hamilton's equations (9) named *Birkhoff's equations*[22]:

$$\dot{a}^i(t) = \omega^{ij}(a)\partial_j H(a(t)) \tag{64}$$

where the generalized fundamental 2-form

$$\Big(\omega^{ij}(a)\Big) = \Big(\partial_i R_j - \partial_j R_i\Big)^{-1} \tag{65}$$

is not the constant (10) and depends on phase space coordinates. It should be noted that the symplectic manifold can be always covered by local charts with constant $\omega(a)$ due to Darboux theorem.

Generalized Poisson brackets then read

$$\{A \hat{,} B\} = \frac{\partial A}{\partial a^i}\omega^{ij}(a)\frac{\partial B}{\partial a^j} \tag{66}$$

and they verify the Lie algebra axioms of antisymmetricity and Jacobi identity. So, fundamental Poisson brackets have the form $\{a^i, a^j\} = \omega^{ij}(a)$. Thus, the Birkhoffian mechanics is a Lie-isotopic lifting of Hamiltonian mechanics due to (cf. (62))

$$\{A, B\} = \partial_i A \omega^{ij} \partial_j B \rightarrow \{A \hat{,} B\} = \partial_i A \omega^{ij}(a)\partial_j B \tag{67}$$

In the case $\omega^{ij}(a) = \omega^{ij} = const$ or, equivalently, $R_i(a) = (\vec{0}, \vec{p})$, Birkhoffian mechanics covers the conventional Hamiltonian one.

One of the particular cases of the Birkhoffian mechanics is provided by choosing $R_i(a) = (0, p_e T^e_f(a))$, where T is a symmetric non-degenerate matrix, $e, f, \ldots = 1, \ldots, n$. This form of $R_i(a)$ obviously provides an off-diagonal form of the symplectic tensor $\omega^{ij}(a)$ so that the action does not depend on the momenta, as it is in the proper Hamiltonian mechanics. Birkhoff's equations (64) are reduced in this case to Hamilton-Santilli equa- tions[23]

$$\dot{a}^i(t) = \omega^{ij} I^k_j(a)\partial_k H(a(t)) \tag{68}$$

where ω^{ij} is defined by (10) and

$$I(a) = diag(I_T, I_T), \quad I_T = (T_{ef} + p_g \partial T_e^g / \partial p_f)^{-1} \tag{69}$$

This example appeares to be a nearest Birkhoffian generalization of the Hamiltonian mechanics which preserves the structure of the latter. However, due to the dependence of the I matrix on phase space coordinates the generalization permits one to incorporate, for instance, *nonlocal* effects leaving the Hamiltonian H to be responsible for the usual potential local forces.

We may conclude that Birkhoffian mechanics is a realization in classical mechanics of the Lie-isotopic algebra. Birkhoffian mechanics is directly universal in the sense of being able to represent much wider class of dynamical systems than the conventional Hamiltonian mechanics does.

It should be noted that the function H entering the Birkhoff's equations (64) does not represent in general the total energy of the system. To avoid confusion Santilli introduced the name *Birkhoffian* for H. We will use the same notation H for Birkhoffian in the next section.

For precise and complete development and numerious physical applications of Birkhoffian mechanics we refer the reader to[20] and references cited therein; see also[26, 27] for a review of Santilli's Lie-isotopic theory.

It should be noted that the representation (65) defines canonical symplectic structure while the symplectic-isotopic structure, on which Hamilton-Santilli and Birkhoff-Santilli mechanics are based, is defined by
$\omega(a)_{ij} T^{jk}(t, a, \dot{a}, \ldots)$, where T^{jk} is a Lie-isotopic element[23]. All possible integral terms, which are responsible for non-local effects, are embedded into the element T^{jk}.

Also, it seems to be interesting to suppose that Lie-isotopic element may include *(pseudo)differential* operators so that the underlying algebra $\hat{\mathcal{A}}$ becomes *non-commutative.* This follows the line of reasoning by Gozzi and Reuter[29] who proposed quantum-deformed exterior differential calculus on the phase space to construct an analogue of the conventional exterior calculus, for quantum mechanics. Formally, starting point of their studies is very similar to the one of Lie-isotopic generalization, with the generalized product being of special type, namely, Moyal star-product[30], $a * b = aTb$, $T = exp[i\hbar\bar{\partial}_i \omega^{ij} \vec{\partial}_j/2]$. The Poisson bracket generalization is due to (62), with the additional factor $1/i\hbar$. Here, pseudodifferential operator T provides nonlocal properties peculiar to quantum mechanics. According to this approach[29], quantum mechanics may be treated as a smooth deformation of the classical one.

4 Path integral approach to Birkhoffian mechanics

To construct path integral approach to Birkhoffian mechanics one need to rerun the procedure made in Sec 2 but starting with Birkhoff's equations (64) instead of Hamiltonian ones. The difference arises due to including the dependence of the 2-form $\omega(a)$ on phase space coordinates[18].

Generating functional Z for Birkhoffian mechanics has the same form as (13) but a^i_{cl} are now solutions of Birkhoff's equations (64). The delta function in (13) then can be rewritten according to (64) as

$$\delta(a^i - a^i_{cl}) = \delta[\dot{a}^i - \omega^{ij}(a)\partial_j H]det\Big(\partial_t \delta^i_j - \partial_k(\omega^{ik}(a)\partial_j H)\Big) \tag{70}$$

So, associated Lagrangian $\mathcal{L}$ in the generating functional (15) takes the form

$$\mathcal{L} = q_i(\dot{a}^i - \omega^{ij}(a)\partial_j H) + i\bar{c}_i\Big(\partial_t \delta^i_j - \partial_k(\omega^{ik}(a)\partial_j H)\Big)c^j \tag{71}$$

It has been proven[18] that the generalized Lagrangian $\mathcal{L}$ is invariant under linear as well as nonlinear canonical transformations. One can easily read off *Birkhoff function* $\mathcal{H}$ from (71)

$$\mathcal{H} = q_i\omega^{ij}(a)\partial_j H + i\bar{c}_i\partial_k(\omega^{ik}(a)\partial_j H)c^j \tag{72}$$

which evidently coincides with the Hamilton function (17) of Hamiltonian mechanics in the case $\omega(a) = \omega$.

4.1 Liouvillian formulation of Birkhoffian mechanics

In terms of Hamiltonian vector field[24, 25]

$$h^i = \omega^{ij}(a)\partial_j H \tag{73}$$

and differential operator realization (27) the Birkhoff function (72) can be rewritten simply as

$$\mathcal{H} = -i\ell_h \tag{74}$$

where ℓ_h is a Lie-isotopic derivative along a vector field h

$$\ell_h = h^i\partial_i + (\partial_k h^i)c^k \frac{\partial}{\partial c^i} \tag{75}$$

This gives a geometrical meaning of the Birkhoff function $\mathcal{H}$, namely, the latter is $(-i)$ times the Lie derivative along the Hamiltonian vector field. Particularly, the fact that the BRS charge Q commute with the Birkhoff function $\mathcal{H}$ is a reflection of the wellknown differential geometrical property that the exterior derivative d always commutes with a Lie derivative, $[d, \ell] = 0$.

Liouville equation (11) for general density $\rho(a, c, t)$ then takes the following generalized form:

$$\partial_t\rho = -\{\rho \hat{,} H\} = -\ell_h\rho \tag{76}$$

This equation coincides with the original Liouville equation (11) when ρ does not depend on ghost variables c^i, and $\omega(a) = \omega$, with the identification of the Liouvillian, $L = \ell_h$.

4.2 Modified anti-BRS charge

One can observe that the anti-BRS charge $\bar{Q}$ defined by (21), unlike the BRS charges Q, is no longer nilpotent and does not commute with the Birkhoff function (72). To regain the supersymmetry for the Birkhoffian generalization, one should try to keep the algebra (26) of $ISp(2)$ group unchanged by appropriate modifying of the expression (21) for the anti-BRS charge. This can be done by direct use of the commutator $[\bar{K}, Q] = \bar{Q}$ of the algebra (26). The result is straightforward and reads[18]

$$\bar{Q} = i\bar{c}_i\omega^{ij}(a)q_j - \frac{1}{2}(\partial_k\omega^{ij}(a))c^k\bar{c}_i\bar{c}_j \tag{77}$$

This new definition of the anti-BRS charge provides all the algebraic and transformational properties of the previous scheme of Sec 2. Particularly, cohomology of the anti-BRS charge is isomorphic to the conventional de Rham cohomology[18].

Dynamical supercharge $\bar{Q}_H$ due to (77) and the definition (34) now reads

$$\bar{Q}_H = i\bar{c}_i\omega^{ij}(a)q_j - \frac{1}{2}(\partial_k\omega^{ij}(a))c^k\bar{c}_i\bar{c}_j + \beta\bar{c}_i\omega^{ij}(a)\partial_j H \tag{78}$$

This charge can be casted into the following form:

$$\bar{Q}_H = \bar{c}_i D^i_+ - \frac{1}{2}f^{kl}{}_m c^m\bar{c}_k\bar{c}_l \tag{79}$$

where

$$f^{ijk} = \omega^{im}\partial_m\omega^{jk} - \omega^{jm}\partial_m\omega^{ik} \tag{80}$$

and the operators D^i_+ are given by the definition (50). In the field theoretical BRS technique, the operators placed similarly as D^i_+ in (79) play the role of the generators of some Lie group characterizing symmetry of the theory. It is easy to check that the operators D^i_+ satisfy the commution rule

$$[D^i_+, D^j_+] = f^{ijk}D^+_k \tag{81}$$

and, owing to the identity $\omega^{im}\partial_m\omega^{jk}+\omega^{km}\partial_m\omega^{ij}+\omega^{jm}\partial_m\omega^{ki} = 0$, the function f^{ijk} satisfies the identity

$$f^{ijk} + f^{kij} + f^{jki} = 0 \tag{82}$$

so that the operators D^i_+ constitute a Lie algebra. Note that the algebra defined by (81) is specific for Birkhoffian mechanics since $f^{ijk} = 0$ in the Hamiltonian case.

4.3 Anti-BRS invariant state

With the modified anti-BRS charge (78) we have a new form of the equation for anti-BRS invariant state while the equation for BRS invariant state remaines the same.

These equations mean, in general, that to describe the ergodic phase the density ρ must be BRS and anti-BRS invariant. However, it should be noted that there are *classes* of the solutions of the equations (42) and (43) that are characterized by different ghost numbers. Indeed, trivial solutions of the equation (42) characterized by ghost number m have due to the nilpotency of Q_H the form $\rho = Q_H\chi$, where χ should have the ghost number $m-1$.

We will assume that two solutions ρ and ρ' of the same ghost number m are equivalent if

$$\rho - \rho' = Q_H \chi \tag{83}$$

so that they belong to the same class of BRS cohomology. There are $2n$ cohomology classes of solutions. Similarly, for the anti-BRS invariance equation (43) we have the anti-BRS cohomology classes of the solutions. It has been shown that the cohomology of the modified anti-BRS operator is isomorphic to de Rham cohomology[18].

The most interesting to identify are 0, 1, and $2n$-ghost states:

(i) *0-ghost state.* 0-ghost (ghost-free) state is anti-BRS invariant because the anti-ghosts entering the anti-BRS operator (78) annihilate any ghost-free state.

(ii) *2n-ghost state.* $2n$-ghost state is BRS invariant since the BRS operator (33) adds an extra ghost to the $2n$-ghost state, and therefore annihilates it.

(iii) *1-ghost state.* Anti-BRS invariance property of the 1-ghost state appears to be related to the operator D^i_+. Indeed, the pair of anti-ghosts in the last term of the anti-BRS operator definition (78) annihilates any 1-ghost state while one anti-ghost entering the first term of the operator never can do it. Therefore, the condition $\bar{Q}_H \rho = 0$ is equivalent to

$$D^i_+ \rho = 0 \tag{84}$$

This means that 1-ghost state is anti-BRS invariant if and only if it is D^i_+-invariant. However, it should be stressed that since it may occur that $\rho = \bar{Q}_H \chi$ (χ has the ghost number two) there is *no* one-to-one corespondence between the space of D^i_+-invariant 1-ghost states and the space of cohomology class of 1-ghost states.

In addition, the ergodicity condition

$$\mathcal{H} \rho = 0 \tag{85}$$

is provided according to (37) by the pair of the equations (42) and (43), and hence the density ρ is time independent due to the generalized Liouville equation (76).

With the use of the definitions (33) and (78) for the charges and the expansion (44) for the density, the equations (42) and (43) take the form[3]

$$Q_H \rho(a,c) = \sum_{k=0}^{2n} c^j c^{i_1} \cdots c^{i_k} D_j^- \rho_{i_1 \dots i_k}(a) = 0 \quad (86)$$

$$\bar{Q}_H \rho(a,c) = \sum_{p=0}^{2n} \sum_{k=0}^{2n} \alpha(p) c^{i_1} \cdots \delta_j^{i_p} \cdots c^{i_k} D_+^j \rho_{i_1 \dots i_k}(a) - \frac{1}{2} c^l \sum_{p \neq q}^{2n} \sum_{k=0}^{2n} \alpha(p)\alpha(q) c^{i_1} \cdots \delta_i^{i_p} \cdots \delta_j^{i_q} \cdots c^{i_k} \omega^{ij}{}_{,l} \rho_{i_1 \dots i_k}(a) = 0 \quad (87)$$

Again, as in Sec 2.5, we are leaved in effect to satisfy identically the first equation (86) because it implies itself physically irrelevant solution characterized by $exp(+\beta H)$. This can be done *only* with the choice $k = 2n$. Analogue of the full analysis of the system (42)-(43) made in Sec 2.5 for Hamiltonian mechanics is complicated for the Birkhoffian generalization due to the presence of the additional derivative term in (87).

(a) *Even-ghost sector.* Nevertheless, one may analyse the whole *even-ghost* sector by means of the representation (52). Indeed, tedious calculations show that all the commutators (57) are still valid in the Birkhoffian case so that in the even ghost sector we have the only meaningful solution, the Gibbs state (60). Furthermore, since K^n, being proportional to the phase space volume form ω^n, is still invariant under Hamiltonian flow the solution (60) with $\omega_{ij} = \omega_{ij}(a)$ is defined only by the factor $exp(-\beta H(a))$ so that it is indeed a Gibbs state (cf.[18]).

(b) *Two-dimensional case.* Calculations in the case of two-dimensional phase space ($n = 1$) show [4] that the general system (86)-(87) has a non-trivial physically meaningful solution only for $2n$-ghost sector of the density $\rho(a,c)$. So, the $2n$-ghost sector seems to be the only ghost sector which is responsible for "stable" BRS and anti-BRS invariant state.

It is remarkable that the KMS condition (61) remains untouched[18] in Birkhoffian mechanics despite the fact that there is the additional derivative term in $\bar{Q}_H$ due to the modified definition (78).

[3] See Appendix A

[4] See Appendix C

Acknowledgments

The author would like to thank R.M.Santilli for helpful suggestions, and E.Gozzi for very useful correspondence.

Appendices

A

Using the anticommutator

$$\bar{c}_j c^i + c^i \bar{c}_j = \delta^i_j$$

we find

$$\bar{c}_j c^{i_1} \cdots c^{i_k} = \sum_{p=0}^{2n} \alpha(p) c^{i_1} \cdots \delta^{i_p}_j \cdots c^{i_k} + c^{i_1} \cdots c^{i_k} \bar{c}_j$$

and

$$\bar{c}_i \bar{c}_j c^{i_1} \cdots c^{i_k} = \sum_{p \neq q}^{2n} \alpha(p)\alpha(q) c^{i_1} \cdots \delta^{i_p}_i \cdots \delta^{i_q}_j \cdots c^{i_k}$$

$$+ \sum_{p=0}^{2n} \alpha(p) c^{i_1} \cdots \delta^{i_p}_j \cdots c^{i_k} \bar{c}_i + \sum_{p=0}^{2n} \alpha(p) c^{i_1} \cdots \delta^{i_p}_i \cdots c^{i_k} \bar{c}_j$$

$$+ c^{i_1} \cdots c^{i_k} \bar{c}_i \bar{c}_j$$

where $\alpha(p) = +1$ (-1) for odd (even) p. The terms in the r.h.s. of the last two equations containing $\bar{c}_i$ annihilate $\rho_{i_1 \ldots i_k}(a)$ identically since it does not depend on ghosts.

B

The system (47)-(48) for arbitrary *non-zero* ghost-dependent parts reduces to

$$D^-_j \rho_{i_1 \ldots i_k}(a) = 0$$

$$D^j_+ \rho_{i_1 \ldots i_k}(a) = 0$$

that is due to the definitions (49) and (50)

$$(\partial_j - \beta \partial_j H)\rho(a) = 0$$

$$\omega^{jk}(\partial_k + \beta \partial_k H)\rho(a) = 0$$

Then, looking for solutions in the form $\rho(a) = \kappa exp(\gamma H)$, where κ and γ are constants, we find that $\kappa = 0$, *i.e.* there is only trivial solution, $\rho(a) = 0$.

C

Let us consider a two-dimensional phase space, $n = 1$. It is characterized by the symplectic tensor $\omega^{12}(a) = -\omega^{21}(a)$. The general expansion (44) then reads

$$\rho(a, c) = \rho_0 + \rho_1 c^1 + \rho_2 c^2 + \rho_{12} c^1 c^2$$

The general system of equations (47)-(48) reduces to

$$c^i D_i^- \rho(a, c) = 0 \qquad (C1)$$

$$(\bar{c}_i D_+^i - \frac{1}{2}\omega^{kl}{}_{,m} c^m \bar{c}_k \bar{c}_l)\rho(a, c) = 0 \qquad (C2)$$

which in turn implies

$$c^1 D_1^- \rho_0 = 0$$

$$c^2 D_2^- \rho_0 = 0$$

$$c^1 c^2 (D_1^- \rho_2 - D_2^- \rho_1) = 0$$

for the equation (C1) and

$$\omega^{12}(D_2^+ \rho_1 - D_1^+ \rho_2) = 0$$

$$c^1(\omega^{12} D_1^+ + \omega^{12}{}_{,1})\rho_{12} = 0$$

$$c^2(\omega^{12} D_2^+ + \omega^{12}{}_{,2})\rho_{12} = 0$$

for the equation (C2). This series of equations for the functions $\rho_0,\ \rho_1,\ \rho_2$ and ρ_{12} implies

$$\rho_0 = \kappa_) e^{+\beta H}$$

$$\rho_1 = \rho_2 = 0$$

$$\rho_{12} = \kappa e^{-\beta H - ln\omega^{12}}$$

We see that 0-ghost sector and all odd-ghost sectors of the two-dimensional case of Birkhoffian mechanics are not relevant while the 2-ghost sector yields good solution.

This properties might go beyond two dimensions so that it may occur that in general the BRS and anti-BRS invariant solutions are non-trivial and physically acceptable only in $2n$-ghost sector, as it does in the Hamiltonian mechanics.

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